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ON THE PARALLELOGRAM LAW AND BOHR'S INEQUALITY IN G-INNER PRODUCT SPACES

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ABSTRACT. In this paper, we give some results which are in connection to the parallelogram law in G-inner product spaces and also prove some results related to Bohr's inequality in G-inner product spaces.

1. Introduction

Let X be a linear space over the complex numbers and $\|\cdot,\cdot\|$ be a real-valued function defined on $X \times X$ and satisfying the following conditions:

$(2N_1)$ $\|a, b\| = 0$ if and only if a and b are linearly dependent,

$(2N_2)$ $\|a, b\| = \|b, a\|$,

$(2N_3)$ $\|\alpha a, b\| = \alpha\|a, b\|$ for any complex number α,

$(2N_4)$ $\|a + a', b\| \leq \|a, b\| + \|a', b\|$.

Then $\|\cdot,\cdot\|$ is called an 2-norm on X and $(X, \|\cdot,\cdot\|)$ is called a linear 2-normed space.

Let X be a linear space over the complex numbers and $(\cdot,\cdot|\cdot,\cdot)$ be a complex-valued function on $X \times X \times X \times X$ satisfying the following conditions:

(GI_1) $(a, b \mid a, b) = 0$ if and only if a and b are linearly dependent,

(GI_2) $(a, b \mid a, b) \geq 0$,

(GI_3) $(a, b \mid a, b) = (b, a \mid b, a)$,

(GI_4) $(a, b \mid c, b) = \overline{(c, b \mid a, b)}$,

(GI_5) $(\alpha a, b \mid c, b) = \alpha(a, b \mid c, b)$ for any complex number α,

(GI_6) $(a + a', b \mid c, b) = (a, b \mid c, b) + (a', b \mid c, b)$ for all $a, a', b, c \in X$.

Then $(\cdot,\cdot|\cdot,\cdot)$ is called an G-inner product on X and $(X, (\cdot,\cdot|\cdot,\cdot))$ is called an G-inner product space.

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Further, some properties of G-inner product follow immediately from the definition of G-inner product as follows:

(GI_5') $(a, b \mid \alpha c, b) = \overline{\alpha}(a, b \mid c, b)$ for any complex number α,

(GI_6') $(a, b \mid c + c', b) = (a, b \mid c, b) + (a, b \mid c', b)$ for all $a, c, b, c' \in X$.

If an G-inner product space $(X, (\cdot, \cdot|\cdot, \cdot))$ is given, then, for any $a, b, c \in X$, we have the following extension of Cauchy-Buniakowski's inequality:

$$|(a, b \mid c, b)| \leq \sqrt{(a, b \mid a, b)}\sqrt{(c, b \mid c, b)}. \tag{1}$$

Moreover, using $(GI_1)\sim(GI_6)$ and (1), it is easy to see that the function $\|\cdot, \cdot\|$ defined on $X \times X$ by

$$\|a, b\| = \sqrt{(a, b \mid a, b)} \tag{2}$$

is 2-norm for the G-inner product space X.

For the 2-norm defined by (2), we have

$$(a, c \mid b, c) = \frac{1}{4}\left[\|a + b, c\|^2 - \|a - b, c\|^2 + i\|a + ib, c\|^2 - i\|a - ib, c\|^2\right] \tag{3}$$

and the following extension of the parallelogram law is also valid:

$$\|a + b, c\|^2 + \|a - b, c\|^2 = 2\left[\|a, c\|^2 + \|b, c\|^2\right]. \tag{4}$$

Further, from (3) and (4), it follows that

$$\|a + b, c\|^2 = \|a, c\|^2 + \|b, c\|^2 + 2Re(a, c \mid b, c) \tag{5}$$

and

$$\|a - b, c\|^2 = \|a, c\|^2 + \|b, c\|^2 - 2Re(a, c \mid b, c). \tag{6}$$

The details on the definitions and results stated above as well as some further results in G-inner product spaces can be found in the book [1].

In this paper, we show that some known results which are in connection to the parallelogram law are also valid in G-inner product spaces and give some related inequalities.

