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FINITE EXTENSIONS OF WEIGHTED WORD L-DELTA GROUPS

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Abstract. The purpose of this paper is to investigate the finite extension of weighted word L-delta groups. The paper revealed that a finite extension of a weighted word L-delta group is a weighted word L-delta group, and an abelian group, in addition, is a weighted word L-delta group and simultaneously a word L-delta group.

1. Introduction

L-delta groups have been an interesting topic in the study of geometric groups in that they include hyperbolic groups which opened the study of finitely generated groups from a geometric viewpoint. Very strong L_δ groups and strong L_δ groups were discussed as subclasses of L_δ groups [2]. We rename these groups respectively by *word L-delta groups*, *weighted word L-delta groups*, and *L-delta groups*. Then, there exists an inclusion chain: Hyperbolic Groups $\subset$ Word L-delta Groups $\subseteq$ Weighted Word L-delta groups $\subseteq$ L-delta Groups.

Extension of groups is a classical topic in the study of groups. Direct products, semidirect products, wreath products, HNN extensions, finite extensions are all instances of extensions of groups. This paper investigates a finite extension of a weighted word L-delta group. In other words, finite-by-L-delta and L-delta-by-finite groups are discussed over the weighted Cayley graph.

It is an interesting problem whether or not the three levels of L-delta groups are actually the same. There are some similarities among word L-delta, weighted word L-delta, and (general) L-delta groups. They are all closed under taking direct product [2]; word L-delta groups are finitely presented and have a quadratic isoperimetric function [4]; and in general L-delta groups are finitely presented and have a quadratic

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isometric function [3]. We here focus on word L-delta groups and weighted word L-delta groups, and pose the question: *Is being a weighted word L-delta group is equivalent to being a word L-delta group?* In the case of abelian groups, the answer to this question is positive.

The organization of this paper follows: Section 2 gives some preliminary definitions and examples. Section 3 includes the main theorems about the finite extension of weighted word L-delta groups [Theorem 3.1, Theorem 3.2]. In Section 4, we then see that a finitely generated abelian group is a weighted word L-delta group and simultaneously a word L-delta group. Further studies are introduced in Section 5.

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2. DEFINITIONS AND EXAMPLES

Suppose that $\mathcal{P}$ is a property of geodesic metric spaces, and call a space with $\mathcal{P}$ a $\mathcal{P}$ geodesic metric space, or more briefly $\mathcal{P}$ space. A finitely generated group G is called a (*geometric*) $\mathcal{P}$ *group* if G acts properly, cocompactly, and by isometries on a $\mathcal{P}$ space. Then an L-delta group is a finitely generated group which acts properly,cocompactly, and by isometries on an L-delta space.

First define the L-delta space. Let (X, d) be a geodesic space and δ be a nonnegative constant. Choose three points x_1, x_2, x_3 in X. The triplet x_1, x_2, x_3 has the *L-delta property* if there exists $t \in X$ such that

$$d(x_i, t) + d(t, x_j) \leq d(x_i, x_j) + \delta$$

where $i, j \in \{1, 2, 3\}$ and $i \neq j$. The point $t \in X$ is called a δ-center for the triplet x_1, x_2, x_3. Then X is called a L-delta space if any triplet in X has the L-delta property. We may assume that δ is an integer since, otherwise, we can choose an integer δ' greater than δ by which t is a δ'-center for a triplet in X. In this sense, we use the term *L-delta* for general δ values rather than $L(\delta)$ or L_δ which is for a specific δ value.

The Cayley graph is a geometric realization of a group, which turns out to be a metric space with the associated word metric. We define the notion of weighted Cayley graph in a natural way. Throughout this section, let G be a finitely generated group and let A be a finite *symmetric* generating set for G. By this, we mean a finite set A with an involution, denoted by $a \mapsto a^{-1}$, and a function $\mu : A \to G$ such that $\mu(a^{-1}) = \mu(a)^{-1}$, and such that the induced homomorphism $A^* \to G$ is onto where A^* is the set of all words in A. This epimorphism $\mu : A^* \to G$ is denoted by $v \mapsto \overline{v}$.

Define a weight function $\omega_A : A \to \mathbb{N}$ so that

- for all $a \in A$, $\omega_A(a) = \omega_A(a^{-1})$, and
- for any edge (g, a) in the Cayley graph $\Gamma(G, A)$, $\omega_A(g, a) = \omega_A(a)$ where $g \in G$ and $a \in A$.

