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AN IMPROVEMENT OF THE MADDOX THEOREM ON THE MATRIX CLASS $(\ell^\infty(X),\, c_0(Y))$

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Abstract. A clear-cut characterization of the matrix class $(\ell^\infty(X),\, c_0(Y))$ is obtained for a very general case.

1. Introduction

For Banach spaces X, Y and a matrix $(f_{ij})_{i,j\in\mathbb{N}}$ in $L(X,Y)$, the family of continuous linear operators from X to Y, let

$$R_{ij} = (f_{ij}, f_{ij+1}, f_{ij+2}, \cdots),$$

$$\|R_{ij}\| = \sup_{p\in\mathbb{N},\|(x_k)\|_\infty\le 1} \left\| \sum_{k=j}^{j+p} f_{ik}(x_k) \right\|, \ \forall i,j\in\mathbb{N} \ \text{(see [3,4])}.$$

Recall that

$$\ell^\infty(X) = \{(x_j)_1^\infty \subset X : (x_j)_1^\infty \text{ is bounded}\}, \quad c_0(Y) = \{(y_j)_1^\infty \subset Y : y_j \to 0\}$$

and we write $(x_j)_1^\infty = (x_j)$, simply. Let

$$(\ell^\infty(X), c_0(Y)) = \left\{ (f_{ij})_{i,j\in\mathbb{N}} : f_{ij} \in Y^X,\ \lim_i \sum_{j=1}^\infty f_{ij}(x_j) = 0,\ \forall\, (x_j) \in \ell^\infty(X) \right\}$$

(see also C. Swartz [2]).

I.J. Maddox [3, 4] gave a characterization of the matrix class

$$(\ell^\infty(X), c_0(Y))|_L = \{(f_{ij})_{i,j\in\mathbb{N}} \in (\ell^\infty(X), c_0(Y)) : \text{each } f_{ij} \text{ is linear and continuous}\}$$

for Banach spaces X and Y as follows.

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Theorem 1 ([3, 4, Theorem 4.8]). *Let X and Y be Banach spaces and $f_{ij} : X \to Y$ is a linear continuous operator for $i, j \in \mathbb{N}$. Then $(f_{ij})_{i,j\in\mathbb{N}} \in (\ell^\infty(X), c_0(Y))|_L$ if and only if*

(1) $\lim_i f_{ij}(x) = 0$ *for each* $j \in \mathbb{N}$ *and* $x \in X$,
(2) $\|R_{i1}\| < +\infty$ *and* $\lim_j \|R_{ij}\| = 0$ *for each* i,
(3) $\lim_j \overline{\lim_i} \|R_{ij}\| = 0$.

This result is restricted to linear continuous operators between Banach spaces. However, Li Ronglu [7] has obtained a series of summability results for matrices of quasi-homogeneous mappings. Note that the family of quasi-homogeneous mappings includes all linear operators and many nonlinear mappings [7, Example 1].

For topological vector spaces X and Y, $\mathfrak{F}_0(X,Y) = \{f \in Y^X : f(0) = 0\}$ is an extremely large subfamily of Y^X, and quasi-homogeneous mappings between X and Y forms a small subfamily of $\mathfrak{F}_0(X,Y)$. In this paper we would like to establish above Maddox theorem for matrices of mappings in $\mathfrak{F}_0(X,Y)$.

First, we improve Theorem 1 of [5] as follows.

Lemma 1. *Let G be an abelian topological group. For every $\Omega \neq \emptyset$ and $\{f_j\} \subset G^\Omega$ the following (i) and (ii) are equivalent.*

(i) $\sum_{j=1}^{\infty} f_j(\omega_j)$ *converges for each* $\{\omega_j\} \subset \Omega$.
(ii) $\sum_{j=1}^{\infty} f_j(\omega_j)$ *converges uniformly with respect to* $\{\omega_j\} \subset \Omega$.

Proof. If (i) holds but (ii) fails, then there exists a neighborhood U of $0 \in G$ and integers $m_1 < n_1 < m_2 < n_2 < \cdots$ and $\{\omega_{ij} \in \Omega : m_i \le j \le n_i, i \in \mathbb{N}\}$ such that

$$\sum_{j=m_i}^{n_i} f_j(\omega_{ij}) \notin U,$$

$i = 1, 2, 3, \cdots$. Pick an $\omega_0 \in \Omega$ and let

$$\omega_j = \begin{cases} \omega_{ij}, & m_i \le j \le n_i,\ i = 1, 2, 3, \cdots, \\ \omega_0, & \text{otherwise}, \end{cases}$$

then $\sum_{j=1}^{\infty} f_j(\omega_j)$ diverges. This contradicts (i) and so (i) implies (ii). □

Now we state the main result as follows.

