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CONVERGENCE THEOREMS ON VISCOSITY APPROXIMATION METHODS FOR FINITE NONEXPANSIVE MAPPINGS IN BANACH SPACES

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ABSTRACT. Strong convergence theorems on viscosity approximation methods for finite nonexpansive mappings are established in Banach spaces. The main theorem generalize the corresponding result of Kim and Xu [10] to the viscosity approximation method for finite nonexpansive mappings in a reflexive Banach space having a uniformly Gâteaux differentiable norm. Our results also improve the corresponding results of [7, 8, 19, 20].

1. INTRODUCTION

Let E be a real Banach space and C be a nonempty closed convex subset of E. Recall that a mapping $f : C \to C$ is a *contraction* on C if there exists a constant $k \in (0,1)$ such that $\|f(x) - f(y)\| \leq k\|x - y\|$, $x, y \in C$. We use Σ_C to denote the collection of mappings f verifying the above inequality. That is, $\Sigma_C = \{f : C \to C \mid f \text{ is a contraction with constant } k\}$. Let $T : C \to C$ be a nonexpansive mapping (recall that a mapping $T : C \to C$ is *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$, $x, y \in C$), and $Fix(T)$ denote the set of fixed points of T, that is, $Fix(T) = \{x \in C : x = Tx\}$.

We consider the Mann iterative scheme for nonexpansive mapping: for T nonexpansive mapping and $\alpha_n \in (0,1)$,

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \quad n \geq 0, \tag{1.1}$$

where the initial guess $x_0 \in C$ is chosen arbitrarily.

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The Mann iterative scheme for nonlinear mappings has extensively been studied over the last forty years for constructions of fixed points of nonlinear mappings and of solutions of nonlinear operator equations involving nonexpansive mappings, monotone, accretive and pseudo-contractive operators and others. (see, e,g., [2, 3, 9, 11]). In particular, the construction of fixed points of nonexpansive mappings by Mann iterative scheme is important and useful in the theory of nonexpansive mappings and its applications in a number of applied areas, for instance, in image recovery and signal processing (see e.q., [1, 14, 15]).

In 2003, Nakajo and Takahashi [13] proposed the following modification of the Mann iterative scheme (1.1) in a Hilbert space H:

$$\begin{cases} x_0 = x \in C, \\ y_n = \alpha_n x_n + (1-\alpha_n)Tx_n, \\ C_n = \{z \in C : \|y_n - z\| \le \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n \rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}(x_0), \end{cases} \tag{1.2}$$

where P_K denotes the nearest point (metric) projection from H onto a closed convex subset K of H. They proved that if the sequence $\{\alpha_n\}$ is bounded above from one, then $\{x_n\}$ generated by (1.2) converges strongly to $P_{Fix(T)}(x_0)$. Their argument does not work outside the Hilbert space setting.

Recently, without any additional projection in the scheme, Kim and Xu [10] provided a simpler modification of Mann iterative scheme (1.1) in a uniformly smooth Banach space as follows:

$$\begin{cases} x_0 = x \in C, \\ y_n = \beta_n x_n + (1-\beta_n)Tx_n, \\ x_{n+1} = \alpha_n u + (1-\alpha_n) y_n, \end{cases} \tag{1.3}$$

where $u \in C$ is an arbitrary (but fixed) element, and $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in (0,1). They proved that $\{x_n\}$ generated by (1.3) converges to a fixed point of T under the control conditions:

(H1) $\lim_{n\to\infty} \alpha_n = 0, \quad \lim_{n\to\infty} \beta_n = 0;$

(H2) $\sum_{n=0}^{\infty} \alpha_n = \infty, (\text{or, equivalently}, \prod_{n=0}^{\infty}(1-\alpha_n) = 0), \quad \sum_{n=0}^{\infty} \beta_n = \infty;$

(H3) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \quad \sum_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < \infty.$

On the other hand, as the viscosity approximation method, Moudafi [12] and Xu [18] considered the iterative scheme: for T a nonexpansive mapping, $f \in \Sigma_C$ and $\alpha_n \in (0,1)$,

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)Tx_n, \quad n \geq 0. \tag{1.4}$$