Then, the weight of a word $v = a_1 a_2 \cdots a_n$ in A^* is defined as the sum of each weight of a_i, $1 \leq i \leq n$. That is,

$$\omega_A(v) = \omega_A(a_1) + \omega_A(a_2) + \cdots + \omega_A(a_n).$$

Consequently, measuring the distance between two points x and y by the least weight on an edge path joining x and y is a typical way to define a metric on a space. Define a distance function $d_A : G \times G \to \mathbb{N}$ by $(x, y) \mapsto d_A(x, y)$ where $x, y \in G$ and

$$d_A(x, y) := \min \left\{ \omega_A(v) \mid v \in A^* \text{ and } \overline{v} = x^{-1} y \right\}.$$

It is obvious that this distance function d_A becomes a metric, and we call it the *weighted word metric* associated with the weighted generating set A. The weighted Cayley graph $\Gamma(G, A)$ is a geodesic space with each edge e isometric to the interval $[0, \omega_A(e)] \subset \mathbb{R}$, and furthermore G acts on $\Gamma(G, A)$ properly, cocompactly, and by isometries [2]. An (unweighted) Cayley graph is a weighted Cayley graph whose edges have the unit weight.

A group G is quasi-isometric to its Cayley graphs [1, 7], and also is quasi-isometric to its weighted Cayley graph [2]. If a geometric property $\mathcal{P}$ is a quasi-isometry invariant, we can handle the $\mathcal{P}$ group on its Cayley graph. For example, hyperbolicity is a quasi-isometry invariant so we do not distinguish hyperbolic groups and word hyperbolic groups. However, the L-delta property is not a quasi-isometry invariant, so we may distinguish (unweighted) word L-delta groups and weighted word L-delta groups from (general) L-delta groups.

Definition 2.1 (weighted word L-delta group, word L-delta group). A finitely generated group G is called a *weighted word L-delta group* if a weighted Cayley graph of G has the L-delta property. G is called an (unweighted) *word L-delta group* if a Cayley graph of G has the L-delta property.

A metric graph $\Gamma = (V, E)$ is not a metric space since the metric is defined only on the vertex set $V(\Gamma)$. Indeed, only $V(\Gamma)$ with the edge metric is a metric space. The *geometric realization* $|\Gamma|$ of Γ is a metric space with the shortest path metric, where the length of a geodesic segment on an edge e is measured by the proportional length to $[0, \omega(e)]$. From this graph-theoretic viewpoint, for the group G with a weighted finite generating set A, its weighted Cayley graph $\Gamma(G, A)$ is the metric space with the least weight path metric, say d, and the vertex set of it is a metric space with the associated word metric d_A for which we use the notation (G, d_A). However, discriminating (G, d_A) and $(\Gamma(G, A), d)$ over the L-delta property is redundant by the following propositions.

Proposition 2.2. *Let (X, d) be a metric space and let S and T be subsets of X such that S is a subset of the k-neighborhood $\mathcal{N}_k(T)$ of T and T is a subset of the k-neighborhood $\mathcal{N}_k(S)$ of S for some constant $k \geq 0$. If S is an L-delta space, then T is also an L-delta space.*

Proof. Let x_1, x_2, x_3 in T and choose respectively $x'_1, x'_2, x'_3 \in S$ so that $d(x_i, x'_i) \leq k$ for $i = 1, 2, 3$. Since (S, d_S) has the L-delta property, there exists a δ-center $s \in S$ for the triplet x'_1, x'_2, x'_3. Choose $t \in T$ so that $d(s, t) \leq k$. Then

$$\begin{aligned}
d(x_i, t) + d(t, x_j) &\leq d(x_i, x'_i) + d(x'_i, s) + d(s, t) + d(t, s) + d(s, x'_j) + d(x'_j, x_j) \\
&\leq d(x'_i, s) + d(s, x'_j) + 4k \\
&\leq d(x'_i, x'_j) + \delta + 4k \\
&\leq d(x'_i, x_i) + d(x_i, x_j) + d(x_j, x'_j) + \delta + 4k \\
&\leq d(x_i, x_j) + \delta + 6k.
\end{aligned}$$

where $i, j \in \{1, 2, 3\}$ and $i \neq j$. Then any triplet $x, y, z \in T$ has a $(\delta + 6k)$-center in T, and so T has the L-delta property. □

