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HYPER ORDER OF SOLUTIONS OF COMPLEX DIFFERENTIAL EQUATIONS IN THE DISC

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ABSTRACT. We investigate the growth of solutions of complex linear differential equations in the unit disc. We obtain properties of solutions of differential equations with entire coefficients. We use the concept of the hyper order to estimate the growth of solutions.

1. INTRODUCTION

We assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna's value distribution theory of meromorphic functions in $\mathbb{C}$ and in $\Delta = \{z : |z| < 1\}$, (e.g. see [3, 7]). In addition, the order of a meromorphic function f in Δ is defined by

$$\sigma(f) = \limsup_{r\to 1^-} \frac{\log^+ T(r,f)}{\log \frac{1}{1-r}},$$

where $T(r,f)$ is the Nevanlinna characteristic function of $f(z)$. For an analytic function f in Δ, we also define

$$\sigma_M(f) = \limsup_{r\to 1^-} \frac{\log^+ \log^+ M(r,f)}{\log \frac{1}{1-r}},$$

where $M(r,f)$ is the maximum value of $|f(z)|$ on $|z| = r$.

We also define the hyper-order of a meromorphic function f in Δ similarly to the plane case

$$\sigma_2(f) = \limsup_{r\to 1^-} \frac{\log^+ \log^+ T(r,f)}{\log \frac{1}{1-r}}.$$

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If f is an analytic function in Δ, we also define

$$\sigma_{M2}(f) = \limsup_{r\to 1^-} \frac{\log^+ \log^+ \log^+ M(r,f)}{\log \frac{1}{1-r}}.$$

Definition 1. A meromorphic function f in Δ is called admissible if

$$\limsup_{r\to 1^-} \frac{T(r,f)}{\log \frac{1}{1-r}} = \infty.$$

And f is called non-admissible if

$$\limsup_{r\to 1^-} \frac{T(r,f)}{\log \frac{1}{1-r}} < \infty.$$

Definition 2. Let f be analytic in Δ and let $q \in [0,\infty)$. Then f is said to belong to the weighted Hardy space H_q^∞ provided that

$$\sup_{z\in\Delta}(1-|z|^2)^q |f(z)| < \infty.$$

We say that f is an $\mathcal{H}$−function when $f \in H_q^\infty$ for some q.

Theorem A ([2]). *Let $A_0(z), \ldots, A_{k-1}(z)$ be the sequence of entire coefficients of the equation*

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_d(z)f^{(d)} + \cdots + A_0(z)f = 0. \tag{1.1}$$

Let $A_d(z)$ be the last transcendental coefficient while $A_{d+1}(z), \ldots, A_{k-1}(z)$ are polynomials. Then (1.1) possesses at most d linearly independent entire solutions of finite order of growth.

J. Heittokangas [4] obtained the following counterpart in Δ to Theorem A.

Theorem B. *Let $A_0(z), \ldots, A_{k-1}(z)$ be the sequence of coefficients of (1.1) analytic in Δ. Let $A_d(z)$ be the last coefficient not being an $\mathcal{H}$−function while the coefficients $A_{d+1}(z), \ldots, A_{k-1}(z)$ are $\mathcal{H}$−functions. Then (1.1) possesses at most d linearly independent analytic solutions of finite order of growth in Δ.*

By Theorems A and B, we see that the equation (1.1) in $\mathbb{C}$ (or in Δ) possesses at least $k-j$ linearly independent solutions of infinite order.

It is natural to ask problems : (1) How can one more precisely estimate the growth of these $k-j$ linearly independent solutions of infinite order?

(2) What can be said if $A_{d+1}, \ldots, A_{k-1}$ only satisfy $\sigma(A_j) < \sigma(A_d)$ $(j = d+1, \ldots, k-1)$?

We use the concept of the hyper order and new methods to answer these two problems, and obtain the following theorem 1.