Under the conditions (H1), (H2) and (H3) on $\{\alpha_n\}$, Xu [18] showed in a uniformly smooth Banach space that $\{x_n\}$ generated by (1.4) converges strongly to a fixed point of T, which solves certain variational inequality. The results of Xu [18] extended the results of Moudafi [12] to a Banach space setting. In 2006, Jung [8] considered the iterative scheme: for $N > 1$, $T_1, T_2, \cdots, T_N$ nonexpansive mappings, $f \in \Sigma_C$ and $\alpha_n \in (0,1)$,

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)T_{n+1}x_n, \quad n \geq 0, \tag{1.5}$$

where $T_n := T_{n \bmod N}$, and extended results of Xu [18] (and Moudafi [12]) to the case of a family of finite nonexpansive mappings. In particular, under the conditions (H1), (H2) on $\{\alpha_n\}$ and the perturbed control condition on $\{\alpha_n\}$;

$$|\alpha_{n+N} - \alpha_n| \leq o(\alpha_{n+N}) + \sigma_n, \quad \sum_{n=0}^{\infty} \sigma_n < \infty, \tag{H4}$$

he obtained the strong convergence of the sequence $\{x_n\}$ generated by (1.5) to a solution in $\bigcap_{i=1}^N Fix(T_i)$ of certain variational inequality in either a reflexive Banach space having a uniformly Gâteaux differentiable norm with the assumption that every weakly compact convex subset of E has the fixed point property for nonexpansive mappings or a reflexive Banach space having a weakly sequentially continuous duality mapping, and gave an example which satisfies the conditions (H1), (H2) and (H4) on $\{\alpha_n\}$, but fails to satisfy the condition (H3) on $\{\alpha_n\}$ for $N > 1$; $\sum_{n=0}^{\infty} |\alpha_{n+N} - \alpha_n| < \infty$.

Very recently, Yao et al. [19] proposed the following modified Mann iterative scheme: for T nonexpansive mapping, $f \in \Sigma_C$ and $\{\alpha_n\}$, $\{\beta_n\} \subset (0,1)$,

$$\begin{cases} x_0 = x \in C, \\ y_n = \beta_n x_n + (1-\beta_n)Tx_n, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)y_n. \end{cases}$$

By using Lemma 2 of Suzuki [17], they studied strong convergence of this iterative scheme in a uniformly smooth Banach space under the following conditions on the parameters $\{\alpha_n\}$ and $\{\beta_n\}$

$$\text{(H5)} \qquad \alpha_n \to 0, \ \sum_{n=0}^{\infty} \alpha_n = \infty, \quad 0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1.$$

In this paper, motivated by [8, 18, 19], as a viscosity approximation method, we introduce modified Mann iterative scheme for finite nonexpansive mappings : for $N > 1$, $T_1, T_2, \cdots, T_N$ nonexpansive mappings, $f \in \Sigma_C$ and $\{\alpha_n\}$, $\{\beta_n\} \subset (0,1)$,

$$\text{(IS)} \qquad \begin{cases} x_0 = x \in C, \\ y_n = \beta_n x_n + (1-\beta_n) T_{n+1} x_n, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) y_n, \end{cases}$$

and establish the strong convergence of the sequence $\{x_n\}$ generated by (IS) in a reflexive Banach space having a uniformly Gâteaux differentiable norm satisfying that every weakly compact convex subset of E has the fixed point property for nonexpansive mappings under certain appropriate conditions on the parameters $\{\alpha_n\}$ and $\{\beta_n\}$ and the sequence $\{x_n\}$. Moreover, we show that this strong limit is a solution of certain variational inequality. The main results improve the recent result of Kim and Xu [10] to the viscosity approximation method for finite nonexpansive mappings. Our results also generalize the corresponding results of [7, 8, 19, 20].

2. Preliminaries and Lemmas

Let E be a real Banach space with norm $\|\cdot\|$ and let E^* be its dual. The value of $f \in E^*$ at $x \in E$ will be denoted by $\langle x, f\rangle$. When $\{x_n\}$ is a sequence in E, then $x_n \to x$ (resp., $x_n \rightharpoonup x$, $x_n \overset{*}{\rightharpoonup} x$) will denote strong (resp., weak, weak*) convergence of the sequence $\{x_n\}$ to x.

The (*normalized*) *duality mapping* J from E into the family of nonempty (by Hahn-Banach theorem) weak-star compact subsets of its dual E^* is defined by

$$J(x) = \{f \in E^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\}$$

for each $x \in E$. It is single valued if and only if E is smooth.

