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SYMPLECTIC DECOMPOSITION OF SYMPLECTIC SUBSPACES

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Abstract. We introduce a decomposition on a symplectic subspace determined by symplectic structure and study its properties. As a consequence, we give an elementary proof of the deformation of the Grassmannians of symplectic subspaces to the complex Grassmannians.

1. Introduction

On a $2n$-dimensional symplectic vector space V with a symplectic structure ω, the set of all $2k$-dimensional symplectic subspaces $S \subset V$ is called the *symplectic Grassmannian* $Gr^{Sp}(2k,2n)$. This is an open subset of the real Grassmannian $Gr(2k,2n)$ and contains the complex Grassmannian $Gr_{\mathbb{C}}(k,n)$ consisted of those subspaces S which are complex for a linear complex structure on V.

In [2], the symplectic Grassmannian $Gr^{Sp}(2k,2n)$ is introduced and identified as

$$Gr^{Sp}(2k,2n) \simeq \frac{Sp(2n,\mathbb{R})}{Sp(2k,\mathbb{R})\,Sp(2n-2k,\mathbb{R})},$$

and we discussed natural inclusions

$$Gr_{\mathbb{C}}(k,n) \subset Gr^{Sp}(2k,2n) \subset Gr(2k,2n).$$

The inclusion $Gr^{Sp}(2k,2n) \subset Gr(2k,2n)$ is the complement of a hyperplane in $\Lambda^{2k}V$, and the inclusion $Gr_{\mathbb{C}}(k,n) \subset Gr^{Sp}(2k,2n)$ is the intersection with $\Lambda^{k,k}_{\mathbb{R}}V$, and topologically it is a deformation retract. In particular, the deformation retract of $Gr^{Sp}(2k,2n)$ is as natural as the deformation from $Sp(2n,\mathbb{R})$ to $U(n)$ which is related to the Siegel upper half space $Sp(2n,\mathbb{R})/U(n)$ representing the space of all compatible complex structures on (V,ω). But the proof in [2] contains tedious issues

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left to the reader, and it turns out that many readers find difficulties to fill in the detail of arguments. The objective of this paper is to show the deformation retract of $Gr^{Sp}(2k, 2n)$ via rather elementary methods.

We introduce a notion of *ω-basis* and *ω-decomposition* of a symplectic subspace S in (V, ω) determined by the symplectic structure ω on V (Definition 4), and show that the decomposition is uniquely determined (Lemma 3). Moreover, we obtain the following key Theorem 7,

Theorem. *For each $2k$-dimensional symplectic subspace S in a $2n$-dimensional Hermitian vector space (V, ω, J, g), there is an 1-parameter family of symplectic subspaces $S(t)$, $t \in [0, 1]$ in V such that $S(0) = S$ and $S(1)$ is a complex subspace in V. Moreover, it is uniquely determined by the ω-decomposition of S.*

As a consequence we show that complex Grassmannian $Gr_{\mathbb{C}}(k, n)$ is a strong deformation retract of the symplectic Grassmannian $Gr^{Sp}(2k, 2n)$ (Theorem 9)

In [2], each element in $Gr^{Sp}(2k, 2n)$ is considered as an element of $\Lambda^{2k}V$ but in this paper we rather consider it as a linear subspace.

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2. PRELIMINARIES

In this section, we recall the definition of symplectic Grassmannians and discuss their basic properties and their relationships with real and complex Grassmannians in [2].

In this article, V always denotes a $2n$-dimensional symplectic vector space with symplectic form ω. Moreover, we consider a Hermitian structure (ω, J, g) on V where J is a complex structure and g is a metric structure on V satisfying

$$\omega(u, v) = g(Ju, v), \; g(Ju, Jv) = g(u, v)$$

for any $u, v \in V$. Note that any two among symplectic, complex and metric structures on V are called compatible with each other if they define the third structure via the relation $\omega(u, v) = g(Ju, v)$ and together they make V into a Hermitian vector space.

Symplectic complement, null space and symplectic rank For any linear subspace P in symplectic vector space (V,ω), we recall the following notions.

