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NUMERICAL RESULTS ON ALTERNATING DIRECTION SHOOTING METHOD FOR NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS

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ABSTRACT. This paper is concerned with the numerical solutions to steady state nonlinear elliptical partial differential equations (PDE) of the form $u_{xx}+u_{yy}+Du_x+Eu_y+Fu=G$, where D, E, F are functions of x, y, u, u_x, and u_y, and G is a function of x and y. Dirichlet boundary conditions in a rectangular region are considered. We propose alternating direction shooting method for solving such nonlinear PDE. Numerical results show that the alternating direction shooting method performed better than the commonly used linearized iterative method.

1. INTRODUCTION AND PRELIMINARIES

This paper is concerned with the numerical solutions to steady state 2-dimensional nonlinear elliptical partial differential equations of the form

$$u_{xx}+u_{yy}+Du_x+Eu_y+Fu=G \tag{1.1}$$

where D, E, F are functions of x, y, u, u_x, and u_y, and G is a function of x and y. Dirichlet boundary conditions in a rectangular region are imposed. Usually, large sparse algebraic systems are generated when a finite difference discretization scheme is applied. Iterative algorithms are often used to obtain solutions to the algebraic systems. When the partial differential equation is linear so is the algebraic system and many iterative algorithms have been developed for symmetric positive definite systems [2, 9, 14], indefinite systems [7, 8, 14], and nonsymmetric systems [6, 10, 14, 15]. However, when the partial differential equation is nonlinear the algebraic equations generated by the discretization process may also be nonlinear. Commonly used iterative algorithms for solving a nonlinear system involve an "inner-outer" iteration [1]. The outer iteration is devised by a method of successive approximations.

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Douglas [5] solved a "mildly" nonlinear elliptical equation of the form

$$u_{xx} + u_{yy} = f(x, y, u) \tag{1.2}$$

in a rectangular region R, where f is a nonlinear function in u. We note that the equation (1.2) is a special case of the equation (1.1). Dirichlet boundary conditions $u = g(x, y)$ are also imposed. Moreover, Douglas assumed that for some constants m and M, $\frac{\partial f}{\partial u}$ is bounded so that $0 < m \leq \frac{\partial f}{\partial u} \leq M < \infty$.

Let us put a mesh on the rectangular region R as shown in figure 1.1 and let the mesh size h in both x and y directions be the same. Using the 5-point stencil (see figure 1.2) and the central difference method, the partial differential equation at the point (i, j) can be approximated as

$$\frac{u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j}}{h^2} = f(x_i, y_j, u_{i,j})$$

and $u_{i,j} = g(x_i, y_j)$ on the boundary of R.

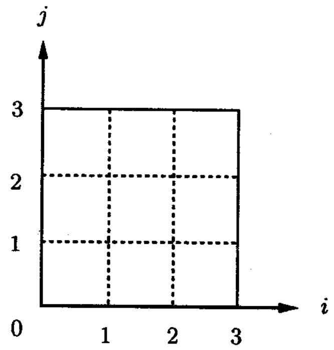

Fig 1.1 Mesh

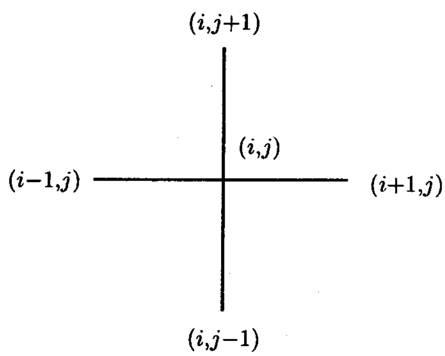

Fig 1.2 Stenci

Instead of using the Picard method for the outer iteration in Douglas' procedure, he used the modified Picard method. A parameter α is introduced in the modified method such that

$$\frac{u_{i+1,j}^{(n+1)} + u_{i-1,j}^{(n+1)} + u_{i,j+1}^{(n+1)} + u_{i,j-1}^{(n+1)} - 4u_{i,j}^{(n+1)}}{h^2} - \alpha u_{i,j}^{(n+1)} = f(x_i, y_j, u_{i,j}^{(n)}) - \alpha u_{i,j}^{(n)} \tag{1.3}$$

The solution of this linear system is accomplished by an inner iteration. This is carried out by means of an alternating direction method [13, 16] that solves linear systems. The optimum value of α, in (1.3), is shown to be [5]

$$\alpha = \frac{1}{2}(M + m).$$

The choice of the procedure carried out in (1.3) improves the outer iteration. It is reported that the number of outer iterations with Picard method is independent of h. In a later section, our numerical results also confirm the analysis.