Proposition 2.3. *Let G be a finitely generated group and A be a weighted finite generating set for G. Then (G, d_A) has the L-delta property if and only if $(\Gamma(G, A), d)$ has the L-delta property.*

Proof. Take $S = G$ and $T = \Gamma(G, A)$ in Proposition 2.2. Then the result follows as required. For the converse, take $S = \Gamma(G, A)$ and $T = G$ in Proposition 2.2. □

By Proposition 2.3, we do not need to check the L-delta property over the whole (weighted) Cayley graph. Instead, it is enough to check the property only on vertex points in the (weighted) Cayley graph. The next definition is alternatively used.

Definition 2.4 (Alternative definition). A group G is a *weighted word L-delta group* if there exists a weighted finite generating set A such that (G, d_A) has the L-delta property. G is a *word L-delta group* if there exists a generating set A with the unit weight such that (G, d_A) has the L-delta property.

In the following examples, we examine the L-delta property for some (weighted) word metrics on the group of integers.

Example 2.5. Let $G = \langle A \rangle = \langle a \mid - \rangle \cong \mathbb{Z}$. Then (G, d_A) has the L-delta property.

Let x_1, x_2 and x_3 be in G. Then each x_i has a normal form. That is, there exist integers m, n, k such that $x_1 = a^m$, $x_2 = a^n$, and $x_3 = a^k$. Without loss of generality we may assume that $m < n < k$. Then the point x_2 in the middle becomes a zero-center for the triplet x_1, x_2, x_3 since it lies on a geodesic from x_1 to x_3. In fact,

$$d_A(x_1, x_2) + d_A(x_2, x_2) = d_A(x_1, x_2),$$
$$d_A(x_2, x_2) + d_A(x_2, x_3) = d_A(x_2, x_3),$$
$$d_A(x_1, x_2) + d_A(x_2, x_3) = d_A(x_1, x_3)$$

whence the required δ is zero.

Example 2.6. If we let $B = \left\{a^2, a^3\right\}$ on Example 2.5 above, B is a finite generating set for $\mathbb{Z}$. Then (G, d_B) has the L-delta property regardless of whether B is weighted or not.

Example 2.5 showed that $G \cong \mathbb{Z}$ with the canonical presentation has the L-delta property, and Example 2.6 showed that $G \cong \mathbb{Z}$ with a non canonical presentation also has the L-delta property. The idea that selecting the middle point is independent of the choice of the weight for generators was used in the proofs of examples. The complete proof of Example 2.6 is left to the readers. These two examples suggest a case that being a weighted word L-delta group is equivalent to being a word L-delta group.

3. Main Theorems

An extension of a group is a classical idea in the study of groups. In this section, finite extensions of weighted word L-delta groups are discussed. There are two cases we can consider. The first case is that the base group is finite and the quotient group is a weighted word L-delta group. This case carries the result that a finite-by-weighted word L-delta group is a weighted word L-delta group for possibly different δ value, and conversely, if a group is a weighted word L-delta group, then its homomorphic image with finite kernel is also a weighted word L-delta group. The other case is that the base group is an weighted word L-delta group of a finite index in the top group. In this case, we require that base group be a central subgroup.

Theorem 3.1. *Let G be a finitely generated group and let N be a finite normal subgroup of G. Then, G is a weighted word L-delta group if and only if G/N is a weighted word L-delta group.*

Proof. Suppose (G, d_A) has the L-delta property and let $\omega_A : A \to \mathbb{Z}$ be a proper weight function. Consider the map $G \to G/N$ by $g \mapsto \bar{g} = gN$. Let $\bar{A} = \{aN \mid a \in A \text{ and } a \notin N\}$. Then $G/N = \langle \bar{A} \rangle$. Define $\omega_{\bar{A}}(\bar{a}) = \omega_A(a)$.