Theorem 1. *Let A_j $(j = 0, \dots, k-1)$ be analytic in Δ. Suppose that there exists some $d \in \{0, \dots, k-1\}$ such that A_d is admissible and $\sigma(A_d) = \sigma_M(A_d) = \mu$, while $\sigma_M(A_j) < \mu$ for $j = d+1, \dots, k-1$; (or $\mu = 0$, while A_j are $\mathcal{H}-$functions for $j = d+1, \dots, k-1$;) and $\sigma_M(A_s) \le \mu$ for $s = 0, \dots, d-1$. Then the equation (1.1) possesses at least $k-d$ linearly independent analytic solutions of the hyper-order $\sigma_2(f) = \mu$ and the order $\sigma(f) = \infty$ (at most d linearly independent analytic solutions of the hyper-order $\sigma_2(f) < \mu$).*

2. Lemmas

The following lemma is due to H. Wittich [6].

Lemma 1. *Let f_j $(j = 1, \dots, k)$ be analytic functions in Δ. Set*

$$\alpha = (f_1, \dots, f_k), \ ||\alpha|| = (\sum_{j=1}^{k} |f_j|^2)^{\frac{1}{2}}.$$

Then we have

(1) $|\frac{d}{dr}||\alpha||| \le ||\frac{d\alpha}{dz}|| = (\sum_{j=1}^{k} |f_j'|^2)^{\frac{1}{2}}$,

(2) $(\sum_{j=1}^{k} |f_j|)^2 \le k \sum_{j=1}^{k} |f_j|^2 = k||\alpha||^2$.

We use H. Wittich's method [6] to prove the following lemma 2.

Lemma 2. *Let A_j $(j = 1, \dots, k-1)$ be analytic functions in Δ with $\sigma_M(A_j) \le \sigma$. Suppose that $f(\not\equiv 0)$ is a solution of the differential equation (1.1). Then we have $\sigma_2(f) \le \sigma$.*

Proof. By the basic theory of differential equations, we know that f is an analytic function in Δ when A_j $(j = 1, \dots, k-1)$ are analytic functions. By $\sigma_M(A_j) \le \sigma$, we know that for any given $\varepsilon(> 0)$, there exists $R \in (0, 1)$, such that

$$M(r, A_j) \le \exp\left\{\left(\frac{1}{1-r}\right)^{\sigma+\varepsilon}\right\} \ (j = 0, \dots, k-1) \tag{2.1}$$

for $|z| = r \in (R, 1)$.

We present (1.1) as a system of equations

$$f = f_1, \ f_j' = f_{j+1} (j = 1, \dots, k-1), \ f_k' = -A_0(z) f_1 - \cdots - A_{k-1}(z) f_k. \tag{2.2}$$

We set

$$\alpha = (f_1, \dots, f_k), \ ||\alpha|| = (\sum_{j=1}^{k} |f_j|^2)^{\frac{1}{2}}.$$

For any fixed ray $\arg z = \theta \in [0, 2\pi)$, by (2.1) and (2.2), we get that as $|z| = r \in (R, 1)$,

$$\begin{aligned} (2.3) \qquad |f_k'(z)| &\le |A_0(z)||f_1(z)| + \cdots + |A_{k-1}(z)||f_k(z)| \\ &\le \exp\left\{ \left(\frac{1}{1-r}\right)^{\sigma+\varepsilon} \right\} \sum_{j=1}^{k} |f_j(z)|. \end{aligned}$$

By (2.2), (2.3) and Lemma 1(2), we know that as $|z| = r \in (R, 1)$,

$$\begin{aligned} (2.4) \qquad \sum_{j=1}^{k} |f_j'(z)|^2 &= \sum_{j=1}^{k-1} |f_j'(z)|^2 + |f_k'(z)|^2 \\ = \sum_{j=1}^{k-1} |f_{j+1}(z)|^2 + |f_k'(z)|^2 &\le ||\alpha||^2 + k \exp\left\{ \frac{2}{(1-r)^{\sigma+\varepsilon}} \right\} ||\alpha||^2 \\ \le \left(1 + k \exp\left\{ \frac{2}{(1-r)^{\sigma+\varepsilon}} \right\}\right) ||\alpha||^2 &\le (1+k) \exp\left\{ \frac{2}{(1-r)^{\sigma+\varepsilon}} \right\} ||\alpha||^2. \end{aligned}$$

