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ON RELATION AMONG COHERENT, DISTORTION AND SPECTRAL RISK MEASURES

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ABSTRACT. In this paper we examine the relation among law-invariant coherent risk measures with the Fatou property, distortion risk measures and spectral risk measures, and give a new proof of the relation among them. It is also shown that the spectral risk measure satisfies the monotonicity with respect to stochastic dominance and the comonotonic additivity.

1. INTRODUCTION

The measurement or quantification of the market risk has been introduced and discussed from theoretical and practical perspectives. Markowitz [9] used the standard deviation to measure the market risk in his portfolio theory but his method doesn't tell the difference between the positive and the negative deviation.

The value at risk (VaR) is widely used by corporate treasurers and fund managers as well as by financial institutions as a quantitative measurement of market risk. VaR is a quantity which is the maximum loss not exceeded with a given confidence level $\lambda\%$ over a given period of time. VaR provides the threshold of the potential loss of the financial position within a given confidence level, but doesn't give any information about the size of a loss when the loss exceeds the VaR. VaR doesn't satisfy the subadditivity property and so is not a convex risk measure. So VaR discourages the diversification.

Artzner et al. [3, 4] proposed a coherent risk measure in an axiomatic approach, and formulated the representation theorems. Rockafella and Uryasev [10, 11], and Acerbi and Tasche [2] studied the coherent risk measure, the average value at risk ($AVaR$), which is also called conditional value at risk ($CVaR$) or expected shortfall

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(ES). $AVaR$ is the most popular alternative to the VaR. $AVaR$ is defined in terms of VaR and is a coherent risk measure which can be viewed as the building blocks for law-invariant convex risk measures on L^∞.

Acerbi [1] proposed the spectral risk measure ρ_ϕ generated by the risk aversion function ϕ. Spectral risk measure is a subset of coherent risk measures satisfying both law-invariance and comonotone additivity properties. Spectral risk measure is reduced to $AVaR$ when the weighting function is taken as $\phi(t) = \frac{1}{\lambda} I_{0\leq t\leq \lambda}$.

Wang [12] introduced a family of risk measure, i.e., distortion risk measures, in terms of distortion function (also see [5]). A coherent risk measure with the Fatou property which is law-invariant can be shown to be identified with the distortion risk measures of the loss with respect to a concave distortion function (refer to [7]).

The contribution of this paper is devoted to giving a new proof on the relation among distortion risk measures, spectral risk measures and law-invariant coherent risk measures with the Fatou property.

This paper is consisted of as follows. In Section 2, the basic properties of coherent risk measure and other risk measures are discussed. In Section 3, the definitions and basic properties of spectral and distortion risk measures are given. In Section 4, the relation on coherent, distortion and spectral risk measure are characterized. It is proven that the spectral risk measure is a coherent risk measure with both the monotonicity with respect to stochastic dominance and the comonotonic additivity.

2. Coherent Risk Measure and Other Risk Measures

The definitions of coherent risk measure and other risk measures such as VaR, $AVaR$, WCE and TCE are given, and the relation of those risk measures are provided.

Definition 2.1. The probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is called *atomless* if there is no set $A \in \mathcal{F}$ such that $\mathbb{P}[A] > 0$ and $\mathbb{P}[B] = 0$ or $\mathbb{P}[A] = \mathbb{P}[B]$ whenever $B \in \mathcal{F}$ is a subset of A.

Notice that probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is atomless if and only if $(\Omega, \mathcal{F}, \mathbb{P})$ supports a random variable with a continuous distribution. Assume that the given probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is atomless.

Definition 2.2. A *coherent risk measure* $\rho : L^\infty(\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R} \cup \{\infty\}$ is a mapping satisfying for X, $Y \in L^\infty(\Omega, \mathcal{F}, \mathbb{P})$

(1) $\rho(X+Y) \leq \rho(X) + \rho(Y)$ (subadditivity),
(2) $\rho(\lambda X) = \lambda\rho(X)$ for $\lambda \geq 0$ (positive homogeneity),
(3) $\rho(X) \geq \rho(Y)$ if $X \leq Y$ (monotonicity),
(4) $\rho(X+m) = \rho(X) - m$ for $m \in \mathbb{R}$ (cash invariance).