For any $g \in G$, choose a word $a_1 \cdots a_r$ of least weight in A^* such that $g = a_1 \cdots a_r$ in G. Then $gN = \bar{a_1} \cdots \bar{a_r}$, so

$$|gN|_{\bar{A}} \leq \sum_{i=1}^{r} \omega_{\bar{A}}(\bar{a}) = \sum_{i=1}^{r} \omega_A(a_i) = |g|_A \,. \tag{3.1}$$

Now choose a word $\bar{a}_1 \cdots \bar{a}_s$ in $\bar{A}^*$ of least weight such that $gN = \bar{a}_1 \cdots \bar{a}_s$ in G/N. Then there exists $n \in N$ such that $g = a_1 a_2 \cdots a_s \cdot n$. Thus

$$|g|_A \leq \sum_{i=1}^{s} \omega_A(a_i) + \omega_A(n) = \sum_{i=1}^{s} \omega_{\bar{A}}(\bar{a}_i) + \omega_A(n) = |gN|_{\bar{A}} + \omega_A(n). \tag{3.2}$$

From (3.1) and (3.2), we obtain the inequality $|gN|_{\bar{A}} \leq |g|_A \leq |gN|_{\bar{A}} + K$ where $K = \max\{|n|_A : n \in N\}$, and consequently, we deduce that for all $g, h \in G$,

$$d_{\bar{A}}(gN, hN) \leq d_A(g, h) \leq d_{\bar{A}}(gN, hN) + K.$$

Choose x_1N, x_2N and x_3N in G/N. Since (G, d_A) has the L-delta property, there exists a δ-center $t \in G$ for the triplet x_1, x_2, x_3 in G. We show that $tN \in G/N$ is a $(K+\delta)$-center for the triplet x_1N, x_2N, x_3N in G/N:

$$\begin{aligned} d_{\bar{A}}(x_iN, tN) + d_{\bar{A}}(tN, x_jN) &\leq d_A(x_i, t) + d_A(t, x_j) \\ &\leq d_A(x_i, x_j) + \delta \\ &\leq d_{\bar{A}}(x_iN, x_jN) + K + \delta \end{aligned}$$

where $i, j \in \{1, 2, 3\}$ and $i \neq j$. Hence $(G/N, d_{\bar{A}})$ has the L-delta property.

For the converse, assume that $G/N = \langle B \rangle$ has the L-delta property. Construct a finite generating set A for G: For each $b \in B$, choose $a_b \in G$ with $\bar{a}_b = b$ and let $A = \{a_b \in G \mid b \in B\} \cup N$. Then define the weights $\omega_A(a_b) = \omega_B(b)$ for $b \in B$ and $\omega_A(n) = 1$ for $n \in N$.

Recalling the inequality (3.1), we get $|\bar{g}|_B \leq |g|_A$ for all $g \in G$. Choose $\bar{g} = gN$ in G/N and write $\bar{g} = b_1 b_2 \cdots b_s$ where $b_i \in B$. Then there exists $n \in N$ such that

$g = a_{b_1} \cdots a_{b_s} \cdot n$. Thus

$$|g|_A \leq \sum_{i=1}^{s} \omega_A(a_{b_i}) + \omega_A(n) = \sum_{i=1}^{s} \omega_B(b_i) + 1 = |\bar{g}|_B + 1, \tag{3.3}$$

and so we obtain $|\bar{g}|_B \leq |g|_A \leq |\bar{g}|_B + 1$, which induces that for all $g, h \in G$,

$$d_B(\bar{g}, \bar{h}) \leq d_A(g, h) \leq d_B(\bar{g}, \bar{h}) + 1.$$

Let x_1, x_2, x_3 be in G. Then $\bar{x}_1 = x_1N, \bar{x}_2 = x_2N, \bar{x}_3 = x_3N$ are in G/N, and since $(G/N, d_B)$ is an L-delta space, there exists a δ'-center $tN \in G/N$ for the triplet x_1N, x_2N, x_3N in G/N. Then,

$$\begin{aligned} d_A(x_i, t) + d_A(t, x_j) &\leq d_B(x_iN, tN) + d_B(tN, x_jN) + 2 \\ &\leq d_B(x_iN, x_jN) + \delta' + 2 \\ &\leq d_A(x_i, x_j) + \delta' + 2 \end{aligned}$$

where $i, j \in \{1, 2, 3\}$ and $i \neq j$. The triplet x_1, x_2, x_3 in G has a $(\delta' + 2)$-center $t \in G$, and hence (G, d_A) has the L-delta property. $\square$

Theorem 3.1 is true for hyperbolic groups which are word L-delta groups. A hyperbolic group G is quasi-isometric to its homomorphic image G/N with N finite. Since hyperbolicity is an quasi-isometry invariant [1, 7], G is hyperbolic if and only if G/N is hyperbolic.