(1) The *symplectic complement* of P is $P^{\omega} := \{u \in V \mid \omega(u,v) = 0 \text{ for all } v \in P\}$
(2) The *null space of* P is $N(P) := P \cap P^{\omega}$ and its dimension is called the nullity $n(P) := \dim N(P)$
(3) The *symplectic rank*, or *simply rank*, of P is $r(P) := \max\{r \in \mathbb{N} : (\omega|_P)^r \neq 0\}$
(4) Any two vectors u and v with $\omega(u,v) = 0$ (resp. $g(u,v) = 0$) are called *ω-orthogonal* (resp. *g-orthogonal*)

The null space $N(P)$ is the largest subset of P where the restriction of ω vanishes, and the rank $r(P)$ is half of the maximal possible dimension among subspaces in P where ω is nondegenerate.

Symplectic complements satisfy the following basic properties:

$$\text{(i) } (P^{\omega})^{\omega} = P, \text{ (ii) } (P \cap Q)^{\omega} = P^{\omega} + Q^{\omega} \text{ and (iii) } P \subset Q \text{ iff } P^{\omega} \supset Q^{\omega}.$$

And we also have $\dim P = 2 \cdot r(P) + n(P)$.

Subspaces in symplectic spaces A linear subspace S of V is called *symplectic* if the restriction of ω to S defines a symplectic structure on S, equivalently $n(S) = 0$. Thus, S must be of even dimension. And a subspace P is called *isotropic* (resp. *coisotropic*) if $P \subset P^{\omega}$ (resp. $P^{\omega} \subset P$), and this condition is equivalent to $\omega|_P = 0$; namely $P = N(P)$ (resp. $P^{\omega} = N(P)$). When a subspace is both isotropic and coisotropic, we call it *Lagrangian*.

In particular, one can obtain the following equivalent statements for symplectic subspaces : (1) P is a symplectic subspace in V; (2) $n(P) = 0$; (3) $\dim P = 2 \cdot r(P)$ and (4) $V = P \oplus P^{\omega}$.

It is well-known that the symplectic complement of a symplectic (resp. isotropic) subspace in (V,ω) is symplectic (resp. coisotropic). As a matter of fact, the similar statements for orthogonal complements are also true (See [2] for detail).

Lemma 1. *Let P be any linear subspace in a Hermitian vector space (V,ω,J,g). For the orthogonal complement $P^{\perp}$ of P in V, we have the following.*

(1) $JN(P) = N\left(P^{\perp}\right)$
(2) *P is symplectic iff $P^{\perp}$ is symplectic.*
(3) *P is coisotropic iff $P^{\perp}$ is isotropic.*

Symplectic Grassmannians

Definition 2. Given any $k \leq n$, the set of all $2k$-dimensional symplectic linear subspaces in a $2n$-dimensional symplectic vector space V is called the *symplectic Grassmannian*, and it is denoted as $Gr^{Sp}(2k, V)$, or simply $Gr^{Sp}(2k, 2n)$.

Because the symplectic condition on a subspace is an open condition, a natural choice of topology for the symplectic Grassmannian is one given by being an open subset of the real Grassmannian $Gr(2k, 2n)$ of all $2k$-dimensional real linear subspaces in V.

We pick a metric g on V so that we obtain orthogonal decomposition $V = P \oplus P^{\perp}$ for each subspace P in V. By considering the orthonormal bases of P and $P^{\perp}$ which also give one on V, we identify $Gr(2k, 2n)$ with a homogeneous space of $O(2n)$ with the isotropy subgroup the product of orthogonal groups of P and $P^{\perp}$. Therefore, we have $Gr(2k, 2n) \simeq O(2n)/O(2k)\,O(2n-2k)$. Moreover, for the same reason, there is also a canonical identification between $Gr^{Sp}(2k, 2n)$ and the homogeneous space $Sp(2n, \mathbb{R})/Sp(2k, \mathbb{R})\,Sp(2n-2k, \mathbb{R})$ and if we equip a complex vector space (V, J) with a Hermitian metric g, then the *complex Grassmannian* $Gr_{\mathbb{C}}(k, n)$ can be identified with $U(n)/U(k)\,U(n-k)$.