The subadditivity and the positive homogeneity can be relaxed to a weaker quantity, i.e., convexity

$$\rho(\lambda X + (1-\lambda)Y) \leq \lambda\rho(X) + (1-\lambda)\rho(Y) \quad \forall\lambda \in [0,1],$$

which means diversification should not increase the risk.

For $t \in (0,1)$, a *t-quantile* of a random variable X on $(\Omega, \mathcal{F}, \mathbb{P})$ is a $q_X(t) \in \mathbb{R}$ such that

$$\mathbb{P}[X < q_X(t)] \leq t \leq \mathbb{P}[X \leq q_X(t)].$$

Notice that the quantile function $q_X : (0,1) \to \mathbb{R}$ is an inverse function of a distribution function F of a random variable X such that

$$F(q_X(t)-) \leq t \leq F(q_X(t)) \quad \text{for all} \quad t \in (0,1).$$

The *upper* and the *lower* quantile functions of X are defined as

$$\begin{aligned} q_X^+(t) &= \inf\{x \in \mathbb{R} \mid \mathbb{P}[X \leq x] > t\} = \sup\{x \in \mathbb{R} \mid \mathbb{P}[X < x] \leq t\}, \\ q_X^-(t) &= \sup\{x \in \mathbb{R} \mid \mathbb{P}[X < x] < t\} = \inf\{x \in \mathbb{R} \mid \mathbb{P}[X \leq x] \geq t\}, \end{aligned}$$

respectively.

Definition 2.3. Let $\lambda \in (0,1)$. The *value at risk* at level λ (VaR) is defined as

$$VaR_\lambda(X) := -q_X^+(\lambda).$$

VaR systematically underestimates the risks of the potential loss by taking out the least dangerous scenario. Notice that

$$\begin{aligned} q_{-X}^-(1-\lambda) &= \sup\{x \in \mathbb{R} \mid \mathbb{P}[X \leq -x] > \lambda\} = -\inf\{x \in \mathbb{R} \mid \mathbb{P}[X \leq x] > \lambda\} \\ &= -q_X^+(\lambda), \\ -q_X^+(\lambda) &= -\sup\{x \in \mathbb{R} \mid \mathbb{P}[X < x] \leq \lambda\} = \inf\{x \in \mathbb{R} \mid \mathbb{P}[X < -x] \leq \lambda\} \\ &= \inf\{x \in \mathbb{R} \mid \mathbb{P}[x + X < 0] \leq \lambda\}. \end{aligned}$$

Example 2.4. Consider a portfolio X with two situations, where the loss of X is $-\$200$ with 5% probability and the profit of X is \$20 with 95% probability. Then

$$VaR_{5\%}(X) = -\sup\{x \mid \mathbb{P}[X < x] \leq 0.05\} = \$200.$$

On the other hand, consider a portfolio Y with three situations, where the loss of X is $-\$300$ with 4% probability, $-\$200$ with 1% probability, and the profit of X is \$20 with 95% probability. Then

$$VaR_{5\%}(Y) = -\sup\{x \mid \mathbb{P}[X < x] \leq 0.05\} = \$200.$$

Easily we can see that the portfolio Y is riskier than the portfolio X, but X and Y have the same VaR.

Average value at risk ($AVaR$) is introduced to remedy underestimation of the risks of the left tail of the distribution.

Definition 2.5. The *average value at risk* of a position X at level $\lambda \in (0,1]$ is defined as

$$AVaR_\lambda(X) := \frac{1}{\lambda}\int_0^\lambda VaR_t(X)\,dt.$$

When $\lambda = 0$, it is defined as

$$AVaR_0(X) := -ess\,inf\,X,$$

which is the same as the worst-case risk measure,

$$\rho_{\max}(X) := \inf\{x\in\mathbb{R} \mid X + x \geq 0 \;\; \mathbb{P} - a.s.\}$$

defined on L^∞.

In the case of Example 2.4,

$$AVaR_{5\%}(X) = \frac{1}{0.05}(0.05 \times 200) = 200,$$
$$AVaR_{5\%}(Y) = \frac{1}{0.05}(0.04 \times 300 + 0.01 \times 200) = 280.$$

So the $AVaR$ shows that the portfolio Y is riskier than X.