The next theorem discusses the finite extension of the base group of weighted L-delta group.

Theorem 3.2. *Let G be a finitely generated group and H be a finite index central subgroup of G. If H is a weighted word L-delta group, then G is a weighted word L-delta group.*

Proof. Let B be a symmetric weighted finite generating set for H such that (H, d_B) has the L-delta property. Let A be a symmetric finite generating set for G such that $B \subseteq A$. Choose a transversal T for H in G. Then for any $g \in G$, $g \in Ht$ for some $t \in T$, i.e., $g = ht$ for some $h \in H$. Note that h and t are dependent only on the choice of a transversal T. Define a weight function $\omega_A : A \to \mathbb{Z}$, for all $a \in A$,

$$\omega_A(a) = \max\left\{\left|tas^{-1}\right|_B : tas^{-1} \in H \text{ and } t, s \in T\right\}$$

where tas^{-1}, called a *pinch*, has the norm $\left|tas^{-1}\right|_B$ as the least weight on a path in (H, d_B) from the identity to tas^{-1}.

We should here check if $|tbs^{-1}|_B = |b|_B$ for any $b \in B$. Note that $tbs^{-1} = ts^{-1}b \in H$, since $\langle B \rangle = H \leq Z(G)$. Then $ts^{-1} \in H$ or $t = s$, and thus

$$|tbs^{-1}|_B = |ts^{-1}b|_B + |ss^{-1}b|_B = |b|_B .$$

It immediately implies that $\omega_A(b) = |b|_B$ for $b \in B$.

Claim that $|h|_A = |h|_B$ for all $h \in H$. On one hand, write $h = a_1 \cdots a_n$, where $a_i \in A$ and $|h|_A = \sum \omega_A(a_i)$. Let $t_0 = 1$ and let t_i be the unique element of T such that $Ha_1 \cdots a_i = Ht_i$, for $1 \leq i \leq n$. Then $t_n = 1$ since $a_1 \cdots a_n = h \in H$. Rewrite $h = a_1 \cdots a_n$ by

$$h = (t_0 a_1 t_1^{-1})(t_1 a_2 t_2^{-1}) \cdots (t_{n-1} a_n t_n^{-1}),$$

and note that each pinch is in H. Thus,

$$|h|_B \leq \sum_{i=1}^{n} |t_{i-1} a_i t_i^{-1}|_B \leq \sum_{i=1}^{n} \omega_A(a_i) = |h|_A . \tag{3.4}$$

On the other hand, write $h = b_1 \cdots b_n$, where $b_i \in B$ and $|h|_B = \sum \omega_B(b_i)$. Then

$$|h|_A \leq \sum_{i=1}^{n} |b_i|_A \leq \sum_{i=1}^{n} \omega_A(b_i) = \sum_{i=1}^{n} |b_i|_B \leq \sum_{i=1}^{n} \omega_B(b_i) = |h|_B . \tag{3.5}$$

From (3.4) and (3.5) above, $|h|_A = |h|_B$, which implies that d_B is the restriction of d_A to $H \times H$. That is, for all h and h' in H,

$$d_B(h, h') = |h^{-1}h'|_B = |h^{-1}h'|_A = d_A(h, h').$$

Choose x_1, x_2, x_3 in G. Then, for some t_1, t_2 and t_3 in T, $x_1 \in Ht_1$, $x_2 \in Ht_2$ and $x_3 \in Ht_3$, i.e., there exists $x_i' \in H$ such that $x_i = x_i' t_i$, $i = 1, 2, 3$. Since (H, d_B) has the weighted L-delta property, there exists δ-center $t \in H$ for the triplet x_1', x_2', x_3'. And the distance between x_i and x_i' is that $d_A(x_i, x_i') = |x_i^{-1} x_i'|_A = |t|_A$ for some $t \in T$. Thus,

$$\begin{aligned} d_A(x_i, t) + d_A(t, x_j) &\leq d_A(x_i, x_i') + d_A(x_i', t) + d_A(t, x_j') + d_A(x_j', x_j) \\ &\leq d_A(x_i', x_j') + \delta + 2\max\{|t|_A : t \in T\} \\ &\leq d_A(x_i', x_i) + d_A(x_i, x_j) + d_A(x_j, x_j') + \delta + 2K \\ &\leq d_A(x_i, x_j) + \delta + 4\max\{|t|_A : t \in T\} \end{aligned}$$

where $i, j \in \{1, 2, 3\}$ and $i \neq j$. Then $t \in H \subset G$ is a $(\delta' + 4K)$-center for the triplet x_1, x_2, x_3 in G where $K = \max\{|t|_A : t \in T\}$, and hence G is a weighted word L-delta group. $\square$