The norm of E is said to be *Gâteaux differentiable* (and E is said to be *smooth*) if

$$\text{(2.1)} \qquad \lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t}$$

exists for each x, y in its unit sphere $U = \{x \in E : \|x\| = 1\}$. The norm is said to be *uniformly Gâteaux differentiable* if for $y \in U$, the limit is attained uniformly for

$x \in U$. The space E is said to have a *uniformly Fréchet differentiable norm* (and E is said to be *uniformly smooth*) if the limit in (2.1) is attained uniformly for $(x, y) \in U \times U$. It is well-known that if E has a uniformly Gâteaux differentiable norm, the duality mapping J is uniformly norm to weak* continuous on each bounded subsets of E ([4, 5]).

Let C be a nonempty closed convex subset of E. C is said to have the *fixed point property* for nonexpansive mappings if every nonexpansive mapping of a bounded closed convex subset D of C has a fixed point in D. Let D be a subset of C. Then a mapping $Q : C \to D$ is said to be a *retraction* from C onto D if $Qx = x$ for all $x \in D$. A retraction $Q : C \to D$ is said to be *sunny* if $Q(Qx + t(x - Qx)) = Qx$ for all $t \geq 0$ and $x + t(x - Qx) \in C$. A sunny nonexpansive retraction is a sunny retraction which is also nonexpansive. Sunny nonexpansive retractions are characterized as follows [6. p. 48]: If E is a smooth Banach space, then $Q : C \to D$ is a sunny nonexpansive retraction if and only if the following condition holds:

$$\langle x - Qx, J(z - Qx)\rangle \leq 0, \quad x \in C, \quad z \in D. \tag{2.2}$$

We need the following lemmas for the proof of our main results, For these lemmas, we refer to [4, 9, 11].

Lemma 2.1. *Let E be a real Banach space. If E is smooth, then one has*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, J(x + y)\rangle, \quad \text{for all } x, y \in E,$$

Lemma 2.2. *Let $\{s_n\}$ be a sequence of non-negative real numbers satisfying*

$$s_{n+1} \leq (1 - \lambda_n)s_n + \lambda_n\delta_n + \gamma_n, \quad n \geq 0,$$

where $\{\lambda_n\}$, $\{\delta_n\}$ and $\{\gamma_n\}$ satisfy the following conditions:

(i) $\{\lambda_n\} \subset [0, 1]$ *and* $\sum_{n=0}^{\infty} \lambda_n = \infty$,

(ii) $\limsup_{n\to\infty} \delta_n \leq 0$ *or* $\sum_{n=1}^{\infty} \lambda_n\delta_n < \infty$,

(iii) $\gamma_n \geq 0$ $(n \geq 0)$, $\sum_{n=0}^{\infty} \gamma_n < \infty$.

Then $\lim_{n\to\infty} s_n = 0$.

Finally, the sequence $\{x_n\}$ in E is said to be *asymptotically regular* if for $N \geq 1$,

$$\lim_{n\to\infty} \|x_{n+N} - x_n\| = 0.$$

3. MAIN RESULTS

Now, we study the strong convergence results for modified Mann iterative scheme (IS) for finite nonexpansive mappings.

We consider N mappings $T_1, T_2, \cdots, T_N$. For $n > N$, set $T_n := T_{n \bmod N}$, where $n \bmod N$ is defined as follows: if $n = kN + l$, $0 \le l < N$, then

$$n \mod N := \begin{cases} l & \text{if } l \neq 0, \\ N & \text{if } l = 0. \end{cases}$$

For any $n \ge 1$, $T_{n+N}T_{n+N-1}\cdots T_{n+1} : C \to C$ is nonexpansive and so, for any $t \in (0,1)$ and $f \in \Sigma_C$, $tf + (1-t)T_{n+N}T_{n+N-1}\cdots T_{n+1} : C \to C$ defines a strict contraction mapping. Thus, by the Banach contraction mapping principle, there exists a unique fixed point $x_t^n(f)$ satisfying

$$x_t^n(f) = tf(x_t^n(f)) + (1-t)T_{n+N}T_{n+N-1}\cdots T_{n+1}x_t^n(f). \tag{A}$$

For simplicity we will write x_t^n for $x_t^n(f)$ provided no confusion occurs.