And by Lemma 1(1) and (2.4), we have

$$\begin{aligned} (2.5) \qquad \frac{d}{dr}||\alpha|| &\le ||\frac{d\alpha}{dz}|| = (\sum_{j=1}^{k} |f_j'(z)|^2)^{\frac{1}{2}} \\ &\le \sqrt{k+1} \exp\left\{ \left(\frac{1}{1-r}\right)^{\sigma+\varepsilon} \right\} ||\alpha||. \end{aligned}$$

Integrating for both sides of (2.5), we get that

$$(2.6) \qquad \log ||\alpha|| \le \sqrt{k+1} \exp\left\{ \left(\frac{1}{1-r}\right)^{\sigma+\varepsilon} \right\}.$$

By (2.6), we know that for all z satisfying $|z| \le r$,

$$||\alpha|| \le \exp\left\{ \sqrt{k+1} \exp\left\{ \left(\frac{1}{1-r}\right)^{\sigma+\varepsilon} \right\} \right\}.$$

Now we take $|z| = r$ and $|f(z)| = M(r, f)$. Then we have

$$\begin{aligned} (2.7) \qquad M(r, f) &\le (|f(z)|^2 + |f'(z)|^2 + \cdots + |f^{(k-1)}(z)|^2)^{\frac{1}{2}} \\ &= (|f_1(z)|^2 + |f_2(z)|^2 + \cdots + |f_k(z)|^2)^{\frac{1}{2}} \\ &= ||\alpha|| \le \exp\left\{ \sqrt{k+1} \exp\left\{ \left(\frac{1}{1-r}\right)^{\sigma+\varepsilon} \right\} \right\}. \end{aligned}$$

Since ε is arbitrary, by (2.7), we get $\sigma_2(f) \le \sigma$.

By using the similar reasoning as in the proof of [5, Lemma 7.7], we can prove the following lemma.

Lemma 3. *Let $f_1, \dots, f_k$ be linearly independent meromorphic solutions of the linear differential equation*

$$g^{(k)} + a_{k-1}g^{(k-1)} + \cdots + a_0 g = 0$$

in Δ, with meromorphic coefficients $a_0, \dots, a_{k-1}$ in Δ. Then we have, for each $j = 0, \dots, k-1$,

$$m(r, a_j) = O\left(\log \frac{1}{1-r} + \log(\max(T(r, g_\nu), \nu = 1, \dots, k))\right), r \notin E,$$

where $E \subset (0,1)$, $\int_E \frac{1}{1-r}dr < \infty$.

Lemma 4. *Let $A(z)$ be a meromorphic function in Δ, with*

$$\limsup_{r \to 1^-} \frac{\log m(r, A)}{\log \frac{1}{1-r}} = \sigma. \tag{2.8}$$

Suppose that a set $E \subset (0,1)$ with $\int_E \frac{1}{1-r}dr < \infty$. Then there is a sequence $\{r_n\} \subset (0,1)\backslash E$, $(r_1 < r_2 < \cdots,\ r_n \to 1^-)$ satisfying

$$\lim_{n \to \infty} \frac{\log m(r_n, A)}{\log \frac{1}{1-r_n}} = \sigma. \tag{2.9}$$

Proof. By (2.8) we see that there exists a sequence $\{r'_n\} \subset (0,1)$, $(r'_1 < r'_2 < \cdots,\ r'_n \to 1^-)$ satisfying

$$\lim_{n \to \infty} \frac{\log m(r'_n, A)}{\log \frac{1}{1-r'_n}} = \sigma.$$

If we set

$$\int_E \frac{1}{1-r}dr = \alpha < \infty,\ \beta = e^{-(\alpha+1)} < \frac{1}{2},\ s(r) = 1 - \beta(1-r),$$