Definition 2.6. The *worst conditional expectation* at level λ (WCE_λ) is defined as

$$WCE_\lambda(X) = \sup\{E[-X|A] \mid A \in \mathcal{F},\, \mathbb{P}(A) > \lambda\}.$$

The $WCE_\lambda(X)$ is a coherent risk measure induced by $\mathbb{Q}$, where $\mathbb{Q}$ is defined as

$$\mathbb{Q} = \{\mathbb{P}[\cdot|A] \mid A \in \mathcal{F},\, \mathbb{P}(A) > \lambda \text{ for some fixed } \lambda \in (0,1)\}.$$

Definition 2.7. Let $\lambda \in (0,1]$ be a given confidence level. The *tail conditional expectation* (TCE_λ) is defined as

$$TCE_\lambda(X) = E[-X \mid -X \leq VaR_\lambda(X)].$$

In general, VaR and TCE doesn't satisfy the subadditivity axiom (see [1, 6]). WCE is a coherent risk measure but not law-invariant (see Definition 3.1).

Proposition 2.8 ([6]). *For all $X \in L^\infty$ and $\lambda \in (0,1]$*

$$AVaR_\lambda(X) \geq WCE_\lambda(X) \geq TCE_\lambda(X) \geq VaR_\lambda(X).$$

If X has a continuous distribution, then the first two inequalities become equal.

3. Spectral and Distortion Risk Measures

In this section, it is shown that the spectral risk measure is a coherent risk measure and $AVaR$ is a special case of the spectral risk measure. It is also shown that the coherent risk measure which is law-invariant and has the Fatou property is represented in terms of $AVaR_\lambda$ and quantiles q_X. The distortion risk measure is defined in terms of distortion function.

Definition 3.1. A risk measure ρ on L^∞ is called *law-invariant* if

$$\rho(X) = \rho(Y)$$

whenever $F_X(\cdot) = F_Y(\cdot)$. Here $F_X(x)$ is a distribution function of X.

The law-invariance of the risk measure is an important property from a practical perspective, which means that it can be estimable from the empirical data.

It is said that ρ has the *Fatou property* if for the bounded sequences (X_n) which converges $\mathbb{P}$-a.s. to some X,

$$\rho(X) \leq \liminf_{n\to\infty} \rho(X_n).$$

Definition 3.2. An element $\phi \in L^1[0,1]$ is called an *admissible risk spectrum* or *risk aversion function* if it satisfies

(1) $\phi \geq 0$

(2) ϕ is non-increasing

(3) $\|\phi\|_{L^1} = \int_0^1 |\phi(t)|\, dt = 1.$

Definition 3.3. For an admissible risk spectrum $\phi \in L^1[0,1]$, the *spectral risk measure* M_ϕ generated by ϕ is defined as

$$M_\phi(X) := -\int_0^1 q_X^+(t)\phi(t)\, dt.$$

The function $\phi : [0,1] \to \mathbb{R}$ can be thought as assigning different weights $\phi(t)dt$ to each $q_X^+(t)$. Notice that $q_X^+(t) = q_X(t)$ for a.e. $t \in (0,1)$.

Example 3.4. The examples of an admissible risk spectrum $\phi \in L^1[0,1]$ are $\phi(t) = \frac{1}{\lambda} I_{0 \le t \le \lambda}$ in which we have $M_\phi(X) = AVaR_\lambda(X)$, and

$$\phi_r(t) = \frac{e^{-t/r}}{r(1-e^{-1/r})}, \quad r \in (0,\infty).$$

Lemma 3.5. *Let $\rho_\lambda(X)$ be a coherent risk measure. Let μ be a measure defined on $[0,1]$ satisfying $\int_0^1 \mu(d\lambda) = 1$. Then*

$$\rho(X) := \int_0^1 \rho_\lambda(X)\, \mu(d\lambda)$$

is also a coherent risk measure. In general, the convex combination of coherent risk measures is a coherent risk measure.