The following result for the existence of $Q(f)$ which solves a variational inequality

$$\langle (I-f)(Q(f)), J(Q(f)-p)\rangle \le 0, \quad f \in \Sigma_C, \quad p \in F := \bigcap_{i=1}^{N} Fix(T_i) \neq \emptyset$$

was obtained by Jung [8].

Theorem J ([8]). *Let E be a reflexive Banach space having a uniformly Gâteaux differentiable norm such that every weakly compact convex subset of E has the fixed point property for nonexpansive mappings. Let C be a nonempty closed convex subset of E and $T_1, \cdots, T_N$ nonexpansive mappings from C into itself with $F := \bigcap_{i=1}^{N} Fix(T_i) \neq \emptyset$ and*

$$F = Fix(T_N\cdots T_1) = Fix(T_1T_N\cdots T_3T_2) = \cdots = Fix(T_{N-1}T_{N-2}\cdots T_1T_N).$$

Then $\{x_t^n\}$ defined by (A) *converges strongly to a point in F. If we define $Q : \Sigma_C \to F$ by*

$$Q(f) := \lim_{t\to 0} x_t^n, \quad f \in \Sigma_C, \tag{3.1}$$

then $Q(f)$ is independent of n and $Q(f)$ is the unique solution of a variational inequality

$$\langle (I-f)(Q(f)), J(Q(f)-p)\rangle \le 0, \quad f \in \Sigma_C, \quad p \in F.$$

Remark 3.1. In Theorem 3.1, if $f(x) = u$, $x \in C$, is a constant, then it follows from (2.2) that (3.1) is reduced to the sunny nonexpansive retraction from C onto F,

$$\langle Qu - u, J(Qu - p)\rangle \leq 0, \quad u \in C, \quad p \in F.$$

Using Theorem J, we have the following result.

Theorem 3.1. *Let E be a reflexive Banach space having a uniformly Gâteaux differentiable norm such that every weakly compact convex subset of E has the fixed point property for nonexpansive mappings. Let C be a nonempty closed convex subset of E and $T_1, \cdots, T_N$ nonexpansive mappings from C into itself with $F := \bigcap_{i=1}^N Fix(T_i) \neq \emptyset$ satisfying the following conditions:*

(i) $$T_N T_{N-1} \cdots T_1 = T_1 T_N \cdots T_3 T_2 = \cdots = T_{N-1} T_{N-2} \cdots T_1 T_N;$$

(ii) $$F = Fix(T_N T_{N-1} \cdots T_1) = Fix(T_1 T_N \cdots T_3 T_2) = \cdots = Fix(T_{N-1} \cdots T_1 T_N).$$

Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $(0,1)$ which satisfy the conditions:
(C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(B1) $\lim_{n\to\infty} \beta_n = 0$.
Let $f \in \Sigma_C$ and let $\{x_n\}$ be the sequence generated by

(IS) $$\begin{cases} x_0 = x \in C, \\ y_n = \beta_n x_n + (1-\beta_n) T_{n+1} x_n, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) y_n, \quad n \geq 0. \end{cases}$$

If $\{x_n\}$ is asymptotically regular, then $\{x_n\}$ converges strongly to $Q(f) \in F$, where $Q(f)$ is the unique solution of a variational inequality

$$\langle (I - f)(Q(f)), J(Q(f) - p)\rangle \leq 0 \quad f \in \Sigma_C, \quad p \in F.$$

Proof. First, we note that by Theorem J, there exists a solution $Q(f)$ of a variational inequality

$$\langle (I - f)(Q(f)), J(Q(f) - p)\rangle \leq 0, \quad f \in \Sigma_C, \quad p \in F,$$

where $Q(f) = \lim_{t\to 0} x_t$, and x_t is defined by $x_t = t(x_t) + (1-t)Sx_t$ for $S = T_N T_{N-1} \cdots T_1$ and $t \in (0,1)$.