2. RASSIAS, DRAGOMIR AND SÁNDOR'S INEQUALITY

The following result was proved in [2] and [3]:

Theorem RDS. *Let $(X, (\cdot|\cdot))$ be pre-Hilbert space (real or complex). If $0 < p \leq 2$, then, for any $x, y \in X$,*

$$\begin{aligned}(\|x\| + \|y\|)^p + |\|x\| - \|y\||^p &\leq \|x + y\|^p + \|x - y\|^p \\ &\leq 2(\|x\|^2 + \|y\|^2)^{\frac{p}{2}}\end{aligned} \tag{7}$$

If $p > 2$ *or* $p < 0$, *then the reverse inequalities in* (7) *hold. Further, for* $p = 0$ *and* 2, *the equalities in* (7) *hold.*

Now, by using some properties of G-inner products and 2-norms, we can extend Theorem RDS to the setting of G-inner product spaces as follows:

Theorem 1. *Let* $(X, (\cdot, \cdot|\cdot, \cdot))$ *be an* G*-inner product space with the norm defined by* (2). *If* $a, b, c \in X$ *and* $0 < p \leq 2$, *then*

$$(8)\quad \begin{aligned}(\|a,c\| + \|b,c\|)^p + |\|a,c\| - \|b,c\||^p &\leq \|a+b,c\|^p + \|a-b,c\|^p \\ &\leq 2(\|a,c\|^2 + \|b,c\|^2)^{\frac{p}{2}}.\end{aligned}$$

If $p > 2$ *or* $p < 0$, *then the reverse inequalities in* (8) *hold. Further, for* $p = 0$ *and* 2, *we have the equalities in* (8).

Proof. As in [2] and [3] (see also [4]), we have by, (5) and (6),

$$(9)\quad \begin{aligned}&\|a+b,c\|^p + \|a-b,c\|^p \\ &= (\|a+b,c\|^2)^{\frac{p}{2}} + (\|a-b,c\|^2)^{\frac{p}{2}} \\ &= (\|a,c\|^2 + \|b,c\|^2 + 2Re(a,c \mid b,c))^{\frac{p}{2}} \\ &\quad + (\|a,c\|^2 + \|b,c\|^2 - 2Re(a,c \mid b,c))^{\frac{p}{2}}.\end{aligned}$$

By the Cauchy-Buniakowsky inequality (1), we have

$$\frac{|(a,c \mid b,c)|}{\|a,c\|\,\|b,c\|} \leq 1.$$

So we can set

$$\cos\phi = \frac{Re(a,c \mid b,c)}{\|a,c\|\,\|b,c\|}.$$

Therefore, we can consider the function

$$\begin{aligned}d(\phi) &= (\|a,c\|^2 + \|b,c\|^2 + 2\|a,c\|\|b,c\|\,\cos\phi)^{\frac{p}{2}} \\ &\quad + (\|a,c\|^2 + \|b,c\|^2 - 2\|a,c\|\|b,c\|\,\cos\phi)^{\frac{p}{2}}\end{aligned}$$

for all $\phi \in [0, 2\pi]$. As in [2] or [3], we have that $d(0)$ is a minimum and $d(\frac{\pi}{2})$ is a maximum of the function d.

In fact, if you put $\|x\| = \|a,c\|$ and $\|y\| = \|b,c\|$ in the above function, we can see that $d(\phi)$ is the same with the function considered in [2] or [3] (see also [4]). Then it was proved that $d(0)$ is a minimum and $d(\frac{\pi}{2})$ is a maximum of the function d. Since

$$d(0) = (\|a,c\| + \|b,c\|)^p + |\|a,c\| - \|b,c\||^p$$

and

$$d\left(\frac{\pi}{2}\right) = 2(\|a,c\|^2 + \|b,c\|^2)^{\frac{p}{2}},$$

it follows from (9) that (8) follows.

If $p \geq 2$, we have (8) with the reverse inequalities. For $p = 2$, we have the equalities in (8), which, in fact, is the parallelogram law (4). This completes the proof. □

Theorem 2. *Let X be as in Theorem 1 and $a, b, c \in X$ with $\|a,c\|$, $\|b,c\| \neq 0$. If $0 < p \leq 1$ or $p \geq 2$, then*

$$\begin{aligned}(10)\qquad &(\|a+b,c\|^2 + \|a-b,c\|^2)^p \\ &\geq 2^p\big((\|a,c\|^p + \|b,c\|^p)^2 + (2^p - 2^2)\|a,c\|^p\,\|b,c\|^p\big).\end{aligned}$$

If $p \leq 0$ or $1 \leq p \leq 2$, then we have the reverse inequality of (10).