Therefrom we have the canonical inclusions

$$Gr_{\mathbb{C}}(k, n) \subset Gr^{Sp}(2k, 2n) \subset Gr(2k, 2n)$$

correspond to the following inclusions of homogeneous spaces

$$\frac{U(n)}{U(k)\,U(n-k)} \subset \frac{Sp(2n, \mathbb{R})}{Sp(2k, \mathbb{R})\,Sp(2n-2k, \mathbb{R})} \subset \frac{O(2n)}{O(2k)\,O(2n-2k)}.$$

3. Retract of Symplectic Grassmannians

By Lemma 1, the symplectic Grassmannians $Gr^{Sp}(2k, 2n)$ and $Gr^{Sp}(2n-2k, 2n)$ are dual to each other. Thus we could restrict our attention to only those subspaces in V which are at most half dimensional. For the rest of this section, we assume that $2k \leq n$.

In [2], $Sp(2n, \mathbb{R})$ acts on $V \simeq \mathbb{R}^{2n}$, it also acts on $Gr(2k, 2n)$ which is the space of linear subspaces of V. we have a disjoint union decomposition

$$Gr(2k, 2n) = \coprod_{r=0}^{k} \mathcal{O}_r \ ,$$

where $\mathcal{O}_r$ is the orbit of rank k subspaces. $\mathcal{O}_k = Gr^{Sp}(2k, 2n)$ is the unique open orbit in $Gr(2k, 2n)$. In particular the complement of $Gr^{Sp}(2k, 2n)$ is a hypersurface in $Gr(2k, 2n)$.

On the other hands, we recall $\frac{1}{k!}\omega^k$ is a calibration satisfying the Wirtinger's inequality which is

$$\left|\frac{1}{k!}\omega^k(\varsigma)\right| \leq 1 \quad \text{for all } \varsigma \in Gr(2k, 2n),$$

and equality sign holds if and only if ς is complex subspace, i.e. $\varsigma \in Gr_{\mathbb{C}}(k, n)$ (see [1]). Here $\omega^k(\varsigma)$ denotes

$$\omega^k(e_1 \wedge e_2 ... \wedge e_{2k})$$

where $e_1, e_2, ..., e_{2k}$ is an oriented orthonormal basis of ς. Equivalently, $\frac{\omega^k}{k!}$ is a calibration with contact set $Gr_{\mathbb{C}}(k, n)$. Note we have

$$0 < \left|\frac{1}{k!}\omega^k(\varsigma)\right| \leq 1 \quad \text{for all } \varsigma \in Gr^{Sp}(2k, 2n).$$

In the below, we show that $Gr_{\mathbb{C}}(k, n)$ is actually a strong deformation retract of $Gr^{Sp}(2k, 2n)$.

ω-basis and ω-decomposition of symplectic subspaces

Lemma 3. *Let (V, ω, J, g) be a $2n$-dimensional Hermitian vector space. For any $2k$-dimensional symplectic subspace S in V, there is an ordered orthonormal basis $\{u_1, v_1, ..., u_k, v_k\}$ on S such that*

$$\omega(u_1, v_1) \geq \omega(u_2, v_2) \geq ... \geq \omega(u_k, v_k) > 0$$

and ω-orthogonal for other pairs in the basis.

Proof. For each oriented pair of orthonormal vectors u and v in S, we consider

$$\omega(u, v) = g(Ju, v).$$

And there is a maximizing oriented orthonormal pair of vectors u_1 and v_1 in S satisfying $0 < g(Ju_1, v_1) \leq \sqrt{g(Ju_1, Ju_1)\, g(v_1, v_1)} = 1$. Here, $g(Ju_1, v_1)$ is nonzero because S is symplectic.

For each unit vector w in $S \cap$ *span* $\{u_1, v_1\}^{\perp}$, we observe that the function

$$f(\theta) := g(Ju_1, \cos\theta v_1 + \sin\theta w)$$

has a maximum value at $\theta = 0$ and hence the derivative of f at $\theta = 0$ vanishes, $0 = f'(0) = g(Ju_1, w)$, i.e. $w \perp Ju_1$. Similarly, we also obtain $w \perp Jv_1$. Therefore any g-orthogonal vector w to $span\,\{u_1, v_1\}$ is also ω-orthogonal, and a maximizing pair (u_1, v_1) is isolated.