We proceed with the following steps:

STEP 1. We show that $\|x_n - p\| \leq \max\{\|x_0 - p\|, \frac{1}{1-k}\|f(p) - p\|\}$ for all $n \geq 0$ and all $p \in Fix(T)$ and so $\{x_n\}$ is bounded. Indeed, let $p \in F$ and $d = \max\{\|x_0 -$

$p\|, \frac{1}{1-k}\|f(p)-p\|\}$. Noting that

$$\|y_n - p\| \le \beta_n\|x_n - p\| + (1-\beta_n)\|T_{n+1}x_n - p\| \le \|x_n - p\|,$$

we have

$$\begin{aligned}\|x_1 - p\| &\le (1-\alpha_0)\|y_0 - p\| + \alpha_0\|f(x_0) - p\| \\ &\le (1-\alpha_0)\|x_0 - p\| + \alpha_0(\|f(x_0) - f(p)\| + \|f(p) - p\|) \\ &\le (1-(1-k)\alpha_0)\|x_0 - p\| + \alpha_0\|f(p) - p\| \\ &\le (1-(1-k)\alpha_0)d + \alpha_0(1-k)d = d.\end{aligned}$$

Using an induction, we obtain $\|x_{n+1} - p\| \le d$. Hence $\{x_n\}$ is bounded, and so are $\{y_n\}$, $\{T_{n+1}x_n\}$ and $\{f(x_n)\}$.

STEP 2. We show that $\lim_{n\to\infty}\|x_n - T_{n+N}T_{n+N-1}\cdots T_{n+1}x_n\| = 0$. Indeed, as a consequence with the control conditions (C1) and (B1), from Step 1 we get

$$\begin{aligned}\|x_{n+1} - T_{n+1}x_n\| &\le \|x_{n+1} - y_n\| + \|y_n - T_{n+1}x_n\| \\ &\le \alpha_n(\|f(x_n)\| + \|y_n\|) + \beta_n(\|y_n\| + \|T_{n+1}x_n\|) \to 0 \text{ (as } n\to\infty).\end{aligned}$$

By using the same method, we have

$$\|x_{n+N} - T_{n+N}\cdots T_{n+1}x_n\| \to 0 \quad (\text{as } n\to\infty).$$

Indeed, noting that each T_i is nonexpansive and using just above fact, we obtain the finite table

$$\begin{aligned} x_{n+N} - T_{n+N}x_{n+N-1} &\to 0, \\ T_{n+N}x_{n+N-1} - T_{n+N}T_{n+N-1}x_{n+N-2} &\to 0, \\ &\vdots \\ T_{n+N}\cdots T_{n+2}x_{n+1} - T_{n+N}\cdots T_{n+1}x_n &\to 0.\end{aligned}$$

Adding up this table yields

$$x_{n+N} - T_{n+N}\cdots T_{n+1}x_n \to 0 \quad (\text{as } n\to\infty).$$

Since $\{x_n\}$ is asymptotically regular, we have

$$\lim_{n\to\infty}\|x_n - T_{n+N}T_{n+N-1}\cdots T_{n+1}x_n\| = 0.$$

STEP 3. We show that $\lim_{n\to\infty}\|x_n - Sx_n\| = 0$ for $S = T_NT_{N-1}\cdots T_1$. Indeed, it is easily to see the following:

If $n \mod N = 1$, then $T_{n+N}T_{n+N-1}\cdots T_{n+1} = T_1T_N\cdots T_2$;

If $n \mod N = 2$, then $T_{n+N}T_{n+N-1}\cdots T_{n+1} = T_2T_1T_N\cdots T_3$;

$\vdots$

If $n \mod N = N$, then $T_{n+N}T_{n+N-1}\cdots T_{n+1} = T_NT_{N-1}\cdots T_1$.

In view of the condition (i)

$$T_N T_{N-1} \cdots T_1 = T_1 T_N \cdots T_3 T_2 = \cdots = T_{N-1} T_{N-2} \cdots T_1 T_N,$$

so we have

$$T_N T_{N_1} \cdots T_1 = T_{n+N} T_{n+N-1} \cdots T_{n+1}, \quad \text{for all } n \geq 1.$$

By Step 2, this implies that

$$\begin{aligned} \|x_n - Sx_n| &= \|x_n - T_N T_{N-1} \cdots T_1 x_n\| \\ &= \|x_n - T_{n+N} T_{n+N-1} \cdots T_{n+1} x_n\| \to 0 \ \ (\text{as } n \to \infty). \end{aligned}$$