Proof. It is straightforward. □

Theorem 3.6 ([6]). *For $\lambda \in (0,1]$, $AVaR_\lambda$ is a coherent risk measure with the Fatou property and is represented as*

$$AVaR_\lambda(X) = \max_{\mathbb{Q}\in\mathcal{Q}_\lambda} E_{\mathbb{Q}}[-X], \quad X \in L^\infty$$

where $\mathcal{Q}_\lambda$ is defined as

$$\mathcal{Q}_\lambda = \Big\{ \mathbb{Q} << \mathbb{P} \;\Big|\; \frac{d\mathbb{Q}}{d\mathbb{P}} \le \frac{1}{\lambda} \quad \mathbb{P}-a.s. \Big\}.$$

Theorem 3.7. *The spectral risk measure $M_\phi(X)$ is a coherent risk measure.*

Proof. This is a modification of the proof in [1]. Define a locally finite positive measure ν on $[0,1]$ as

$$\nu((t,1]) = \int_{(t,1]} \nu(d\lambda) := \phi(t), \quad 0 \le t < 1 \quad \text{and}$$
$$\nu(\{0\}) = 0.$$

So by the Fubini's theorem the spectral risk measure M_ϕ is expressed as

$$\begin{aligned} M_\phi(X) &= -\int_0^1 q_X^+(t)\Big(\int_t^1 \nu(d\lambda)\Big)\,dt = -\int_0^1 \Big(\int_0^\lambda q_X^+(t)\,dt\Big)\nu(d\lambda) \\ &= -\int_0^1 AVaR_\lambda(X)\,\lambda\,\nu(d\lambda). \end{aligned}$$

Moreover, we have

$$\int_0^1 \lambda\, \nu(d\lambda) = \int_0^1 \Big(\int_0^\lambda dt \Big) \nu(d\lambda) = \int_0^1 \Big(\int_t^1 \nu(d\lambda) \Big) dt$$
$$= \int_0^1 \phi(t)\, dt = 1.$$

Clearly, $\lambda\, \nu(d\lambda)$ is positive measure on $[0,1]$. Since $AVaR$ is a coherent risk measure by Theorem 3.6, it is done by Lemma 3.5. □

Let $\mathcal{M}_1(\mathbb{P}) := \mathcal{M}_1(\Omega, \mathcal{F}, \mathbb{P})$ be the set of all probability measures on $(\Omega, \mathcal{F})$ which are absolutely continuous with respect to $\mathbb{P}$.

The law-invariant coherent risk measures $\rho : L^\infty(\mathbb{P}) \to \mathbb{R}$ with the Fatou property is represented in terms of $AVaR_\lambda$ and quantile functions q_X.

Theorem 3.8 ([6]). *The law-invariant coherent risk measures $\rho : L^\infty(\mathbb{P}) \to \mathbb{R}$ with the Fatou property is represented as*

$$\rho(X) = \sup_{m \in \mathcal{M}} \Big(\int_{[0,1]} AVaR_\lambda(X)\, m(d\lambda) \Big)$$
$$= \sup_{\mathbb{Q} \in \mathcal{M}_1(\mathbb{P})} \Big(\int_0^1 q_{-X}(t) q_\varphi(t)\, dt \Big)$$

for some set $\mathcal{M} \subset \mathcal{M}_1([0,1])$ where $\varphi := \frac{d\mathbb{Q}}{d\mathbb{P}}$.

Definition 3.9. Let $\psi : [0,1] \to [0,1]$ be increasing function with $\psi(0) = 0$ and $\psi(1) = 1$, which is called a distortion function. The set function

$$c_\psi(A) := \psi(\mathbb{P}(A)), \quad A \in \mathcal{F}$$

is called *distortion* of $\mathbb{P}$ with respect to the distortion function ψ.

Definition 3.10. A set function $c : \mathcal{F} \to [0,1]$ is called *monotone* if

$$c(A) \subset c(B) \quad \text{for } A \subset B$$

and *normalized* if $c(\emptyset) = 0$ and $c(\Omega) = 1$.