Theorem 2. *Let X, Y be topological vector spaces where Y is separated and $f_{ij} \in \mathfrak{F}_0(X,Y)$ for all $i,j \in \mathbb{N}$. Then $(f_{ij})_{i,j\in\mathbb{N}} \in (\ell^\infty(X), c_0(Y))$ if and only if*

(1) $\lim_i f_{ij}(x) = 0$ *for each* $j \in \mathbb{N}$ *and* $x \in X$,

(4) *for every bounded* $B \subset X$, $\sum_{j=1}^{\infty} f_{ij}(x_j)$ *converges uniformly with respect to both* $i \in \mathbb{N}$ *and* $\{x_j\} \subset B$.

Proof. Suppose that $(f_{ij})_{i,j\in\mathbb{N}} \in (\ell^\infty(X), c_0(Y))$. Since each $f_{ij} \in \mathfrak{F}_0(X,Y)$ and $(0,\cdots,0,\overset{(j)}{x},0,0,\cdots) \in \ell^\infty(X)$ when $j \in \mathbb{N}$ and $x \in X$,

$$\lim_i f_{ij}(x) = \lim_i \left[\sum_{k=1}^{j-1} f_{ik}(0) + f_{ij}(x) + \sum_{k=j+1}^{\infty} f_{ik}(0)\right] = 0.$$

So (1) holds for $(f_{ij})_{i,j\in\mathbb{N}}$.

If (4) fails to hold for some bounded $B \subset X$, then there is a neighborhood V of $0 \in Y$ such that

$$(*)\ \forall m_0 \in \mathbb{N}\, \exists m > m_0 \text{ and } i \in \mathbb{N} \text{ and } \{x_j\} \subset B \text{ for which } \sum_{j=m}^{\infty} f_{ij}(x_j) \notin V.$$

Pick a neighborhood W of $0 \in Y$ such that $W + W \subset V$. By $(*)$ there exist integers $m_1 > 1$, $i_1 \in \mathbb{N}$ and $\{x_{1j}\} \subset B$ such that

$$\sum_{j=m_1}^{\infty} f_{i_1 j}(x_{1j}) \notin V$$

but

$$\sum_{j=n_1+1}^{\infty} f_{i_1 j}(x_{1j}) \in W$$

for some $n_1 > m_1$ so

$$\sum_{j=m_1}^{n_1} f_{i_1 j}(x_{1j}) \notin W.$$

Since $\sum_{j=1}^{\infty} f_{ij}(x_j)$ converges for each $\{x_j\} \subset B$ and $i \in \mathbb{N}$, it follows from Lemma 1 that there is an integer $m_0 > n_1$ such that

$$\sum_{j=m}^{\infty} f_{ij}(x_j) \in V \text{ for every } m > m_0,\ 1 \le i \le i_1 \text{ and } \{x_j\} \subset B.$$

Then $(*)$ shows that there exist integers $m_2 > m_0$, $i_2 > i_1$ and $\{x_{2j}\} \subset B$ such that $\sum_{j=m_2}^{\infty} f_{i_2 j}(x_{2j}) \notin V$ and so $\sum_{j=m_2}^{n_2} f_{i_2 j}(x_{2j}) \notin W$ for some $n_2 > m_2$.

Proceeding inductively we have integer sequences $i_1 < i_2 < i_3 \cdots$, $m_1 < n_1 < m_2 < n_2 < \cdots$ and $\{x_{kj} \in B : k, j \in \mathbb{N}\}$ such that

$$(**) \quad \sum_{j=m_k}^{n_k} f_{i_k j}(x_{kj}) \notin W, \ k = 1, 2, 3, \cdots .$$