By Lemma 1, $S\cap\, span\,\{u_1, v_1\}^{\perp}$ is symplectic. Thus we can repeat the above process for $S\cap\, span\,\{u_1, v_1\}^{\perp}$ and obtain an ordered orthonormal basis $\{u_1, v_1, ..., u_k, v_k\}$ of S such that

$$\omega\,(u_1, v_1) = g\,(Ju_1, v_1) \geq g\,(Ju_2, v_2) \geq ... \geq g\,(Ju_k, v_k) > 0,$$

and each pair of vectors in the basis is ω-orthogonal except $(u_i, v_i)\ \ i = 1, ..., k.$ $\square$

Because the value of $g\,(Ju_i, v_i)$ can be repeated, we give a refined index to the basis in the above Lemma 3 so as to have the following definition.

Definition 4. Let (V, ω, J, g) be a $2n$-dimensional Hermitian vector space. For any $2k$-dimensional symplectic subspace S in V, an ordered orthonormal basis

$$\left\{u_1^1, v_1^1, ..., u_{a_1}^1, v_{a_1}^1, u_1^2, v_1^2, ..., u_{a_2}^2, v_{a_2}^2, ..., u_{a_m}^m, v_{a_m}^m\right\}$$

on S such that

$$1 \geq \lambda_1 > \lambda_2 > ... > \lambda_m > 0$$

where $\lambda_i := \omega\left(u_1^i, v_1^i\right) = ... = \omega\left(u_{a_i}^i, v_{a_i}^i\right),\ i = 1, ..., m$, and ω-orthogonal for other pairs in the basis is called a *ω-basis.* Moreover, we call each

$$S_{\lambda_i} := span\left\{u_1^i, v_1^i, ..., u_{a_i}^i, v_{a_i}^i\right\}$$

in S a λ_i-component of S and $S = S_{\lambda_1} \oplus ... \oplus S_{\lambda_m}$ a ω-decomposition (or symplectic decomposition) of S.

Remark. The referee comments that the ω-decomposition is simply the decomposition of the eigenspaces of A^2, where A is the non-degenerate skew-symmetric matrix associated to the symplectic form ω.

It is natural to ask the uniqueness of ω-decomposition of a symplectic subspace S in (V, ω, J, g). At first, we consider the following Lemma.

Lemma 5. *Let S be a $2k$-dimensional symplectic subspace in a $2n$-dimensional Hermitian vector space (V, ω, J, g) and let*

$$\mathcal{B} := \left\{u_1^1, v_1^1, ..., u_{a_1}^1, v_{a_1}^1, u_1^2, v_1^2, ..., u_{a_2}^2, v_{a_2}^2, ..., u_{a_m}^m, v_{a_m}^m\right\}$$

be a ω-basis on S as in Definition 4. If u, v are ordered orthonormal vectors in S with $\omega\,(u, v) = \lambda_1,$ then $span\,\{u, v\} \subset S_{\lambda_1}.$

Proof. By applying the ω-decomposition of S given by the basis $\mathcal{B}$, u can be written as $u = a_0u_0 + a_+u_+$ where u_0 and v_+ are orthonormal vectors in S_{λ_1} and $\bigoplus_{i>1} S_{\lambda_i}$, respectively. Since u is a unit vector, the coefficients a_0 and a_+ satisfy $a_0^2 + a_+^2 = 1$. Similarly, v can be written as $v = b_0v_0 + b_+v_+$, and we have $b_0^2 + b_+^2 = 1$.

Since S_{λ_1} and $\bigoplus_{i>1} S_{\lambda_i}$ are also ω-orthogonal, we get $\omega(u_0, v_+) = \omega(v_0, u_+) = 0$, and we obtain

$$\begin{aligned}\lambda_1^2 = |\omega(u,v)|^2 &= |a_0b_0\omega(u_0,v_0) + a_+b_+\omega(u_+,v_+)|^2\\ &\le \left(a_0^2+a_+^2\right)\left((b_0\omega(u_0,v_0))^2 + (b_+\omega(u_+,v_+))^2\right)\\ &\le \left((b_0\lambda_1)^2 + (b_+\lambda_2)^2\right) \le \left(b_0^2+b_+^2\right)\lambda_1^2 = \lambda_1^2.\end{aligned}$$

Here we use the fact that the value of ω on each S_{λ_i} is in $[-\lambda_i, \lambda_i]$ so that we have $\omega^2(u_0, v_0) \le \lambda_1$ and $\omega^2(u_+, v_+) \le \lambda_2$. To get the equality in the above, we need $a_+ = b_+ = 0$ so that u and v are in S_{λ_1}. This gives the Lemma 5. $\square$

By applying Lemma 5 inductively, we conclude the following Theorem.