In this paper, we propose the shooting method in alternating directions for the inner iteration. Generally, shooting methods are techniques for solving ordinary differential equations with boundary values.

Numerical results show that the alternating direction shooting method performed better than the commonly used linearized iterative method. Moreover, the greatest advantage of using the alternating shooting method is that the computation is totally parallel making it suitable for parallel computation resulting in greater speedup. However, we note that the shooting method suffers from stability problems in the same way it does in ordinary differential equations. A possible remedy for these limitations is the consideration of multiple shooting methods [11].

In Section 2, the shooting method as well as their implementations is introduced to solve (1.1). In Section 3, convergence analysis for perturbed linear and nonlinear differential equations are discussed. Theoretical proofs indicate that the convergence is linear. Numerical results agree with the theory. In Section 4, numerical experiments are carried out for the numerical methods described in Section 2 and comparisons are made. Finally, conclusions are given in Section 5.

2. Numerical Methods and Their Implementations

In this section, we describe the numerical methods and their implementations, which are used in this paper, for solving a 2 space dimensional nonlinear partial differential equation (1.1) with Dirichlet boundary conditions.

(A) Shooting Method

A general nonlinear ordinary differential equation with boundary conditions may be written in the form

$$u'' = f(x, u, u'), \ a \le x \le b, \ u(a) = \alpha, \ u(b) = \beta \tag{2.1}$$

where f satisfies the conditions in the following theorem. The theorem, which was proven by Keller [11], assures the existence and uniqueness of the solution to the boundary value problem (2.1).

Theorem 1. *Suppose the function f in the boundary-value problem* (2.1) *is conti-*

-nuous on the set $D = \{(x, u, u') \mid a \le x \le b, -\infty < u < \infty, -\infty < u' < \infty\}$ *and that the partial derivatives* f_u *and* $f_{u'}$ *are also continuous on* D*. If*

(i) $f_u(x, u, u') > 0$, *for all* $(x, u, u') \in D$, *and*

(ii) *a constant* M *exists, with*

$$| f_{u'}(x, u, u') | \le M, \text{ for all } (x, u, u') \in D,$$

then the boundary-value problem has a unique solution.

The shooting method is a numerical technique that solves the nonlinear second-order boundary value problem (2.1). Detailed description can be found in [3]. The procedure is to approximate the solution to the boundary-value problem by using the solution to a sequence of initial-value problems involving a parameter t. The initial value problems have the form

$$u'' = f(x, u, u'), \; a \le x \le b, \; u(a) = \alpha, \; u'(a) = t. \tag{2.2}$$

We solve this by choosing a sequence of the parameters $t = t_k$ so that

$$\lim_{k \to \infty} u(b, t_k) = u(b) = \beta \tag{2.3}$$

where $u(x, t_k)$ denotes the solution to the initial-value problem (2.2) with $u'(a) = t_k$. Since we assume that the boundary value problem satisfies the conditions in Theorem 1, from the uniqueness property the solution $u(x, t_k)$, which satisfies (2.3), is the solution of the boundary value problem (2.1).