STEP 4. We show that $\limsup_{n\to\infty}\langle (I-f)(Q(f)), J(Q(f)-x_n)\rangle \leq 0$. To prove this, let a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ be such that

$$\limsup_{n\to\infty}\langle (I-f)(Q(f)), J(Q(f)-x_n)\rangle = \lim_{j\to\infty}\langle (I-f)(Q(f)), J(Q(f)-x_{n_j})\rangle$$

and $x_{n_j} \rightharpoonup p$ for some $p \in E$. Since

$$x_t - x_n = (1-t)(Sx_t - x_n) + t(f(x_t) - x_n),$$

by Lemma 2.1, we have

$$\|x_t - x_n\|^2 \leq (1-t)^2\|Sx_t - x_n\|^2 + t\langle f(x_t) - x_n, J(x_t - x_n)\rangle.$$

Putting

$$a_j(t) = (1-t)^2\|Sx_{n_j} - x_{n_j}\|(2\|x_t - x_{n_j}\| + \|Sx_{n_j} - x_{n_j}\|) \to 0 \ (\text{as } j \to \infty)$$

by Step 3 and using Lemma 2.1, we obtain

$$\begin{aligned} \|x_t - x_{n_j}\|^2 &\leq (1-t)^2\|Sx_t - x_{n_j}\|^2 + 2t\langle f(x_t) - x_{n_j}, J(x_t - x_{n_j})\rangle \\ &\leq (1-t)^2(\|Sx_t - Sx_{n_j}\| + \|Sx_{n_j} - x_{n_j}\|)^2 \\ &\qquad + 2t\langle f(x_t) - x_t, J(x_t - x_{n_j})\rangle + 2t\|x_t - x_{n_j}\|^2 \\ &\leq (1-t)^2\|x_t - x_{n_j}\|^2 + a_j(t) \\ &\qquad + 2t\langle f(x_t) - x_t, J(x_t - x_{n_j})\rangle + 2t\|x_t - x_{n_j}\|^2. \end{aligned}$$

The last inequality implies

$$\langle x_t - f(x_t), J(x_t - x_{n_j})\rangle \leq \frac{t}{2}\|x_t - x_{n_j}\|^2 + \frac{1}{2t}a_j(t).$$

It follows that

$$\limsup_{j\to\infty}\langle x_t - f(x_t), J(x_t - x_{n_j})\rangle \leq \frac{t}{2}M, \tag{3.2}$$

where $M > 0$ is a constant such that $M \geq \|x_t - x_n\|^2$ for all $n \geq 0$ and $t \in (0,1)$. Taking the lim sup as $t \to 0$ in (3.2) and noticing the fact that the two limits are interchangeable due to the fact that J is uniformly continuous on bounded subsets of E from the strong topology of E to the weak* topology of E^*, we have

$$\limsup_{j\to\infty}\langle (I-f)(Q(f)), J(Q(f)-x_{n_j})\rangle \le 0.$$

STEP 5. We show that $\lim_{n\to\infty}\|x_n - Q(f)\| = 0$. By using (IS), we have

$$x_{n+1} - Q(f) = \alpha_n(f(x_n) - Q(f)) + (1-\alpha_n)(y_n - Q(f)).$$

Applying Lemma 2.1, we obtain

$$\begin{aligned}
&\|x_{n+1} - Q(f)\|^2 \\
&\le (1-\alpha_n)^2\|y_n - Q(f)\|^2 + 2\alpha_n\langle f(x_n) - Q(f), J(x_{n+1} - Q(f))\rangle \\
&\le (1-\alpha_n)^2\|x_n - Q(f)\|^2 + 2\alpha_n\langle f(x_n) - f(Q(f)), J(x_{n+1} - Q(f))\rangle \\
&\quad + 2\alpha_n\langle f(Q(f)) - Q(f), J(x_{n+1} - Q(f))\rangle \\
&\le (1-\alpha_n)^2\|x_n - Q(f)\|^2 + 2k\alpha_n\|x_n - Q(f)\|\|x_{n+1} - Q(f)\| \\
&\quad + 2\alpha_n\langle f(Q(f)) - Q(f), J(x_{n+1} - Q(f))\rangle \\
&\le (1-\alpha_n)^2\|x_n - Q(f)\|^2 + k\alpha_n(\|x_n - Q(f)\|^2 + \|x_{n+1} - Q(f)\|^2) \\
&\quad + 2\alpha_n\langle f(Q(f)) - Q(f), J(x_{n+1} - Q(f))\rangle.
\end{aligned}$$