4. Abelian Groups

It is interesting to look into the similarities and/or differences among word L-delta groups, weighted word L-delta groups, and L-delta groups. Recall the problem in Introduction section: *Is being a weighted word L-delta group equivalent to being a word L-delta group?* In this section, we discuss this problem on abelian groups. The section begins with an observation that $\mathbb{Z}^n$ is a weighted word L-delta group. Choosing the middle point as seen in Example 2.5 works as a basic idea.

Lemma 4.1. *Let E be any weighted basis for $\mathbb{Z}^n$. Then $(\mathbb{Z}^n, d_E)$ has the L-delta property for $\delta = 0$.*

Proof. Let $E = \{e_1, e_2, \ldots, e_n\}$ be a basis of $\mathbb{Z}^n$ and assign proper weights $\omega(e_i)$ for all $e_i \in E$. Choose three elements $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n)$, and $z = (z_1, z_2, \ldots, z_n)$ in $\mathbb{Z}^n$. Their geodesic normal forms on E are $x = \sum x_i e_i$, $y = \sum y_i e_i$, and $z = \sum z_i e_i$, and then their weights are $\omega(x) = \sum x_i \omega(e_i)$, $\omega(y) = \sum y_i \omega(e_i)$, and $\omega(z) = \sum z_i \omega(e_i)$. The distance between x and y is measured by the weighted word metric d_E:

$$d_E(x,y) = |x-y|_E = \sum_{i=1}^{n} |x_i - y_i|\, \omega(e_i).$$

Let $t = (t_1, \cdots, t_n) = \sum t_i e_i$ in $\mathbb{Z}^n$ where each t_i is the middle one in $\{x_i, y_i, z_i\}$. Then,

$$\begin{aligned} d_E(x,t) + d_E(t,y) &= \sum |x_i - t_i|\, \omega(e_i) + \sum |t_i - y_i|\, \omega(e_i) \\ &= \sum \left(|x_i - t_i| + |t_i - y_i|\right) \omega(e_i) \\ &= \sum |x_i - y_i|\, \omega(e_i) \\ &= d_E(x,y). \end{aligned}$$

Likewise, $d_E(y,t) + d_E(t,z) = d_E(y,z)$ and $d_E(x,t) + d_E(t,z) = d_E(x,z)$. So, $t \in \mathbb{Z}^n$ is a zero-center for the triplet x, y, z and hence $(\mathbb{Z}^n, d_E)$ has the L-delta property for $\delta = 0$. □

It is easy to see that $\mathbb{Z}^n$ is a word L-delta group by the choice of the unit weight function in Lemma 4.1. We now show that abelian groups are weighted word L-delta groups as well as (unweighted) word L-delta groups.

Theorem 4.2. *A finitely generated abelian group G is a weighted word L-delta group.*

Proof. By the fundamental theorem of the finitely generated abelian group, $G \cong \mathbb{Z}^n \oplus H$ where H is finite. By Lemma 4.1, $\mathbb{Z}^n$ is a weighted word L-delta group. Then we regard G as a finite-by-weighted L-delta group, and so, by Theorem 3.1, G is a weighted word L-delta group. □

In the proof of Theorem 4.2, G can be also regarded as being wighted L-delta-by-finite with the centrality condition. Then G is, by Theorem 3.2, a weighted word L-delta group.

Theorem 4.3. *A finitely generated abelian group G has an (unweighted) generating set E such that (G, d_E) has the L-delta property for some $\delta \geq 0$.*

Proof. By the fundamental theorem of finitely generated abelian groups, there exist a finite generating set $E = A \cup B$ for G so that $A = \{a_1, \dots, a_m\}$ spans a torsion-free direct factor $F \cong \mathbb{Z}^m$ and $B = \{b_1, \dots, b_k\}$ spans the torsion direct factor H. That is, $G \cong \mathbb{Z}^m \oplus H$ where $\langle a_1, \dots, a_m \rangle \cong \mathbb{Z}^m$ and $H = \langle b_1 \rangle \oplus \langle b_2 \rangle \oplus \cdots \oplus \langle b_k \rangle \cong \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_1} \oplus \ldots \oplus \mathbb{Z}_{n_k}$. Then H has $n_1 n_2 \cdots n_k$ elements and all elements of H are realized by ordered k-tuples from $(0, \dots, 0)$ to $(n_1 - 1, n_2 - 1, \dots, n_k - 1)$ over $\mathbb{Z}$.