Proof. For any $s, t > 0$, define

$$f_p(s,t) = (s^2 + t^2)^p - (s^p + t^p)^2 - (2^p - 2^2)(st)^p.$$

Then $f_p(s,t) \geq 0$ if $0 \leq p \leq 1$ or $p \geq 2$ and $f_p(s,t) \leq 0$ if $1 \leq p \leq 2$ or $p \leq 0$ (see [4]) and so, by the parallelogram law (4), we have the conclusion. This completes the proof. □

Remark 1. Theorem 2 is a generalization of Theorem 8 in [4].

3. Klamkin's Inequality

Klamkin [5] has proved the following inequality:

$$(11)\qquad \tilde{S_\lambda} \equiv \left(\sum |\pm V_1 \pm \cdots \pm V_n|^\lambda\right)^{\frac{1}{\lambda}} \geq 2^{\frac{n}{\lambda}}\left(\sum_{i=1}^{n} |V_i|^2\right)^{\frac{1}{2}},$$

where $\lambda > 2$, each V_i is vector in R^s and summation on the left-hand side is taken over all 2^n possible choices of $\pm$ signs. The inequality is reversed for $\lambda < 2\,(\lambda \neq 0)$, while, for $\lambda = 0$ and 2, we have the equality in (11).

Klamkin also has proved that, for $\lambda \geq 2$,

$$(12)\qquad S_\lambda^\lambda \geq 2^n \sum_{i=1}^{n} |V_i|^\lambda.$$

The generalizations of Klamkin's results were given in [6] and [7].

Now, we shall give some generalizations of such results in G-inner product spaces. First, we shall give a generalization of the parallelogram law (4).

Let $x_i, a \in X$ for $i = 1, \cdots, m$. Then we have

$$\sum \|x_1 \pm \cdots \pm x_m, a\|^2 = 2^{m-1} \sum_{i=1}^{m} \|x_i, a\|^2, \tag{13}$$

where the summation on the left-hand side is taken over all 2^{m-1} possible choices of $\pm$ signs.

By induction, we shall give the proof of (13). In fact, for $m = 2$, the equality (13) is the parallelogram law (4). Suppose that (13) is true. Then we have the following generalization of results of the results of Klamkin [5], Pečarić and Janić [7] (see also [4]):

$$\begin{aligned}
&\sum \|x_1 \pm \cdots \pm x_m \pm x_{m+1}, a\|^2 \\
&= \sum (\|(x_1 \pm \cdots \pm x_m) + x_{m+1}, a\|^2 + \|(x_1 \pm \cdots \pm x_m) - x_{m+1}, a\|^2) \\
&= 2 \sum (\|x_1 \pm \cdots \pm x_m, a\|^2 + \|x_{m+1}, a\|^2) \\
&= 2 \sum \|x_1 \pm \cdots \pm x_m, a\|^2 + 2^m \|x_{m+1}, a\|^2 \\
&= 2^m \sum_{i=1}^{m+1} \|x_i, a\|^2.
\end{aligned}$$

The equality (13) can be rewritten in the form:

$$\sum \| \pm x_1 \pm \cdots \pm x_m, a\|^2 = 2^m \sum_{i=1}^{m} \|x_i, a\|^2.$$

If we use the notations:

$$S_\lambda \equiv \left(\sum \| \pm x_1 \pm \cdots \pm x_n, a\|^\lambda\right)^{\frac{1}{\lambda}}$$

and

$$Q_p = \left(\sum_{i=1}^{m} \|x_i, a\|^p\right)^{\frac{1}{p}},$$

then, as in [7], we can prove the following generalization of the result of Pečarić and Janić [7], which is in fact a generalization of Klamkin's inequality:

Theorem 3. *Let X be a G-inner product space with the norm defined by (2), S_λ, Q_p be defined as above and let $x_1, \cdots, x_m, a \in X$.*

(1) If $\lambda > 2$, then

$$S_\lambda \geq 2^{\frac{m}{\lambda}} Q_2, \tag{14}$$

while, for $\lambda < 2$ $(\lambda \neq 0)$, the reverse inequality holds. For $\lambda = 2$, we have the equality in (14).