It then follows that

$$\begin{aligned}
\|x_{n+1} - Q(f)\|^2 &\le \frac{1-(2-k)\alpha_n + \alpha_n^2}{1-k\alpha_n}\|x_n - Q(f)\|^2 \\
&\quad + \frac{2\alpha_n}{1-k\alpha_n}\langle (I-f)(Q(f)), J(Q(f) - x_{n+1})\rangle \\
&\le \frac{1-(2-k)\alpha_n}{1-k\alpha_n}\|x_n - Q(f)\|^2 + \frac{\alpha_n^2}{1-k\alpha_n}M \\
&\quad + \frac{2\alpha_n}{1-k\alpha_n}\langle (I-f)(Q(f)), J(Q(f) - x_{n+1})\rangle,
\end{aligned} \tag{3.3}$$

where $M = \sup_{n\ge 0}\|x_n - Q(f)\|^2$. Put

$$\lambda_n = \frac{2(1-k)\alpha_n}{1-k\alpha_n} \quad \text{and} \quad \delta_n = \frac{M\alpha_n}{2(1-k)} + \frac{1}{1-k}\langle (I-f)(Q(f)), J(Q(f) - x_{n+1})\rangle.$$

From the condition (C1) on $\{\alpha_n\}$ and Step 4, it follows that $\lambda_n \to 0$, $\sum_{n=0}^{\infty}\lambda_n = \infty$ and $\limsup_{n\to\infty}\delta_n \le 0$. Since (3.3) reduces to

$$\|x_{n+1} - Q(f)\|^2 \le (1-\lambda_n)\|x_n - Q(f)\|^2 + \lambda_n\delta_n,$$

from Lemma 2.2 with $\gamma_n = 0$, we conclude that $\lim_{n\to\infty}\|x_n - Q(f)\| = 0$. This completes the proof. □

Remark 3.2. If $\{\alpha_n\}$ and $\{\beta_n\}$ in Theorem 3.1 satisfy conditions:

(B2) $\sum_{n=0}^{\infty} |\beta_{n+N} - \beta_n| < \infty$,

(C1) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$ and

(C2) $\sum_{n=0}^{\infty} |\alpha_{n+N} - \alpha_n| < \infty$, or,

(C3) $|\alpha_{n+N} - \alpha_n| \leq o(\alpha_{n+N}) + \sigma_n$, $\sum_{n=0}^{\infty} \sigma_n < \infty$ (the perturbed control condition),

then the sequence $\{x_n\}$ generated by (IS) is asymptotically regular.

Now we give only the proof in the case when $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy the conditions (B2), (C1)) and (C3). Indeed, by Step 1 in the proof of Theorem 3.1, there exists a constant $L > 0$ such that

$$L = \max\left\{ \sup_{n\geq 0}\{\|f(x_n)\| + \|T_{n+1}x_n\|\},\ \sup_{n\geq 0}\{\|x_n\| + \|T_{n+1}x_n\|\} \right\}.$$

Since for all $n \geq 1$, $T_{n+N} = T_n$, we have

$$\begin{aligned} &x_{n+N} - x_n \\ &= (\alpha_{n+N-1} - \alpha_{n-1})(f(x_{n-1}) - T_n x_{n-1}) + (1 - \alpha_{n+N-1})\beta_{n+N-1}(x_{n+N-1} - x_{n-1}) \\ &\quad + [(\beta_{n+N-1} - \beta_{n-1})(1 - \alpha_{n+N-1}) - (\alpha_{n+N-1} - \alpha_{n-1})\beta_{n-1}](x_{n-1} - T_n x_{n-1}) \\ &\quad + (1 - \beta_{n+N-1})(1 - \alpha_{n+N-1})(T_{n+N}x_{n+N-1} - T_{n+N}x_{n-1}) \\ &\quad + \alpha_{n+N-1}(f(x_{n+N-1}) - f(x_{n-1})), \end{aligned}$$

and so

$$\begin{aligned} &\|x_{n+N} - x_n\| \\ &\leq (1 - \beta_{n+N-1})(1 - \alpha_{n+N-1})\|x_{n+N-1} - x_{n-1}\| \\ &\quad + k\alpha_{n+N-1}\|x_{n+N-1} - x_{n-1}\| + (1 - \alpha_{n+N-1})\beta_{n+N-1}\|x_{n+N-1} - x_{n-1}\| \\ &\quad + |\beta_{n+N-1} - \beta_{n-1}|L + 2|\alpha_{n+N-1} - \alpha_{n-1}|L \\ &\leq (1 - (1-k)\alpha_{n+N-1})\|x_{n+N-1} - x_{n-1}\| + |\beta_{n+N-1} - \beta_{n-1}|L \\ &\quad + 2(o(\alpha_{n+N-1}) + \sigma_{n-1})L. \end{aligned}$$