then, for each r'_n, we have

$$\int_{[r'_n,\ s(r'_n)]} \frac{1}{1-r}dr = \alpha + 1.$$

Thus there is a point $r_n \in [r'_n,\ s(r'_n)]\backslash E$. Since

$$\frac{\log m(r_n, A)}{\log \frac{1}{1-r_n}} \ge \frac{\log m(r'_n, A)}{\log \frac{1}{1-s(r'_n)}} = \frac{\log m(r'_n, A)}{\log \frac{1}{1-r'_n} + (\alpha+1)},$$

we get that

$$\liminf_{n\to\infty} \frac{\log m(r_n, A)}{\log \frac{1}{1-r_n}} \geq \liminf_{n\to\infty} \frac{\log m(r_n', A)}{\log \frac{1}{1-r_n'} + (\alpha+1)} = \lim_{n\to\infty} \frac{\log m(r_n', A)}{\log \frac{1}{1-r_n'}} = \sigma.$$

Hence (2.9) holds.

Lemma 5 ([1]). *Let $g(r)$ and $h(r)$ be monotone increasing and real valued functions on $[0,1)$ such that $g(r) < h(r)$ possibly outside an exceptional set $E \subset (0,1)$, for which $\int_E \frac{1}{1-r}dr < \infty$. Then there exists a constant $b \in (0,1)$ such that if $s(r) = 1 - b(1-r)$, then $g(r) \leq h(s(r))$ for all $r \in [0,1)$.*

3. Proof of Theorem 1

Proof. Assume that $\{f_1, \ldots, f_k\}$ is a solution base of the equation (1.1). By Lemma 2, we know that $\sigma_2(f_j) \leq \mu$ $(j = 1, \ldots, k)$.

If f_j $(j = 1, \ldots, k)$ are all of $\sigma_2(f_j) < \mu$, then there exist constants μ_1 $(< \mu)$ and R $(0 < R < 1)$, such that for all z satisfying $R \leq |z| = r < 1$,

$$T(r, f_j) \leq \exp\left\{\frac{1}{(1-r)^{\mu_1}}\right\}. \tag{3.1}$$

By Lemma 3 and (3.1), we get that

$$m(r, A_d) \leq M\left(\log\frac{1}{1-r} + \log\left\{\exp\frac{1}{(1-r)^{\mu_1}}\right\}\right) \quad (r \notin E) \tag{3.2}$$

where $M(> 0)$ is some constant and a set $E \subset (0,1)$ with $\int_E \frac{1}{1-r}dr < \infty$. By (3.2), we have $\sigma(A_d) \leq \mu_1 < \mu$. This contradicts the hypothesis $\sigma(A_d) = \mu$. Hence there is at least one f_j satisfying $\sigma_2(f_j) = \mu$.

If $d = k-1$, then Theorem 1 holds by above result. Now assume that $0 \leq d \leq k-2$.

Suppose that $f_1, \ldots, f_{d+1}$ are $d+1$ linearly independent analytic solutions of (1.1), with $\sigma_2(f_j) < \mu, (j = 1, \ldots, d+1)$.

We now apply the order reduction procedure. For convenience, we use the notation ν_0 instead of f, $A_{0,0}, \ldots,$ and $A_{0,k-1}$ instead of $A_0, \ldots, A_{k-1}$, respectively. Set

$$A_{0,k} \equiv 1, \ \nu_1 = \frac{d}{dz}\frac{\nu_0}{f_1}, \ \nu_1^{(-1)} = \frac{\nu_0}{f_1}.$$

Then we have $(\nu_1^{(-1)})' = \nu_1,\ \nu_0 = f_1\nu_1^{(-1)}$, and

$$f^{(j)} = \nu_0^{(j)} = \sum_{m=0}^{j} C_j^m f_1^{(m)} \nu_1^{(j-1-m)},\ (j = 0, \dots, k) \tag{3.3}$$

where C_j^m is the usual notation for the binomial coefficients. Substituting (3.3) into (1.1), we obtain that

$$\sum_{m=0}^{k} C_k^m f_1^{(m)} \nu_1^{(k-1-m)} + \sum_{l=1}^{k-1} A_{0,l} \sum_{m=0}^{l} C_l^m f_1^{(m)} \nu_1^{(l-1-m)} + A_{0,0} f_1 \nu_1^{(-1)} = 0. \tag{3.4}$$