We start with a parameter t_0 that determines the initial slope, we then solve the initial value problem (2.2) with $u'(a) = t_0$. After getting the solution $u(x, t_0)$ we check whether $u(b, t_0)$ is sufficiently close to β. If not, we then correct the slope by choosing

$$t_{k+1} = t_k - \frac{g(t_k)}{g'(t_k)}, \; k = 0, 1, 2, \cdots$$

where $g(t_k) = u(b, t_k) - \beta$ and $g'(t_k) = \frac{\partial}{\partial t} u(b, t_k)$. Since we do not have an explicit function for $g'(t)$, we must find an approximation by solving the initial value problem

$$z''(x, t) = f_u(x, u, z) z(x, t) + f_{u'}(x, u, z) z'(x, t), \; a \le x \le b \tag{2.4}$$

with the initial conditions

$$z(a, t) = 0 \text{ and } z'(a, t) = 1$$

where $z(x, t) = \frac{\partial}{\partial t} u(x, t)$.

We then use the Runge Kutta method of order 4 to solve the initial value problem (2.4) for any given t_k to obtain $z(x, t_k)$ and evidently we obtain $z(b, t_k) = \frac{\partial}{\partial t} u(b, t_k)$ that is an approximation of $g'(t_k)$.

(B) ALTERNATING DIRECTION PROCEDURE

We noted that the shooting method is implemented in alternating direction manner. We now describe the procedure for the alternating direction for the equation (1.1).

The alternating direction procedure for the equation is as follows: treat the partial differential equation as an ordinary differential equation by assuming that one of the variables and all corresponding partial derivatives are constants. Therefore equation (1.1) becomes an ordinary differential equation in the variable x

$$u_{xx} + Du_x + Fu = G(x, y) - (u_{yy} + Eu_y) \tag{2.5}$$

where the value of y is fixed and u_y and u_{yy} are approximated by either a finite difference scheme or by solving (1.1) for $u_{yy} + Eu_y$. We note that if D and E consist of u_y, we do the same approximation. Similarly, equation (1.1) becomes an ordinary differential equation in the variable y

$$u_{yy} + Eu_y + Fu = G(x, y) - (u_{xx} + Du_x), \tag{2.6}$$

where the value of x is fixed u_x and u_{xx} are approximated by either a finite difference scheme or by solving equation (1.1) for $u_{xx} + Du_x$. We note that if D and E consist of u_y, then we perform the same approximation.

When the solution u changes, we notice that the approximations in the right side of equations (2.5) and (2.6) are altered. It is assumed that the sequence of changes is getting smaller, and that thus the limit of the approximating solutions approaches the exact solution.

The alternating direction procedure for solving the equation (1.1) is described below. We impose a rectangular mesh on the region as given in Fig. 1.1, where the equation is defined. For each j, the value of y is fixed, thus we solve the equation (2.5) for all j to obtain a half step estimation of the solution to the equation (1.1). Then we use the estimation to approximate the right side of equation (2.6) in which for each i, the value of x is fixed. After solving the equation (2.6) for all i, we obtain a full step estimation of the solution to the equation (1.1). This alternating direction procedure is as follows: the equation (2.5) is solved using x as the variable and then the equation (2.6) is solved using y as the variable. The alternating direction procedure was shown to converge for linear partial differential equations [13, 16]. The algorithm is given below.

Algorithm 1.

(1) *Choose an initial guess* $u^{(0)}$ *and a small value* $\epsilon > 0$;
(2) *Approximate the right hand side of* (2.5) *using* $u^{(0)}$;
(3) *Use the shooting method to solve* (2.5) *to obtain an estimation* $u^{(1/2)}$;
(4) *Approximate the right hand side of* (2.6) *using* $u^{(1/2)}$;
(5) *Use the shooting method to solve* (2.6) *to obtain an estimation* $u^{(1)}$;
(6) *Check* $\| u^{(1)} - u^{(0)} \| < \varepsilon$, *if yes, exit;*
(7) *If no, let* $u^{(0)} = u^{(1)}$ *go to* (2).

(C) APPROXIMATION SCHEMES FOR "CONSTANT VARIABLE"

As mentioned in (B), we derive two ordinary differential equations (2.5) and (2.6) from (1.1) in the variables x and y, respectively. In order to solve (2.5) using the shooting method, we need an approximation for u_y and u_{yy}. We use two different schemes in our study.