Consider the matrix $\left[\sum_{j=m_k}^{n_k} f_{i_p j}(x_{kj})\right]_{p,k\in\mathbb{N}}$. As was proved above, $(f_{ij})_{i,j\in\mathbb{N}}$ satisfies the condition (1) and so

$$\lim_p \sum_{j=m_k}^{n_k} f_{i_p j}(x_{kj}) = \sum_{j=m_k}^{n_k} \lim_p f_{i_p j}(x_{kj}) = 0$$

for each $k \in \mathbb{N}$. If $k_1 < k_2 < \cdots$ in $\mathbb{N}$ and

$$x_j = \begin{cases} x_{k_v j}, & m_{k_v} \le j \le n_{k_v}, \ v = 1, 2, 3, \cdots, \\ 0, & \text{otherwise}, \end{cases}$$

then $\{x_j\} \subset B \cup \{0\}$ so $\{x_j\} \in \ell^\infty(X)$, it follows from $(f_{ij})_{i,j\in\mathbb{N}} \in (\ell^\infty(X), c_0(Y))$ and $f_{ij} \in \mathfrak{F}_0(X, Y)$ that

$$\lim_p \sum_{v=1}^{\infty} \sum_{j=m_{k_v}}^{n_{k_v}} f_{i_p j}(x_{k_v j}) = \sum_{j=1}^{\infty} f_{i_p j}(x_j) = 0.$$

Then the Antosik-Mikusinski theorem [2, 6, 8] implies that

$$\lim_k \sum_{j=m_k}^{n_k} f_{i_k j}(x_{kj}) = 0.$$

This contradicts $(**)$ and so (4) holds for $(f_{ij})_{i,j\in\mathbb{N}}$.

Conversely, suppose that (1) and (4) hold for $(f_{ij})_{i,j\in\mathbb{N}}$. For every $\{x_j\} \in \ell^\infty(X)$, $\sum_{j=1}^{\infty} f_{ij}(x_j)$ converges uniformly with respect to $i \in \mathbb{N}$ and so

$$\lim_i \sum_{j=1}^{\infty} f_{ij}(x_j) = \lim_i \lim_n \sum_{j=1}^{n} f_{ij}(x_j) = \lim_n \lim_i \sum_{j=1}^{n} f_{ij}(x_j) = \lim_n \sum_{j=1}^{n} \lim_i f_{ij}(x_j) = 0.$$

Thus, $(f_{ij})_{i,j\in\mathbb{N}} \in (\ell^\infty(X), c_0(Y))$. □

Let E be a vector space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. A function $\|\cdot\| : E \to [0, +\infty)$ is a paranorm if $\|0\| = 0$, $\|-x\| = \|x\|$, $\|x+y\| \le \|x\| + \|y\|$ and $\|t_n x_n - tx\| \to 0$ when $\|x_n - x\| \to 0$ and $t_n \to t$ in $\mathbb{K}$, and Fréchet spaces are just separated complete paranormed spaces [1, p. 56]. A paranorm $\|\cdot\| : E \to [0, +\infty)$ is a seminorm if $\|tx\| = |t|\|x\|$ for $t \in \mathbb{K}$ and $x \in E$.

Corollary 1. *Let X be a topological vector space and $(Y, \|\cdot\|)$ a Fréchet space. Let $f_{ij} \in \mathfrak{F}_0(X,Y)$ for $i,j \in \mathbb{N}$. Then $(f_{ij})_{i,j\in\mathbb{N}} \in (\ell^\infty(X), c_0(Y))$ if and only if*

(1) $\lim_i \|f_{ij}(x)\| = 0$, $\forall x \in X, j \in \mathbb{N}$,

(4′) *for every bounded $B \subset X$ and $\varepsilon > 0$ there is an $m_0 \in \mathbb{N}$ such that*

$$\left\| \sum_{j=m}^{n} f_{ij}(x_j) \right\| < \varepsilon, \ \forall n \ge m > m_0, \ i \in \mathbb{N}, \ \{x_j\} \subset B.$$

Proof. (4)$\Longrightarrow$(4′): Let $B \subset X$ be bounded and $\varepsilon > 0$. By (4) there is an $m_0 \in \mathbb{N}$ such that

$$\left\| \sum_{j=m}^{\infty} f_{ij}(x_j) \right\| < \varepsilon/2, \ \forall m > m_0, \ i \in \mathbb{N}, \ \{x_j\} \subset B.$$

Hence

$$\begin{aligned} \left\| \sum_{j=m}^{n} f_{ij}(x_j) \right\| &= \left\| \sum_{j=m}^{\infty} f_{ij}(x_j) - \sum_{j=n+1}^{\infty} f_{ij}(x_j) \right\| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \quad \forall n \ge m > m_0, \ i \in \mathbb{N}, \ \{x_j\} \subset B. \end{aligned}$$