By taking $s_{n+1} = \|x_{n+N} - x_n\|$, $\lambda_n = (1-k)\alpha_{n+N-1}$, $\lambda_n\delta_n = 2o(\alpha_{n+N-1})L$ and $\gamma_n = |\beta_{n+N-1} - \beta_{n-1}|L + 2\sigma_{n-1}L$, we have

$$s_{n+1} \leq (1 - \lambda_n)s_n + \lambda_n\delta_n + \gamma_n.$$

Hence, by (B2), (C1), (C3) and Lemma 2.2,

$$\lim_{n\to\infty} \|x_{n+N} - x_n\| = 0.$$

In view of this observation, we have the following result.

Corollary 3.1. *Let E be a uniformly smooth Banach space. Let C be a nonempty closed convex subset of E and $T_1, \cdots, T_N$ nonexpansive mappings from C into itself with $F := \bigcap_{i=1}^{N} Fix(T_i) \neq \emptyset$ satisfying the following conditions:*

(i) $$T_N T_{N-1} \cdots T_1 = T_1 T_N \cdots T_3 T_2 = \cdots = T_{N-1} T_{N-2} \cdots T_1 T_N;$$

(ii) $$F = Fix(T_N T_{N-1} \cdots T_1) = Fix(T_1 T_N \cdots T_3 T_2) = \cdots = Fix(T_{N-1} \cdots T_1 T_N).$$

Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $(0,1)$ which satisfy the conditions

(C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(C3) $|\alpha_{n+N} - \alpha_n| \leq o(\alpha_{n+N}) + \sigma_n, \quad \sum_{n=0}^{\infty} \sigma_n < \infty$;

(B1) $\lim_{n\to\infty} \beta_n = 0$;

(B2) $\sum_{n=0}^{\infty} |\beta_{n+N} - \beta_n| < \infty$,

(or the conditions (C1), (C2), (B1), *and* (B2) *in Remark* 3.2*). Let $f \in \Sigma_C$ and let $\{x_n\}$ be the sequence generated by*

$$\begin{cases} x_0 = x \in C, \\ y_n = \beta_n x_n + (1-\beta_n) T_{n+1} x_n, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) y_n, \quad n \geq 0. \end{cases}$$

Then $\{x_n\}$ converges strongly to $Q(f) \in F$, where $Q(f)$ is the unique solution of a variational inequality

$$\langle (I-f)(Q(f)), J(Q(f)-p) \rangle \leq 0, \quad f \in \Sigma_C, \quad p \in F$$

Proof. Since E is a uniformly smooth Banach space, E is reflexive and the norm is uniformly Gâteaux differentiable norm and its every nonempty weakly compact convex subset of E has the fixed point property for nonexpansive mappings. Thus the conclusion of Corollary 3.1 follows from Theorem 3.1 and Remark 3.2 immediately. □

Remark 3.3. (1) Theorem 3.1 improves the corresponding results of Jung [7, 8], Kim and Xu [10] and Zhou et al. [20] (that is, Theorem 10 of [7], Theorem 2 in [8], Theorem 1 in [10], and Theorem 6 and Theorem 10 in [20]) in several aspects.

(2) Theorem 3.1 also extends Theorem 1 of Yao et al. [19] to the case of a family of finite mappings in more general Banach spaces under the different conditions on the parameter $\{\beta_n\}$ and the sequence $\{x_n\}$.

(3) Corollary 3.1 also improves Theorem 1 of Kim and Xu [10] to the viscosity approximation method for finite nonexpansive mappings together with the condition (C3) different from the condition (C2) on $\{\alpha_n\}$.

(4) Even the case of $\beta_n = 0$, Corollary 3.1 generalizes Theorem 10 of Jung [7] and Theorem 2 of Jung [8] since the assumption of the weakly sequentially continuous duality mapping was removed.

(5) Even the case of $f(x) = u$, $x \in C$, a constant, Theorem 3.1 and Corollary 3.1 also work in a Banach space setting as opposed to iterative scheme of Nakajo and Takahashi [13], which works in only in the framework of Hilbert spaces.

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