(1) Finite difference scheme: Based on the mesh and the stencil given in Fig. 1.1 and 1.2, we let $u_y = \dfrac{u_{i,j+1} - u_{i,j-1}}{2h}$ and $u_{yy} = \dfrac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{h^2}$. Then (2.5) becomes

$$u_{xx} + Du_x + Fu = G(x,y) - \left(\frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{h^2} + E\frac{u_{i,j+1} - u_{i,j-1}}{2h}\right).$$

We note that if the function E depends on u_y, then we also approximate E accordingly. We approximate the equation (2.6) in a similar way.

(2) Getsum scheme: One can also try to approximate $u_{yy} + Eu_y$ by solving $u_{yy} + Eu_y$ from (1.1) which gives

$$u_{yy} + Eu_y = getsum(x, y, u, u_x, u_{xx}) = G(x,y) - u_{xx} - Du_x - Fu$$

where $u_x = \dfrac{u_{i+1,j} - u_{i-1,j}}{2h}$ and $u_{xx} = \dfrac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{h^2}$. Therefore, in this way, (2.5) becomes

$$u_{xx} + Du_x + Fu = G(x,y) - getsum(x, y, u, u_x, u_{xx}).$$

Similarly, we can do the same for (2.6) and we will get

$$u_{yy} + Eu_y + Fu = G(x,y) - \left(\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} + D\frac{u_{i+1,j} - u_{i-1,j}}{2h}\right),$$

and

$$u_{yy} + Du_y + Fu = G(x,y) - getsum(x, y, u, u_y, u_{yy}),$$

where $u_{xx} + Du_x = getsum(x, y, u, u_y, u_{yy}) = G(x, y) - u_{yy} - Eu_y - Fu$.

The computational experiments and the findings resulting from the application of the methods described will be depicted in the following sections.

3. Convergence for Perturbed Linear & Nonlinear D.E.

As we have noted in the preceding section, the right sides of equations (2.5) and (2.6) are changing during the alternating direction procedure; and it is hoped that the changes are getting smaller. In this section, we investigate the convergence in case the right hand sides of the equations (2.5) and (2.6) are perturbed.

We first consider the linear problem without any perturbation as the exact problem (EP)

$$(\text{EP}) \left\{ \begin{array}{l} u'' + p(x)u' + q(x)u = f \\ u(a) = \alpha, \; u(b) = \beta \end{array} \right.$$

where p and q are continuous functions and $q < 0$ for all x in $[a, b]$, and the problem with perturbation ε in the right side as the approximate problem (AP)

$$(\text{AP}) \left\{ \begin{array}{l} u'' + p(x)u' + q(x)u = f + \varepsilon \\ u(a) = \alpha, \; u(b) = \beta \end{array} \right. .$$

If $\bar{u}$ is the solution of (EP) that indicates the exact solution and u^* is the solution of (AP) which is an approximation of (EP), then it is hoped that when $\varepsilon \to 0$ also $u^* \to \bar{u}$. Without loss of generality, we may assume that $[a, b] = [0, 1]$. Let $w = \bar{u} - u^*$, then w satisfies the modified problem (MP)

$$(\text{MP}) \left\{ \begin{array}{l} w'' + p(x)w' + q(x)w = \varepsilon \\ w(0) = w(1) = 0 \end{array} \right. .$$

We claim that the solution w of (MP) satisfies

$$\| w \| < k \mid \epsilon \mid . \tag{3.1}$$

As $\varepsilon \to 0$ and $w \to 0$ for some constants $k > 0$, that implies $\bar{u} - u^* \to 0$ and evidently $u^* \to \bar{u}$. The relation (3.1) indicates the convergence is linear.