Rearranging the sums of (3.4), we get that

$$f_1\nu_1^{(k-1)} + (kf_1' + A_{0,k-1}f_1)\nu_1^{(k-2)} + \sum_{j=0}^{k-3}\Big(\sum_{m=0}^{k-j-1} C_{j+1+m}^m A_{0,j+1+m} f_1^{(m)}\Big)\nu_1^{(j)} \tag{3.5}$$
$$+\nu_1^{(-1)}(f_1^{(k)} + A_{0,k-1}f_1^{(k-1)} + \cdots + A_{0,0}f_1) = 0.$$

Since $f_1(\not\equiv 0)$ is a solution of (1.1), by (3.5), we obtain that

$$\nu_1^{(k-1)} + A_{1,k-2}(z)\nu_1^{(k-2)} + \cdots + A_{1,d-1}(z)\nu_1^{(d-1)} + \cdots + A_{1,0}(z)\nu_1 = 0, \tag{3.6}$$

where

(3.7)

$$A_{1,j} = A_{0,j+1} + \sum_{m=1}^{k-j-1} C_{j+1+m}^m A_{0,j+1+m} \frac{f_1^{(m)}}{f_1},\ (j = 0, \dots, k-2);\ A_{1,k-1}(z) \equiv 1.$$

Setting

$$\max\{\sigma_M(A_{0,j})\ (j = d+1, \dots, k-1);\ \sigma_2(f_m)\ (m = 1, \dots, d+1)\} = \delta, \tag{3.8}$$

since

$$\sigma_M(A_j) = \sigma_M(A_{0,j}) < \mu\ (j = d+1, \dots, k-1),\ \sigma_2(f_m) < \mu\ (m = 1, \dots, d+1),$$

we see that

$$\delta < \mu. \tag{3.9}$$

For any given ε $(0 < 3\varepsilon < \mu - \delta)$, by (3.8) and (3.9), there exists $R \in (0,1)$, such that as $|z| = r \in (R,1)$,

$$m(r, A_{0,j}) = m(r, A_j) \le \left(\frac{1}{1-r}\right)^{\delta+\varepsilon},\ (j = d+1, \dots, k-1), \tag{3.10}$$

$$\log T(r, f_m) \le \left(\frac{1}{1-r}\right)^{\delta+\frac{\varepsilon}{2}},\ (m = 1, \dots, d+1). \tag{3.11}$$

And there exists a set $E_1 \subset (0,1)$, such that $\int_{E_1} \frac{1}{1-r} dr < \infty$ and
(3.12)

$$m\left(r, \frac{f_1^{(s)}}{f_1}\right) = O\left(\log T(r, f_1) + \log \frac{1}{1-r}\right) \leq \left(\frac{1}{1-r}\right)^{\delta+\varepsilon}, \ (s \geq 1, \ r \notin E_1).$$

By (3.7) and (3.10)-(3.12), we get that for $|z| = r \in (R,1)\backslash E_1$

$$m(r, A_{1,j}) = O\left(\left(\frac{1}{1-r}\right)^{\delta+\varepsilon}\right), \ (j = d, \dots, k-2).$$

And by (3.7), we know that

$$A_{1,d-1} = A_{0,d} + \sum_{m=1}^{k-d-2} C_{d+m}^{m} A_{0,d+m} \frac{f_1^{(m)}}{f_1}. \tag{3.13}$$

Since $\sigma(A_{0,d}) = \sigma_M(A_{0,d}) = \mu$, by (3.8)-(3.13), we deduce that

$$\limsup_{r \to 1^-} \frac{\log m(r, A_{1,d-1})}{\log \frac{1}{1-r}} = \mu.$$

Similarly, we get that

$$\limsup_{r \to 1^-} \frac{\log m(r, A_{1,s})}{\log \frac{1}{1-r}} \leq \mu, \ (s = 0, \dots, d-2).$$