Definition 3.11. Let $c : \mathcal{F} \to [0,1]$ be monotone and normalized set function. The *Choquet integral* or *concave distortion risk measure* of X with respect to c is defined as

$$\int X\, dc := \int_{-\infty}^0 (c(X > x) - 1)\, dx + \int_0^\infty c(X > x)\, dx, \quad X \in L^\infty(\Omega, \mathcal{F}).$$

Define the right-hand derivative ψ'_+ of the concave function ψ as

$$\psi'_+(t) = \int_{(t,1]} \frac{1}{s}\, \mu(ds), \quad 0 < t < 1. \tag{3.1}$$

Then it can be shown that there exists one-to-one correspondence between probability measures μ on $[0,1]$ and increasing concave functions $\psi : [0,1] \to [0,1]$ with $\psi(0) = 0$ and $\psi(1) = 1$ [6]. Here $\mu(\{0\}) = \psi(0+)$.

4. CHARACTERIZATION OF COHERENT, DISTORTION AND SPECTRAL RISK MEASURE

The relation of coherent, distortion and spectral risk measure is given in this section. Moreover, the spectral risk measure also characterized in terms of the monotonicity with respect to stochastic dominance and comonotonic additivity.

Definition 4.1. Two measurable functions X and Y on $(\Omega, \mathcal{F})$ are called *comonotone* if there exists a measurable function Z on $(\Omega, \mathcal{F})$ and increasing functions f and g on $\mathbb{R}$ such that

$$X = f(Z) \text{ and } Y = g(Z).$$

A risk measure ρ on L^∞ is called *comonotonic* if

$$\rho(X+Y) = \rho(X) + \rho(Y)$$

whenever X and Y are comonotonic.

Consider the risk measure ρ_m expressed in terms of $AVaR_\lambda$ and a probability measure m on $[0,1]$, i.e.,

$$\rho_m(X) := \int AVaR_\lambda(X)\, m(d\lambda). \tag{4.2}$$

The following theorems 4.2, 4.3 and 4.4 are quoted from [6](also refer to [8]).

Theorem 4.2. *On an atomless probability space, the class of risk measures*

$$\rho_m(X) := \int AVaR_\lambda(X)\, m(d\lambda), \quad m \in \mathcal{M}_1([0,1]),$$

is precisely the class of all law-invariant convex risk measures on L^∞ that are comonotonic. In particular, any convex risk measures that is law-invariant and comonotonic is also coherent risk measure with the Fatou property.

Theorem 4.3. *Let m be a probability measure on $[0,1]$ and ψ be the concave distortion function defined in* (3.1). *For $X \in L^\infty$,*

$$\rho_m(X) = \int (-X)\, dc_\psi = \sup_{\mathbb{Q} \in \mathcal{Q}_m} E_{\mathbb{Q}}[-X] \tag{4.3}$$

where c_ψ is the distortion of $\mathbb{P}$ with respect to ψ and $\mathcal{Q}_m$ is defined as

$$\mathcal{Q}_m = \left\{ \mathbb{Q} \in \mathcal{M}_1(\mathbb{P}) \;\middle|\; \int_t^1 q_\varphi(s)\, ds \le \psi(1-t) \text{ for } t \in (0,1), \varphi := \frac{d\mathbb{Q}}{d\mathbb{P}} \right\}.$$

For $\lambda > 0$, if the probability measure m and the concave distortion function ψ are respectively taken as

$$m(ds) = \delta_\lambda(ds) \quad \text{and} \quad \psi(t) = \begin{cases} \frac{t}{\lambda}, & t \le \lambda \\ 1, & t > \lambda \end{cases},$$

then the equation (4.3) becomes

$$\begin{aligned} AVaR_\lambda(-X) &= \frac{1}{\lambda}\int_0^\infty \mathbb{P}[X > x] \wedge \lambda\, dx \\ &= \int X\, dc_\psi. \end{aligned}$$

Some distortion risk functions used for insurance risk pricing are

(1) $\psi(t) = 1 - (1-t)^\nu, \quad \nu \ge 1$: Dual power function,
(2) $\psi(t) = t^{1/\gamma}, \gamma \ge 1$: Proportional hazard transform,
(3) $\psi_\alpha(t) = \Phi[\Phi^{-1}(t) + \alpha], t \in (0,1)$ where Φ is standard normal distribution: Wang's distribution function.