To prove (3.1), we first assume that the general solution to the homogeneous equation (HP)

$$(\text{HP}) \; w'' + p(x)w' + q(x)w = 0, \; w(0) = w(1) = 0$$

is of the form

$$w(x) = c_1 f(x) + c_2 g(x)$$

where f and g are linearly independent functions and c_1, c_2 are constants. To solve (MP), we use the method of variations of parameters, the particular solution is

$$w_p(x) = A(x)f(x) + B(x)g(x).$$

By taking derivatives and plugging into (MP), we obtain a system of algebraic equations

$$\begin{cases} f(x)A'(x) + g(x)B'(x) = 0 \\ f'(x)A'(x) + g'(x)B'(x) = \varepsilon \end{cases}$$

Let $\Delta(x) = \begin{vmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{vmatrix}$. Since f and g solves the (HP) and are linearly independent, $\Delta(x)$ is never 0. The solutions for $A'(x)$ and $B'(x)$ are $A'(x) = -\dfrac{\varepsilon g(x)}{\Delta(x)}$ and $B'(x) = \dfrac{\varepsilon f(x)}{\Delta(x)}$.
Consequently we obtain

$$A(x) = \int_0^x -\frac{\varepsilon g(t)}{\Delta t} dt \text{ and } B(x) = \int_0^x \frac{\varepsilon f(t)}{\Delta t} dt.$$

Thus the particular solution to (MP) is

$$w_p(x) = f(x)\int_0^x \frac{-\varepsilon g(t)}{\Delta t} dt + g(x)\int_0^x \frac{\varepsilon f(t)}{\Delta t} dt = \varepsilon \int_0^x \frac{g(x)f(t) - f(x)g(t)}{f(t)g'(t) - g(t)f'(t)} dt.$$

We let

$$I(x) = \int_0^x \frac{g(x)f(t) - f(x)g(t)}{f(t)g'(t) - g(t)f'(t)} dt.$$

The general solution is

$$w(x) = c_1 f(x) + c_2 g(x) + \varepsilon I(x).$$

Applying the boundary conditions we obtain c_1 and c_2 as

$$c_1 = \frac{\varepsilon g(0)I(1)}{f(0)g(1) - f(1)g(0)} \quad \text{and} \quad c_2 = \frac{-\varepsilon f(0)I(1)}{f(0)g(1) - f(1)g(0)}.$$

Therefore the general solution $w(x)$ for the (MP) can be written as

$$w(x) = \varepsilon \left[\frac{g(0)I(1)}{f(0)g(1) - f(1)g(0)} f(x) - \frac{f(0)I(1)}{f(0)g(1) - f(1)g(0)} g(x) + I(x) \right].$$

It clearly show that $w(x) \to 0$ as $\varepsilon \to 0$ linearly; which concludes the proof for the linear case. An alternate proof is given in [4].

As for nonlinear case, we assume the exact problem is

$$\text{(ENP) } u'' = F(u, u', f(x)), \; u(a) = \alpha \text{ and } u(b) = \beta$$

and assume that the approximated problem

$$u'' = F(u, u', f(x)) + \varepsilon,\ \ u(a) = \alpha \text{ and } u(b) = \beta.$$

Suppose that $\bar{u}$ and u^* are the solutions to (ENP) and (ANP) respectively. Let $w = u^* - \bar{u}$. Then using perturbation techniques, it can be shown that w is small and $| w | \ll 1$.

We substitute $u^* = \bar{u} + w$ into (ANP); we obtain

$$\begin{cases} \bar{u}'' + w'' = F(\bar{u} + w, \bar{u}' + w', f(x)) + \varepsilon \\ \bar{u}(a) + w(a) = \alpha \text{ and } \bar{u}(b) + w(b) = \beta \end{cases}.$$

Since w and w' are small, we use Taylor series expansion for $F(\bar{u} + w, \bar{u}' + w', f)$ and keep only the linear terms, we get

$$F(u + w, u' + w', f) = F(u, \bar{u}, f) + \frac{\partial F}{\partial u} w + \frac{\partial F}{\partial u} w'.$$

Thus

$$\bar{u}'' + w'' = F(\bar{u} + \bar{u}', f) + \frac{\partial F}{\partial u} w + \frac{\partial F}{\partial u'} w' + \varepsilon. \tag{3.2}$$

Subtract the (ENP) from the equation (3.2), we obtain

$$w'' = \frac{\partial F}{\partial u} w + \frac{\partial F}{\partial u'} w' + \varepsilon, \quad w(a) = 0 \text{ and } w(b) = 0.$$

Then we have

$$w'' - \frac{\partial F}{\partial u'} w' - \frac{\partial F}{\partial u} w = \varepsilon, \quad w(a) = 0 \text{ and } w(b) = 0.$$

Let $p(x) = -\frac{\partial F}{\partial u'}$ and $q(x) = -\frac{\partial F}{\partial u}$. Then follow the analysis for linear case, we obtain $w \to 0$ as $\varepsilon \to 0$ linearly; which concludes the proof for nonlinear case.