If $\lambda \geq 2$, then

$$S_\lambda^\lambda \geq 2^m \sum_{i=1}^{m} \|x_i, a\|^\lambda. \tag{15}$$

(2)

(i) *If $p, \lambda \geq 2$, then*

$$2^{\frac{m}{\lambda}} Q_p \leq S_\lambda \leq m^{\frac{1}{2}-\frac{1}{p}} 2^{\frac{m-1}{2}+\frac{1}{\lambda}} Q_p. \tag{16}$$

For $0 < p$ and $\lambda \leq 2$, the reverse inequalities in (16) are valid.

(ii) *If $0 < \lambda \leq 2$ and $p \geq 2$, then*

$$2^{\frac{m-1}{2}+\frac{1}{\lambda}} Q_p \leq S_\lambda \leq m^{\frac{1}{2}-\frac{1}{p}} 2^{\frac{m}{\lambda}} Q_p. \tag{17}$$

For $0 < p \leq 2$ and $\lambda \geq 2$, the reverse inequalities in (17) are valid.

(iii) *If $\lambda > 0$ and $p < 0$, then we have*

$$S_\lambda \geq m^{\frac{1}{2}-\frac{1}{p}} 2^{\frac{m-1}{2}+\frac{1}{\lambda}} Q_p. \tag{18}$$

(iv) *If $\lambda > 2$ and $p < 0$, then we have*

$$S_\lambda \geq m^{\frac{1}{2}-\frac{1}{p}} 2^{\frac{m}{\lambda}} Q_p. \tag{19}$$

For $\lambda < 0$ and $p > 2$, we have the reverse inequality in (19).

(v) *If $\lambda < 0$ and $0 < p < 2$, then we have*

$$S_\lambda \leq 2^{\frac{m}{\lambda}} Q_p. \tag{20}$$

4. Bohr's Inequality

Some generalizations of the parallelogram law (4) were obtained in [16]. For example, the following identity is valid:

$$(m-2)\sum_{k=1}^{m} \|x_k, a\|^2 + \left\| \sum_{k=1}^{m} x_k, a \right\|^2 = \sum_{1 \leq i < j \leq m} \|x_i + x_j, a\|^2. \tag{21}$$

Now, we shall give some inequalities of Bohr's type in G−inner product spaces. Bohr has proved the following [8] (see also [5]):

Theorem B. *If z_1, z_2 are complex numbers and c is a positive number, then*

$$|z_1 + z_2|^2 \leq (1+c)|z_1|^2 + \left(1 + \frac{1}{c}\right)|z_2|^2$$

with the equality if and only if $z_1 = z_2$.

Various generalizations of this inequality for the norms are given in [5] and [9]. The following generalization of Borh's inequality was obtained by Pečarić and Rassias [10], which is a further generalization of the result in Rassias [11]:

Theorem R. *Let* $x_i\,(i = 1, \cdots, n)$ *be elements of an unitary vector space* X *and* $a_{ij}\,(1 \le i < j \le n)$ *be positive numbers. Then*

$$\left\|\sum_{i=1}^{n} x_i\right\|^2 \le \sum_{k=1}^{n} \|x_k\|^2 \left(1 + \sum_{j=k+1}^{n} a_{kj} + \sum_{i=1}^{k-1} \frac{1}{a_{ik}}\right). \tag{22}$$

In this section, we shall extend Theorem R to the setting of $G-$inner product spaces.

Let $b_1, \cdots, b_m, a \in X$ and $p_1, \cdots, p_m$ be nonnegative numbers. A simple consequence of $(2N_3)$ and $(2N_4)$ is

$$\left\|\sum_{i=1}^{m} p_i b_i, a\right\| \le \sum_{i=1}^{m} p_i \|b_i, a\|. \tag{23}$$

Set $P_m \equiv \sum_{i=1}^{m} p_i$. Then, by Jensen's inequality for nondecreasing convex function $f : R^+ \to R^+$, we have

$$f\left(\frac{1}{P_m}\left\|\sum_{i=1}^{m} p_i b_i, a\right\|\right) \le \frac{1}{P_m} \sum_{i=1}^{m} p_i f(\|b_i, a\|). \tag{24}$$

Moreover, if f is a nondecreasing concave function such that $f(0) = 0$ and $p_i \ge 1\,(i = 1, \cdots, m)$, then (24) and Petrović's inequality give

$$f\left(\left\|\sum_{i=1}^{m} p_i b_i, a\right\|\right) \le \sum_{i=1}^{m} p_i f(\|b_i, a\|). \tag{25}$$

Note that the last two inequalities are generalizations of the results from [13] and [14].