Theorem 4.4. *Let m be a probability measure on $[0,1]$ and let ψ be the concave distortion function defined in (3.1), and c_ψ be the distortion of $\mathbb{P}$ with respect to ψ. Then, for $X \in L^\infty$,*

$$\begin{aligned} \rho_m(-X) &= \psi(0+)AVaR_0(-X) + \int_0^1 q_X(t)\psi'(1-t)\, dt \\ &= \int X\, dc_\psi. \end{aligned}$$

Theorem 4.5. *Let m be a probability measure on $[0,1]$ satisfying $m(\{0\}) = 0$ and ψ be the concave distortion function as in the equation (3.1), then*

$$\rho_m(X) = -\int_0^1 q_X(t)\psi'(t)\, dt. \tag{4.4}$$

Moreover, if we take the spectrum $\phi(t)$ as $\phi(t) = \psi'(t)$, then we have

$$\int (-X)\, dc_\psi = -\int_0^1 q_X(t)\phi(t)\, dt := M_\phi(X).$$

That is, the Choquet integral of $-X$ is equal to the spectral risk measure M_ϕ.

Proof. Since $0 = m(\{0\}) = \psi(0+)$, Theorem 4.4 implies that

$$\rho_m(-X) = \int_0^1 q_X(t)\psi'(1-t)\,dt = \int X\,dc_\psi.$$

By changing variable by $1 - t = s$, and the relation $q_X(t) = -q_{-X}(1-t)$ *a.s.* $t \in (0,1)$,

$$\begin{aligned}\int_0^1 q_X(t)\psi'(1-t)\,dt &= -\int_0^1 q_{-X}(1-t)\psi'(1-t)\,dt \\ &= -\int_0^1 q_{-X}(s)\psi'(s)\,ds.\end{aligned}$$

So we have the equation (4.4). Let us take $\phi(t) = \psi'(t)$. Then, by the definition of $\psi'(t)$, clearly ϕ are nonnegative and nonincreasing and also satisfies

$$\|\phi\|_{L^1} = \int_0^1 \phi(t)\,dt = 1$$

since

$$\int_0^1 \phi(t)\,dt = \int_0^1 \psi'(t)\,dt = \int_{(0,1]} \frac{1}{s}\int_0^1 I_{\{t<s\le 1\}}\,dt\,m(ds) = m((0,1]) = 1, \text{ and}$$
$$m(\{0\}) = 0.$$

So ϕ is the risk aversion function. Thus we have

$$\int (-X)\,dc_\psi = M_\phi(X).$$

□

From Theorem 4.5, we see that spectral risk measures are expressed in the class of coherent risk measures $\rho_m(X)$ defined as in (4.2).

Definition 4.6. Let X and Y be random variables. It is said that Y *dominates* X in stochastic dominance sense, denoted $X \preceq_{st} Y$ if the distribution functions $F_X(x)$ of X and $F_Y(x)$ of Y satisfy

$$F_X(x) \ge F_Y(x), \quad x \in \mathbb{R}.$$

Definition 4.7. The risk measure $\rho : L^\infty \to \mathbb{R}$ is said to be *monotonic* with respect to stochastic dominance if

$$\rho(X) \ge \rho(Y) \quad \text{whenever } X \preceq_{st} Y.$$

The spectral risk measure also characterized as in the following theorem. Refer to [1, 6] for the details.

Theorem 4.8. *The spectral risk measures M_ϕ with $\phi(t) = \psi'(t)$ are all the coherent risk measures which satisfy both the monotonicity with respect to stochastic dominance and comonotonic additivity.*

Proof. Theorem 4.5 implies that a spectral risk measure is a coherent risk measure.

$$\begin{aligned} X \preceq_{st} Y \quad &\Longrightarrow VaR_\lambda(X) \geq VaR_\lambda(Y) \\ &\Longrightarrow AVaR_\lambda(X) \geq AVaR_\lambda(Y) \Longrightarrow \rho_m(X) \geq \rho_m(Y). \end{aligned}$$

So by theorem (4.5) we have

$$M_\phi(Y) = \int (-Y)\, dc_\psi \leq \int (-X)\, dc_\psi = M_\phi(X).$$

Therefore, the spectral risk measure satisfies the monotonicity with respect to stochastic dominance. The comonotonic additivity holds by Theorem 4.2. □

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