4. Numerical Experiments

In this section we present numerical results on model problems using the three numerical methods that were introduced in Section 2. The model problems that involve two dimensional nonlinear partial differential equations are described below.

Model Problem 1:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + u\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} - u = (1 + x)(y^2 - y - 1), \quad [0, 1] \times [0, 1],$$
$$u(0, y) = y, \quad u(1, y) = 2y, \quad u(x, 0) = 0, \quad u(x, 1) = 1 + x.$$

The exact solution is $u(x, y) = (1 + x)y$.

Model Problem 2:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + 2x\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} - u^2 = -1 - x - (xy)^2, \quad [0,1] \times [0,1],$$
$$u(0,y) = 1 = u(x,0), \quad u(1,y) = 1 + y, \quad u(x,1) = 1 + x.$$

The exact solution is $u(x,y) = 1 + xy$.

Model Problem 3:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + u\frac{\partial u}{\partial x} + \cos y\frac{\partial u}{\partial y} - u = x(1 - 2\sin y), \quad [0,1] \times [0,1],$$
$$u(0,y) = 0 = u(x,0), \quad u(1,y) = \sin y, \quad u(x,1) = x\sin 1.$$

The exact solution is $u(x,y) = x\sin y$.

Model Problem 4:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + u\frac{\partial u}{\partial x} - (\frac{\partial y}{\partial u})^2 - u = -e^{-2x} - e^{-x}\sin y, \quad [0,1] \times [0,1],$$
$$u(0,y) = \sin y, \quad u(1,y) = e^{-1}\sin y, \quad u(x,0) = 0, \quad u(x,1) = e^{-x}\sin 1.$$

The exact solution is $u(x,y) = e^{-x}\sin y$.

4.1. Nonlinear Shooting Method

4.1.1. Getsum Scheme for the Shooting Method

We first study the nonlinear shooting method using the second approximation scheme discussed in 2.C on these four model problems with the uniform mesh sizes $h = \frac{1}{25}, \frac{1}{50}, \frac{1}{100}, \frac{1}{200}$, and $\frac{1}{400}$ on both the x and the y directions. Since the exact solutions are provided for all model problems, we undertake a twofold error analysis: (i) the numerical solution is compared to the exact one, and (ii) the error between two successive approximations. These are shown in Tables 1 and 2, respectively. The errors are computed by the two-norm. The stopping procedure used in these model problems was to end the iteration process whenever successive iterates differed by less than 10^{-6}.

The initial guess of the numerical solution for this scheme is chosen to be the value of the numerical solution in the previous space level. For example, when we shoot in the level $j = 1$, then the approximate solution at $j = 0$, which is the boundary, is the initial guess.

Table 1(a-c) gives the result when the nonlinear shooting method converges or the maximum number of iteration is met in the x direction alone, in the y direction alone, and in the x direction then in the y direction, respectively. From these tables,

(a) in x		Mesh Size h				
		1/25	1/50	1/100	1/200	1/400
Model Problem	1	5.116550E-07	5.557681E-07	6.013077E-07	6.285047E-07	6.359865E-07
	2	6.764964E-07	5.876031E-07	5.120789E-07	4.427052E-07	3.813978E-07
	3	1.879890E-07	1.756370E-07	1.586023E-07	4.508507E-07	3.787524E-07
	4	1.284277E-01	2.602919E-01	5.240115E-01	1.051448E+00	2.106317E+00