For $p_i \equiv 1$ and $f(x) = x^r$, where $r \ge 0$, we can get a generalization of Delbosco's inequality [12] as follows:

$$\|b_1 + \cdots + b_m, a\|^r \le C_{r,m}(\|b_1, a\|^r + \cdots + \|b_m, a\|^r), \tag{26}$$

where $C_{r,m} = m^{r-1}\,(r \ge 1)$ and $C_{r,m} = 1\,(0 \le r < 1)$. If we put

$$f(x) = x^r,\ b_i = \frac{x_i}{p_i},\ p_i = q_i^{\frac{1}{1-r}}$$

for any $r \geq 1$, it follows from (24) that

$$\left\| \sum_{i=1}^{m} x_i, a \right\|^r \leq \left(\sum_{i=1}^{m} q_i^{\frac{1}{1-r}} \right)^{r-1} \sum_{i=1}^{m} q_i \|x_i, a\|^r \tag{27}$$

and, for $0 \leq r < 1$ with $q_i \geq 1$, $(i = 1, \cdots, m)$, from (25) that

$$\left\| \sum_{i=1}^{m} x_i, a \right\|^r \leq \sum_{i=1}^{m} q_i \|x_i, a\|^r. \tag{28}$$

In a special case, (27) gives Bohr's inequality for norms, i.e., if c is a positive number and $x_1, x_2, a \in X$, then

$$\|x_1 + x_2, a\|^2 \leq (1+c)\|x_1, a\|^2 + \left(1 + \frac{1}{c}\right)\|x_2, a\|^2. \tag{29}$$

By the substitutions $q_i \to \frac{1}{p_i}$ $(i = 1, \cdots, m)$, since we have, for $1 < r \leq 2$,

$$\left(\sum_{i=1}^{m} p_i^{\frac{1}{r-1}} \right)^{r-1} \leq \sum_{i=1}^{m} p_i,$$

it follows from (27) that the inequality holds:

$$\frac{\left\| \sum_{i=1}^{m} x_i, a \right\|^r}{\left(\sum_{i=1}^{m} p_i \right)} \leq \frac{\sum_{i=1}^{m} \|x_i, a\|^r}{p_i}, \tag{30}$$

where $p_i > 0\,(i = 1, \cdots, m)$.

Moreover, if $p_1 > 0$, $p_i \leq 0\,(i = 2, \cdots, m)$, $P_m > 0$ and $f : R^+ \to R$ is a nondecreasing convex function, then we have, as in [14],

$$f\left(\frac{1}{P_m} \left\| \sum_{i=1}^{m} p_i x_i, a \right\| \right) \geq \frac{1}{P_m} \sum_{i=1}^{m} p_i f(\|x_i, a\|), \tag{31}$$

which is a simple consequence of (24) if we use the substitutions:

$$p_1 \to P_m,\ p_i \to -p_i,\ (i = 2, \cdots, m),$$
$$x_1 \to \frac{1}{P_m} \sum_{i=1}^{m} p_i x_i\ (x_i \to x_i,\ i = 2, \cdots, m).$$

In (31), if we put

$$f(x) = x^r\,(1 \leq r \leq 2),\ x_i \to \frac{x_i}{p_i},\ p_i |p_i|^{-r} \to q_i,$$

then we get

$$\left\| \sum_{i=1}^{m} x_i, a \right\|^r \geq \left(\sum_{i=2}^{m} q_i |q_i|^{\frac{1}{1-r}} \right)^{1-r} \sum_{i=1}^{m} q_i \|x_i, a\|^r, \tag{32}$$

where

$$0 < q_1 \le \left(\sum_{i=2}^{m} |q_i|^{\frac{1}{1-r}}\right)^{1-r}$$

and $q_i \le 0\,(i = 2, \cdots, m)$. If we now set $q_i \to p_i^{-1}\,(i = 1, \cdots, m)$ and use the following inequality (see [4]):