Now we consider meromorphic functions

$$\nu_{1,m}(z) = \frac{d}{dz}\left(\frac{f_m(z)}{f_1(z)}\right), \ (m = 2, \dots, k).$$

Since

$$\sigma_2(f_j) < \mu \ (j = 1, \dots, d+1), \ \sigma_2(f_s) \leq \mu \ (s = d+2, \dots, k),$$

we get that

$$\sigma_2(\nu_{1,j}) < \mu \ (j = 2, \dots, d+1), \ \sigma_2(\nu_{1,s}) \leq \mu \ (s = d+2, \dots, k). \tag{3.14}$$

Suppose that $c_2, \cdots, c_k$ are constants such that

$$c_2 \nu_{1,2} + \cdots + c_k \nu_{1,k} = c_2 (\frac{f_2}{f_1})' + \cdots + c_k (\frac{f_k}{f_1})' = 0. \tag{3.15}$$

Then, by integrating both sides of (3.15), we get that

$$c_2 f_2 + \cdots + c_k f_k + c_1 f_1 = 0, \tag{3.16}$$

where c_1 is some constant. Since $f_1, \dots, f_k$ are linearly independent, we have $c_1 = c_2 = \cdots = c_k = 0$ by (3.16). Hence $\nu_{1,2}, \dots, \nu_{1,k}$ are linearly independent, i.e., $\{\nu_{1,2}, \dots, \nu_{1,k}\}$ is a solution base of the equation (3.6).

We continuously proceed the same order reduction procedure as above. Set

(3.17)
$$\nu_i(z) = \frac{d}{dz}\left(\frac{\nu_{i-1}(z)}{\nu_{i-1,i}(z)}\right),\ \nu_{i,s_i}(z) = \frac{d}{dz}\left(\frac{\nu_{i-1,s_i}(z)}{\nu_{i-1,i}(z)}\right), (i = 2, \ldots, d;\ s_i = i+1, \ldots, k),$$
$$\nu_i^{(k-i)} + A_{i,k-i-1}(z)\nu_i^{(k-i-1)} + \cdots + A_{i,d-i}(z)\nu_i^{(d-i)} + \cdots + A_{i,0}(z)\nu_i = 0.$$

After d steps, we get

(3.18)
$$\nu_d^{(k-d)} + A_{d,k-d-1}(z)\nu_d^{(k-d-1)} + \cdots + A_{d,0}(z)\nu_d = 0.$$

Using the reasoning as above, we know that $\{\nu_{i,i+1}, \ldots, \nu_{i,k}\}$ and $\{\nu_{d,d+1}, \ldots, \nu_{d,k}\}$ are solution bases of (3.17) and (3.18) respectively. Since $\nu_{0,1}, \ldots, \nu_{0,d+1}$ satisfy $\sigma_2(\nu_{0,j}) < \mu$ $(j = 1, \ldots, d+1)$, by (3.14), we see that $\nu_{2,3}, \ldots, \nu_{2,d+1}$ satisfy $\sigma_2(\nu_{2,j}) < \mu$ $(j = 3, \ldots, d+1)$; $\ldots$; $\nu_{i,i+1}, \ldots, \nu_{i,d+1}$ satisfy

(3.19)
$$\sigma_2(\nu_{i,j}) < \mu\ (i = 1, \ldots, d-1;\ j = i+1, \ldots, d+1);$$

$\nu_{d,d+1}$ satisfies

(3.20)
$$\sigma_2(\nu_{d,d+1}) < \mu.$$

Using the similar reasoning as (3.12), by (3.19), we have

(3.21)
$$m\left(r, \frac{\nu_{i,i+1}^{(s)}}{\nu_{i,i+1}}\right) \le \left(\frac{1}{1-r}\right)^{\delta+\varepsilon},\ (1 \le s \le k-2;\ i = 1, \ldots, d-1; r \notin E_1).$$

Now we consider the growth order of solutions of (3.18). To work out the growth order of solutions ν_d of (3.18), we need the coefficients $A_{d,j}$ $(j = 0, \ldots, k-d-1)$ in more detailed form. Suppose