Theorem 6. *For each $2k$-dimensional symplectic subspace S in a $2n$-dimensional Hermitian vector space (V,ω,J,g), the ω-decomposition of S is uniquely determined.*

Now, for each symplectic subspace S, we define an 1-paramameter family of symplectic subspaces which in fact gives a path from $[S]$ in $Gr^{Sp}(2k,2n)$ to an element in $Gr_{\mathbb{C}}(k,n) \subset Gr^{Sp}(2k,2n)$.

Theorem 7. *For each $2k$-dimensional symplectic subspace S in a $2n$-dimensional Hermitian vector space (V,ω,J,g), there is an 1-parameter family of symplectic subspaces $S(t)$, $t \in [0,1]$ in V such that $S(0) = S$ and $S(1)$ is a complex subspace in V. Moreover, it is uniquely determined by the ω-decomposition of S.*

Proof. By Lemma 3, we obtain a ω-basis $\{u_1, v_1, ..., u_k, v_k\}$ of S such that

$$g(Ju_1, v_1) \ge g(Ju_2, v_2) \ge ... \ge g(Ju_k, v_k) > 0,$$

and $S = span\{u_1, v_1\} \oplus ... \oplus span\{u_k, v_k\}$. And for each $0 \le t \le 1$ and $i = 1, ..., k$, we define

$$\begin{aligned}U_i(t) &:= \cos\left(\frac{\pi}{2}t\right)u_i + \sin\left(\frac{\pi}{2}t\right)\frac{v_i + Ju_i}{\sqrt{g(v_i+Ju_i, v_i+Ju_i)}}\\ V_i(t) &:= \cos\left(\frac{\pi}{2}t\right)v_i + \sin\left(\frac{\pi}{2}t\right)\frac{-u_i + Jv_i}{\sqrt{g(-u_i+Jv_i, -u_i+Jv_i)}}\end{aligned}$$

and consider

$$S(t) := span\{U_1(t), V_1(t), ..., U_k(t), V_k(t)\}.$$

One can check that $S(t)$ is symplectic for each t, and $S(0) = S$ and $S(1)$ is a complex subspace in V.

To show the family is uniquely determined by the ω-decomposition of S, we need to show that the construction of symplectic subspace $S(t)$ is independent of the choice of the ω-basis on S.

First, we observe $\{U_1(t), V_1(t), ..., U_k(t), V_k(t)\}$ is a ω-basis of $S(t)$.

By applying

$$g(v_i + Ju_i, v_i + Ju_i) = g(-u_i + Jv_i, -u_i + Jv_i) = 2 + 2\omega(u_i, v_i) \text{ and}$$
$$\omega(U_i(t), V_i(t)) = \cos^2\left(\frac{\pi}{2}t\right)\omega(u_i, v_i) + \sin^2\left(\frac{\pi}{2}t\right) > 0,$$

one can check that

(i) $\{U_1(t), V_1(t), ..., U_k(t), V_k(t)\}$ is an orthonormal basis of $S(t)$,

(ii) $\omega(U_1(t), V_1(t)) \geq \omega(U_2(t), V_2(t)) \geq ... \geq \omega(U_k(t), V_k(t)) > 0$, and

(iii) ω-orthogonal for other pairs in $\{U_1(t), V_1(t), ..., U_k(t), V_k(t)\}$.

Thus $\{U_1(t), V_1(t), ..., U_k(t), V_k(t)\}$ is a ω-basis of $S(t)$, indeed.

It is useful to note that if $\omega(u_i, v_i) = \omega(u_j, v_j)$, then $\omega(U_i(t), V_i(t)) = \omega(U_j(t), V_j(t))$. Therefore if $span\{u_i, v_i\}$ and $span\{u_j, v_j\}$ are in the same component of ω-decomposition of S, $span\{U_i(t), V_i(t)\}$ and $span\{U_j(t), V_j(t)\}$ must be in the same component of ω-decomposition of $S(t)$ for each t.

Second, we want to show that all ω-basis on S produces the same $S(t)$ at each t having the same ω-decomposition of $S(t)$.