$$\left(p_1^{\frac{1}{r-1}} - \sum_{2}^{m} |p_i|^{\frac{1}{r-1}}\right)^{r-1} \ge p_1 - \sum_{2}^{m} |p_i| = \sum_{1}^{m} p_i,$$

then we have, for $1 \le r \le 2$,

$$\frac{\left\| \sum_{i=1}^{m} x_i, a \right\|^r}{\left(\sum_{i=1}^{m} p_i\right)} \ge \frac{\sum_{i=1}^{m} \|x_i, a\|^r}{p_i}, \tag{33}$$

where $p_1 > 0$, $p_i \le 0\,(i = 1, \cdots, m)$ and $P_m > 0$. From (30) and (33), for $m = 2$, it follows that

$$\frac{\|x_1 + x_2, a\|^r}{u + v} \le \frac{\|x_1, a\|^r}{u} + \frac{\|x_2, a\|^r}{v}$$

if $uv(u + v) > 0$.

The reverse inequality holds if $uv(u + v) < 0$, where $x_1, x_2, a \in X$ and $1 \le r \le 2$.

Remark 2. The last result for $r = 2$ in the case of complex number was proved by Bergström [15] (see also [4]).

For example, the following result can be proved (see Theorem 4 from [4]):

Let f be a strictly concave function on $I\,(= [0, +\infty))$, $f(uv) \le f(u)f(v)$ for any $u, v \in I$ and

$$\lim_{t \to 0+} \frac{f(t)}{t} = 0, \quad \lim_{t \to +\infty} \frac{f(t)}{t} = +\infty.$$

If $x_i \in X$ (X is a linear 2-normed space), $i = 1, \cdots, m$, $a \in X$ and p_i are positive numbers for $i = 1, \cdots, m$, then

$$f\left(\left\|\sum_{i=1}^{m} x_i, a\right\|\right) \le g\left(\sum_{i=1}^{m} \frac{1}{g^{-1}(p_i)}\right) \sum_{i=1}^{m} p_i f(\|x_i, a\|),$$

where $g(t) = \frac{f(t)}{t}$.

Further, some generalizations of Bohr's type inequalities can be obtained analogously to the results given in [13]. Finally, we shall prove the following result which is in connection to inequalities of Hlawka's and Bohr's type, which is in fact a generalization of the Rassias's inequality [10]:

Theorem 4. *Let X be G-inner product space with the norm defined by* (2), $a, x_i \in X\,(i = 1, \cdots, m)$ *and* $a_{ij}\,(1 \leq i < j \leq m)$ *is positive numbers. Then*

$$\left\|\sum_{i=1}^{m} x_i, a\right\|^2 \leq \sum_{k=1}^{m} \|x_k, a\|^2 \left(1 + \sum_{j=k+1}^{m} a_{kj} + \sum_{i=1}^{k-1} \frac{1}{a_{ik}}\right). \tag{34}$$

Proof. It is clear that the identity (21) is equivalent to

$$\left\|\sum_{k=1}^{m} x_k, a\right\|^2 - \sum_{k=1}^{m} \|x_k, a\|^2 = \sum_{1\leq i<j\leq m} \left(\|x_i + x_j, a\|^2 - \|x_i, a\|^2 - \|x_j, a\|^2\right).$$

Applying (29) to $\|x_i + x_j, a\|^2$, then we obtain

$$\begin{aligned}&\left\|\sum_{k=1}^{m} x_k, a\right\|^2 - \sum_{k=1}^{m} \|x_k, a\|^2 \\ &\leq \sum_{1\leq i<j\leq m} \left((1 + a_{ij})\|x_i, a\|^2 + \left(1 + \frac{1}{a_{ij}}\right)\|x_j, a\|^2 - \|x_i, a\|^2 - \|x_j, a\|^2\right),\end{aligned}$$

i.e.,

$$\left\|\sum_{k=1}^{m} x_k, a\right\|^2 - \sum_{k=1}^{m} \|x_k, a\|^2 \leq \sum_{1\leq i<j\leq m} \left(a_{ij}\|x_i, a\|^2 + \frac{1}{a_{ij}}\|x_j, a\|^2\right),$$

which is equivalent to (34). This completes the proof. □

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