(b) in y		Mesh Size h				
		1/25	1/50	1/100	1/200	1/400
Model Problem	1	6.187057E-07	7.045971E-07	1.641077E-07	1.677084E-07	9.185222E-08
	2	1.319260E-07	3.395989E-07	2.179453E-07	1.046749E-07	2.386439E-07
	3	6.022762E-01	1.223231E+00	2.465115E+00	4.948870E+00	9.916375E+00
	4	6.885316E-01	1.395721E+00	2.810055E+00	5.638702E+00	1.129598E+01

(c) in both		Mesh Size h				
		1/25	1/50	1/100	1/200	1/400
Model Problem	1	8.291146E-08	2.281750E-07	1.287741E-07	4.151793E-07	1.150775E-07
	2	1.413502E-07	5.192428E-08	1.929839E-07	2.215420E-07	2.605715E-07
	3	4.338306E-01	8.801644E-01	1.772726E+00	3.557795E+00	7.127906E+00
	4	4.520911E-01	9.167890E-01	1.846354E+00	3.705572E+00	*7.143374E+00

*Overflow after 4 iterations.

Table 1. Errors comparing with the exact solution

we see that the approximated solutions are close to the exact solution in model problems 1 and 2 in which u_{xx} and u_{yy} are zero.

However in the model problem 3, u_{xx} is zero but u_{yy} is not. When the shooting method is used in the x direction alone one can obtain good results. However, when we shoot in the y direction, the approximation obtained is not quite satisfactory. In model problem 4, with both u_{xx} and u_{yy} not zero, then the numerical solutions obtained by the shooting method in both the x direction and in the y direction are not satisfactory. From Table 1(c) we see that even if shooting in the x direction yields a good result, it can be spoiled by including the method in the y direction,

		Mesh Size h				
		1/25	1/50	1/100	1/200	1/400
Model Problem	1	1.8688263E-13	1.0214385E-12	6.1235617E-12	3.5131736E-11	7.8140087E-10
	2	1.8448553E-13	9.4572467E-13	1.5090459E-11	8.4224632E-11	4.9085748E-10
	3	5.1451364E-04	3.3564739E-04	5.1227495E-04	5.6648101E-03	8.4927635E-02
	4	1.1584041E-03	9.2020224E-04	2.1187400E-03	1.7685128E-02	9.7496142E-02

Table 2. Errors comparing with the exact solution; Nonlinear Shooting Method

		Mesh Size h				
		1/25	1/50	1/100	1/200	1/400
Model Problem	1	1.8708629E-13	1.0216000E-12	6.1235491E-12	3.5131688E-11	7.8140115E-10
	2	1.8420899E-13	9.4538760E-13	1.5087117E-11	8.4217847E-11	4.9084806E-10
	3	1.5730486E-03	9.2711797E-04	7.05511021E-04	5.9091536E-03	8.5045667E-02
	4	1.9894474E-03	1.6433187E-03	5.7063497E-03	3.0328584E-02	0.15446275265

Table 3. Errors comparing with the previous iteration; Nonlinear Shooting Method

		Mesh Size h				
		1/25	1/50	1/100	1/200	1/400
Model Problem	1	1	1	1	1	1
	2	1	1	1	1	1
	3	15(0.001)	13(0.0005)	10(0.001)	20(0.04)	16(0.05)
	4	15(0.01)	23(0.01)	21(0.24)	30(0.24)	28(0.5)

Table 4. Number of Outer Iterations

because u_{yy} is not zero. This phenomenon suggests that the nonlinear shooting method is not as effective when either of the second partial derivatives is not zero.

4.1.2. Finite Difference Scheme for Shooting Method

The initial guess for the finite difference scheme is chosen to be an average of the two boundary values for each space level. For example, for $j = 1$, assuming the left end boundary to be α and the right end boundary value to be β, then at each grid point i the approximation solution is $\alpha + i * h * \frac{\beta - \alpha}{b - a}$ where $h = \frac{b - a}{n}$.

In this section we repeat the numerical experiments that have been done in the previous section. We use a finite difference scheme to approximate the right sides of the equations (2.5) and (2.6).