Suppose $\{u_1, v_1, ..., u_m, v_m\}$ in the given basis $\{u_1, v_1, ..., u_k, v_k\}$ forms the λ_1-component of S, i.e.

$$\lambda_1 = \omega(u_1, v_1) = ... = \omega(u_m, v_m)$$

and

$$S_{\lambda_1} = span\{u_1, v_1, ..., u_m, v_m\}.$$

For another ω-basis $\mathcal{B}'$ on S, by Lemma 5 $\mathcal{B}'$ has an ordered orthonormal pair u, v contained in S_{λ_1} with $\omega(u, v) = \lambda_1$. As the above, we define $U(t)$ and $V(t)$

(A-1)
$$U(t) := \cos\left(\frac{\pi}{2}t\right)u + \sin\left(\frac{\pi}{2}t\right)\frac{v + Ju}{\sqrt{g(v + Ju, v + Ju)}}$$
$$V(t) := \cos\left(\frac{\pi}{2}t\right)v + \sin\left(\frac{\pi}{2}t\right)\frac{-u + Jv}{\sqrt{g(-u + Jv, -u + Jv)}}$$

and we want to show that $U(t)$ and $V(t)$ are in $span\{U_1(t), V_1(t), ..., U_m(t), V_m(t)\}$ for each t.

Since u and v are in S_{λ_1}, we write u and v as

$$u = x_1u_1 + y_1v_1 + ... + x_mu_m + y_mv_m$$
$$v = z_1u_1 + w_1v_1 + ... + z_mu_m + w_mv_m,$$

where the coefficients are real numbers. Since u and v are unit vectors, we have

$$\text{(A-2)} \qquad \sum_{i=1}^{m}\left(x_i^2 + y_i^2\right) = 1 = \sum_{i=1}^{m}\left(z_i^2 + w_i^2\right)$$

and because $\omega(u, v) = \lambda_1$, we get

$$\text{(A-3)} \qquad \begin{aligned} \lambda_1 = \omega(u, v) &= \omega\left(\sum_{i=1}^{m}(x_iu_i + y_iv_i), \sum_{i=1}^{m}(z_iu_i + w_iv_i)\right) \\ &= \sum_{i=1}^{m}(x_iw_i - y_iz_i)\,\omega(u_i, v_i) = \lambda_1\sum_{i=1}^{m}(x_iw_i - y_iz_i). \end{aligned}$$

Here we use the fact that $\{u_1, v_1, ..., u_m, v_m\}$ is a subset of the ω-basis producing λ_1-component in S. By combining above three equations in $(A$-2) and $(A$-3), we obtain

$$\begin{aligned} 0 &= \sum_{i=1}^{m}\left(x_i^2 + y_i^2\right) + \sum_{i=1}^{m}\left(z_i^2 + w_i^2\right) - 2\sum_{i=1}^{m}(x_iw_i - y_iz_i) \\ &= \sum_{i=1}^{m}\left((x_i - w_i)^2 + (y_i + z_i)^2\right) \end{aligned}$$

which implies $x_i = w_i$ and $y_i = -z_i$ for each i, and that u and v can be written

$$\text{(A-4)} \qquad u = \sum_{i=1}^{m}(x_iu_i + y_iv_i),\ v = \sum_{i=1}^{m}(-y_iu_i + x_iv_i).$$

Now, $U(t)$ and $V(t)$ can be written

$$\text{(A-5)} \qquad \begin{aligned} U(t) &= \cos\left(\frac{\pi}{2}t\right)u + \sin\left(\frac{\pi}{2}t\right)\frac{v + Ju}{\sqrt{g(v + Ju, v + Ju)}} \\ &= \sum_{i=1}^{m} x_i\left(\cos\left(\frac{\pi}{2}t\right)u_i + \sin\left(\frac{\pi}{2}t\right)\frac{v_i + Ju_i}{\sqrt{g(v_i + Ju_i, v_i + Ju_i)}}\right) \end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=1}^{m} y_i \left(\cos\left(\frac{\pi}{2}t\right) v_i + \sin\left(\frac{\pi}{2}t\right) \frac{-u_i + Jv_i}{\sqrt{g\left(v_i + Ju_i, v_i + Ju_i\right)}} \right) \\
&= \sum_{i=1}^{m} x_i U_i(t) + \sum_{i=1}^{m} y_i V_i(t) \\
V\left(t\right) &= \cos\left(\frac{\pi}{2}t\right) v + \sin\left(\frac{\pi}{2}t\right) \frac{-u + Jv}{\sqrt{g\left(-u + Jv, -u + Jv\right)}} \\
&= \sum_{i=1}^{m} x_i V_i(t) - \sum_{i=1}^{m} y_i U_i(t)
\end{aligned}
$$