The Cayley graph $\Gamma(G, E)$ seems to have a structure that every vertex of H has a copy of $F \cong \mathbb{Z}^m$ and every vertex of F has a copy of H. F and H intersects the group identity. In $\Gamma(G, E)$, it is observed that $ewe^{-1} = w$ for any path w in any copy of F and for any generator $e \in E$. It implies that a copy of F, including F itself, is isometrically embedded into $\Gamma(G, E)$, i.e., $d_A(x, y) = d_E(x, y)$ for all $x, y \in \mathbb{Z}^m$. We then see that every vertex g in G has a projective image g_o in F, and the distance $d_E(x, y)$ between them is no greater than the maximal norm K over H, i.e., $K = \max\{|h|_B : h \in H\}$.

Choose x, y, z in (G, d_E) and check the existence of a δ-center for the triplet x, y, z. There exist F-projections x_o of x, y_o of y, and z_o of z, and distances between them $d_E(x, x_o), d_E(y, y_o)$ and $d_E(z, z_o)$ are all no greater than K. We know that, by Lemma 4.1, F has the L-delta property for $\delta = 0$. Let t_o be a zero-center for the triplet x_o, y_o, z_o in F. Thus,

$$\begin{aligned} d_E(x, t_o) + d_E(t_o, y) &\leq d_E(x, x_o) + d_E(x_o, t_o) + d_E(t_o, y_o) + d(y_o, y) \\ &\leq d_A(x_o, t_o) + d_A(t_o, y_0) + 2K \\ &\leq d_A(x_o, y_o) + 2K \\ &\leq d_E(x_o, x) + d_E(x, y) + d_E(y, y_o) + 2K \\ &\leq d_E(x, y) + 4K. \end{aligned}$$

The other two cases are alike:

$$d_E(y, t_o) + d_E(t_o, z) \leq d_E(y, z) + 4K$$

and

$$d_E(x, t_o) + d_E(t_o, z) \leq d_E(x, z) + 4K.$$

Therefore, $t_o \in G$ is a $4K$-center for the triplet x, y, z in (G, d_E), and so (G, d_E) has the L-delta property. $\square$

Theorem 4.2 and Theorem 4.3 tell us that a finitely generated abelian group G is a weighted word L-delta group and simultaneously a word L-delta group. From now on, it is not necessary to tell whether an abelian group G is an word L-delta group or an weighted word L-delta group.

5. Further Studies

Some interesting problems remain. First, as mentioned in Introduction section, one of the big problems on L-delta groups is to determine whether or not three different levels of L-delta groups are actually the same. Theorem 3.1 and Theorem 3.2 discussed cases when a finite extension of a weighted L-delta group can become a weighted L-delta group. It is easy to see these are true for an L-delta group by taking the unit weight function. This is a similarity between word L-delta group and weighted word L-delta group. However, it is unknown if Theorem 3.1 and Theorem 3.2 hold for L-delta groups in general.

Another topic for future research is on L-zero groups which are a special case when $\delta = 0$. Example 2.5 and Lemma 4.1 showed that $\mathbb{Z}$ and $\mathbb{Z}^n$ are L-zero groups. We observe that all paths in L-zero spaces are geodesic, and consequently, using this fact we see that L-zero groups are closed under taking direct product. L-zero groups may give a way to describe CAT(0) cubical groups. Interestingly, both L-zero groups and the 1-skeleton of CAT(0) cubical complexes have the median structure and the loop shortening property. Some results dealing with median structure or loop shortening property will be shown in a separated papers.

Last, we, in Section 4, saw that all finitely generated abelian groups are word L-delta groups . Using the concept of the finite index central extension of weighted L-delta groups, we see that there exists a nilpotent group of class c whose weighted Cayley graph has the L-delta property. It is interesting to see how the nilpotence and the L-delta property are related.

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