Table 2 and 3 are the results that show how close the approximated solutions are to the exact solution, and how close two successive iterations are, respectively. Instead of shooting in a separate direction alone as we have done in the previous section, we use Algorithm 1 which is based on the alternating direction principle. We notice that the results for model problems 1 and 2 are very good. However, the results for model problems 3 and 4 are not as good as those for model problems 1 and 2, even though they are acceptable.

In the actual numerical experiments, the error of 10^{-3} or 10^{-4} given in the parentheses could be obtained within a certain number of outer iterations, the numbers are given in Table 4.

Table 4 shows that an accurate solution was reached in one outer iteration for the model problems 1 and 2. Because the error term of the central difference scheme involves the fourth partial derivatives of the solutions, we conjecture that the solution obtained in one outer iteration comes from the fact that all the fourth order partial derivatives of the exact solutions for the problems are zero.

4.2. Finite Difference Method: Linearized Iterative Method

The initial guess for this method is chosen to be a constant throughout all the grid points. In this section, we show the results on our model problems using the linearized iterative finite difference method. Table 5 shows the errors comparing with the exact solution for each mesh h. Table 6 shows the results of the errors comparing with the previous iteration.

We notice that the linearized iterative method is the most commonly used algorithm for solving nonlinear partial differential equations. Most cases considered in our study did not yield results that are as accurate as those discussed in the previous sections. From Table 6, we see that the linearized iterative algorithm reached convergence (the error of two consecutive approximation solutions is small enough) but the approximated value is not close enough to the exact solution. Moreover, the algorithm suffers from roundoff errors. This is shown in Table 6; the decrease of h is associated with a decrease of accuracy.

		Mesh Size h				
		1/25	1/50	1/100	1/200	1/400
Model Problem	1	7.040610E-13	1.0216000E-05	3.633131E-02	1.034148E-01	6.11999737
	2	3.088148E-04	1.872736E-03	1.191016E-02	6.454872E-02	10.21935416
	3	2.482891E-01	4.970955E-01	9.944290E-01	1.98888951	3.17761121
	4	5.735738E-04	3.916668E-04	2.325716E-03	9.550406E-03	1.31617791

Table 5. Errors comparing with the exact solution ; Linearized Iterative Method

		Mesh Size h				
		1/25	1/50	1/100	1/200	1/400
Model Problem	1	8.541305E-8	1.707771E-07	3.461992E-07	7.033384E-07	3.466550E-08
	2	5.523015E-07	6.094083E-08	1.203735E-07	2.435605E-07	4.931379E-07
	3	3.007410E-08	5.897043E-08	1.2056622E-07	2.330766E-07	1.451794E-07
	4	1.417153E-07	2.405810E-07	4.328954E-07	8.469618E-07	6.747030E-07

Table 6. Errors comparing with the previous iteration; Linearized Iterative Method

5. Conclusions

As we described in Section 1, numerical algorithms with inner-outer iterations are often used for solving nonlinear partial differential equations. In this research, we proposed the shooting method for the inner iteration. Even though these method is well known in other applications, it is not applied in the numerical solution of nonlinear partial differential equations. The main idea in this work is based on the alternating direction procedure. It is assumed that we obtain a better approximation after each iteration. Convergence analysis for perturbed linear and nonlinear cases shows that the convergence is linear.

From this study, we also found out that the efficiency of the methods may depend on the vanishing of the fourth order partial derivatives. Hence, the numerical results seem to imply that if all fourth order partial derivatives of the exact solution of a nonlinear partial differential equation are zero (within discretization errors), then the shooting finite difference scheme can yield an accurate solution in one outer iteration.

The linearized iterative methods are the most commonly used numerical methods for solving nonlinear partial differential equations. In this research, we should note that the shooting method is very well suited for a parallel computer because for each level, either in i(x direction) or in j(y direction), their computations are totally independent of other levels. Therefore we may obtain very high speedup. In theory, we have an n processor machine and we have n levels of grid lines then the limit of speedup is n, which means that if n seconds are required to obtain a result in one processor machine, then we may obtain the same result in one second in an n processor machine.

On the final note to this paper, multiple shooting method may be able to compensate the stability problem for the shooting method.

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