which shows that $U\left(t\right)$ and $V\left(t\right)$ are in $span\left\{U_1\left(t\right), V_1\left(t\right), ..., U_m\left(t\right), V_m\left(t\right)\right\}$. Here we use

$$
\begin{aligned}
g\left(v + Ju, v + Ju\right) &= g\left(-u + Jv, -u + Jv\right) = 2 + 2\omega\left(u, v\right) = 2 + 2\omega\left(u_i, v_i\right) \\
&= g\left(v_i + Ju_i, v_i + Ju_i\right) = g\left(-u_i + Jv_i, -u_i + Jv_i\right)
\end{aligned}
$$

for $i = 1, ..., m$.

By applying this procedure inductively, we can conclude that all ω-basis on S produces the same $S\left(t\right)$ at each t having the same ω-decomposition of $S(t)$. $\square$

From the proof of the Theorem 7, we observe that the expression of u and v in (A-2) is related to $Sp\left(2k, \mathbb{R}\right) \cap O\left(2k\right) = U\left(k\right)$. Therefore, the set of all the ω-bases on a $2k$-dimensional symplectic subspace S is acted by block diagonal unitary matrices in $U(k) \subset O\left(2k\right)$. Thus we have the following corollary.

Corollary 8. *Let S be a $2k$-dimensional symplectic subspace in a $2n$-dimensional Hermitian vector space $\left(V, \omega, J, g\right)$. Suppose S has a ω-basis as in Definition 4, then the set of all the ω-bases on S is acted by $U\left(a_1, a_2, ..., a_m\right)$ which is the set of block diagonal unitary matrices in $U\left(k\right) = Sp\left(2k, \mathbb{R}\right) \cap O\left(2k\right)$ whose blocks have sizes $\left(2a_1, 2a_2, ..., 2a_m\right)$ as real matrices.*

Remark. From (A-4) and (A-5) in Theorem 7, we conclude that the action of block diagonal unitary matrices $U\left(a_1, a_2, ..., a_m\right)$ on the set of all the ω-bases on S and the set of all the ω-bases on $S\left(t\right)$ is compatible via the definition of vectors at t in (A-1).

Now we show that $Gr_{\mathbb{C}}\left(k, n\right)$ is a strong deformation retract of $Gr^{Sp}\left(2k, 2n\right)$, indeed.

Theorem 9. *Let (V, ω, J, g) be a $2n$-dimensional Hermitian vector space. Then the complex Grassmannian $Gr_{\mathbb{C}}(k, n)$ is a strong deformation retract of the symplectic Grassmannian $Gr^{Sp}(2k, 2n)$.*

Proof. First, we take a ω-basis on S for each $[S]$ in $Gr^{Sp}(2k, 2n)$. By applying Theorem 7 for S and the ω-basis, we obtain an 1-parameter family $S(t)$ of symplectic subspaces which is uniquely determined by ω-decomposition of S. As in Theorem 7, the construction of symplectic subspace $S(t)$ is independent of the choice of the ω-basis on S. Therefore we have the following well defined map

$$\begin{array}{ccc} Gr^{Sp}(2k, 2n) \times [0,1] & \longrightarrow & Gr^{Sp}(2k, 2n) \\ (S, t) & \mapsto & S(t). \end{array}$$

which can be easily seen to be a strong deformation retract from $Gr^{Sp}(2k, 2n)$ to $Gr_{\mathbb{C}}(k, n)$. □

Remark. 1. If we vary the complex structure J on V, then the corresponding $Gr_{\mathbb{C}}(k, n)$ will move inside $Gr^{Sp}(2k, 2n)$ and cover the whole symplectic Grassmannian (see [2]).

2. Thus $Gr_{\mathbb{C}}(k, n)$ and $Gr^{Sp}(2k, 2n)$ are homotopically equivalent to each other.

3. The dimensions of λ_i-spaces S_{λ_i} in a ω-decomposition of S is preserved under deformation.

References

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