(4′)$\Longrightarrow$(4): Let $B \subset X$ be bounded. By (4′), $\left\{ \sum_{j=1}^{k} f_{ij}(x_j) \right\}_{k=1}^{\infty}$ are uniformly Cauchy with respect to both $i \in \mathbb{N}$ and $\{x_j\} \subset B$. Since $(Y, \|\cdot\|)$ is complete, $\sum_{j=1}^{\infty} f_{ij}(x_j)$ converge uniformly with respect to both $i \in \mathbb{N}$ and $\{x_j\} \subset B$. □

Corollary 2. *Let X be a seminormed space and Y a Banach space. Let $f_{ij} : X \to Y$ be a linear continuous operator for all $i,j \in \mathbb{N}$. Then $(f_{ij})_{i,j\in\mathbb{N}} \in (\ell^\infty(X), c_0(Y))$ if and only if*

(1) $\lim_i f_{ij}(x) = 0$ *for each $j \in \mathbb{N}$ and $x \in X$,*

(5) $\lim_j \|R_{ij}\| = 0$ *uniformly with respect to $i \in \mathbb{N}$, that is,* $\lim_j \sup_i \|R_{ij}\| = 0$.

Proof. $(4')\Longrightarrow(5)$: Let $B_1 = \{x \in X : \|x\| \le 1\}$ and $\varepsilon > 0$. Thus, by $(4')$ there is an $j_0 \in \mathbb{N}$ such that

$$\|R_{ij}\| = \sup_{p\in\mathbb{N},\{x_k\}\subset B_1} \left\| \sum_{k=j}^{j+p} f_{ik}(x_k) \right\| \le \varepsilon$$

for all $j > j_0$ and $i \in \mathbb{N}$.

$(5)\Longrightarrow(4')$: Let $B \subset X$ be bounded and $\varepsilon > 0$. Then $B \subset n_0 B_1$ for some $n_0 \in \mathbb{N}$. By (5) there is a $j_0 \in \mathbb{N}$ such that $\|R_{ij}\| < \frac{\varepsilon}{n_0}$ for all $j > j_0$ and $i \in \mathbb{N}$.

Let $j > j_0$ and $i, p \in \mathbb{N}$. For every $\{x_k\} \subset B$, $\{\frac{1}{n_0} x_k\} \subset B_1$ and

$$\begin{aligned}
\left\| \sum_{k=j}^{j+p} f_{ik}(x_k) \right\| &= n_0 \left\| \sum_{k=j}^{j+p} f_{ik}(\frac{1}{n_0} x_k) \right\| \\
&\le n_0 \sup_{\{u_k\}\subset B_1} \left\| \sum_{k=j}^{j+p} f_{ik}(u_k) \right\| \\
&\le n_0 \sup_{q\in\mathbb{N},\{u_k\}\subset B_1} \left\| \sum_{k=j}^{j+p} f_{ik}(u_k) \right\| \\
&= n_0 \|R_{ij}\| < n_0 \frac{\varepsilon}{n_0} = \varepsilon.
\end{aligned}$$

□

It is also worthwhile observing that in the condition (2) of the Maddox theorem (Theorem 1 above) the condition "$\|R_{i1}\| < +\infty,\ \forall i \in \mathbb{N}$" can be omitted because the condition "$\lim_j \|R_{ij}\| = 0,\ \forall i \in \mathbb{N}$" of (2) implies "$\|R_{i1}\| < +\infty,\ \forall i \in \mathbb{N}$". In fact, since all f_{ij} are linear and continuous [4, p. 51, 53, Theorem 4.7, 4.8] and

$$\lim_j \|R_{ij}\| = 0, \quad \|R_{ij_0}\| < 1$$

for some $j_0 > 1$ and

$$\begin{aligned}
\|R_{i1}\| &= \sup_{p\in\mathbb{N},\|\{x_j\}\|_\infty\le 1} \left\| \sum_{j=1}^{p} f_{ij}(x_j) \right\| \\
&\le \sup_{k\in\mathbb{N},\|\{x_j\}\|_\infty\le 1} \left(\sum_{j=1}^{j_0-1} \|f_{ij}\| + \left\| \sum_{j=j_0}^{j_0+k-1} f_{ij}(x_j) \right\| \right) \\
&= \sum_{j=1}^{j_0-1} \|f_{ij}\| + \|R_{ij_0}\| < +\infty.
\end{aligned}$$

Hence our Theorem 2 is a substantial improvement of the Maddox theorem, and Corollary 2 is just a clear-cut version of the Maddox theorem.

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