(3.22)
$$\varphi = \varphi(A_{0,d+1}, \ldots, A_{0,k-1}, \frac{f_1^{(s)}}{f_1}; \frac{\nu_{1,2}^{(s)}}{\nu_{1,2}}; \ldots; \frac{\nu_{d-1,d}^{(s)}}{\nu_{d-1,d}}\ (1 \le s \le k-2))$$

denotes a linear combination of $A_{0,d+1}, \ldots, A_{0,k-1}$, with the coefficients

$$\frac{f_1^{(s)}}{f_1}; \frac{\nu_{1,2}^{(s)}}{\nu_{1,2}}; \ldots; \frac{\nu_{d-1,d}^{(s)}}{\nu_{d-1,d}},\ (1 \le s \le k-2).$$

By (3.10), (3.12), (3.21) and (3.22), we get that

(3.23)
$$m(r, \varphi) \le \left(\frac{1}{1-r}\right)^{\delta+\varepsilon},\ (r \in (R, 1)\backslash E_1).$$

By (3.7), we can write

(3.24)
$$A_{1,j}(z) = A_{0,j+1} + \varphi_{1,j}(A_{0,j+2}, \ldots, A_{0,k-1}, \frac{f_1^{(s)}}{f_1}\ (1 \le s \le k-2))\ (j = 0, \ldots, k-2),$$

where $\varphi_{1,j}(A_{0,j+2},\ldots,A_{0,k-1},\frac{f_1^{(s)}}{f_1}\ (1\le s\le k-2))$ is a linear combination of $A_{0,j+2},\ \ldots,\ A_{0,k-1}$, with the coefficients $\frac{f_1^{(s)}}{f_1}\ (1\le s\le k-2)$. Particularly, by (3.22) and (3.24), we get that

$$\begin{aligned}A_{1,j}(z) &= \varphi,\ (j=d,\ldots,k-2);\\ A_{1,d-1}(z) &= A_{0,d}+\varphi_{1,d-1}(A_{0,d+1},\ldots,A_{0,k-1},\frac{f_1^{(s)}}{f_1}\ (1\le s\le k-2))\\ &= A_{0,d}+\varphi,\end{aligned}$$

and

$$\begin{aligned}A_{2,j}(z) &= \varphi,\ (j=d-1,\ldots,k-3);\\ A_{2,d-2}(z) &= A_{1,d-1}+\varphi_{2,d-2}(A_{1,d},\ldots,A_{1,k-2},\frac{\nu_{1,2}^{(s)}}{\nu_{1,2}}\ (1\le s\le k-2))\\ &= A_{0,d}+\varphi+\varphi_{2,d-2}(A_{0,d+1},\ldots,A_{0,k-1},\frac{f_1^{(s)}}{f_1};\ \frac{\nu_{1,2}^{(s)}}{\nu_{1,2}}\ (1\le s\le k-2))\\ &= A_{0,d}+\varphi.\end{aligned}$$

Similarly, we can deduce that

$$A_{d,j}(z) = \varphi,\ (j=1,\ldots,k-d-1); \tag{3.25}$$

(3.26)

$$\begin{aligned}A_{d,0} &= A_{0,d}+\varphi_{d,0}(A_{0,d+1},\ldots,A_{0,k-1},\ \frac{f_1^{(s)}}{f_1};\ \frac{\nu_{1,2}^{(s)}}{\nu_{1,2}};\ldots;\ \frac{\nu_{d-1,d}^{(s)}}{\nu_{d-1,d}}\ (1\le s\le k-2))\\ &= A_{0,d}+\varphi.\end{aligned}$$

By (3.23) and (3.25), we see that for above given ε,

$$m(r,A_{d,j})=O\left\{\left(\frac{1}{1-r}\right)^{\delta+\varepsilon}\right\},\ (j=1,\ldots,k-d-1;\ r\in(R,1)\backslash E_1). \tag{3.27}$$

By (3.26), (3.27) and Lemma 5, we see that

$$\limsup_{r\to1^-}\frac{\log m(r,A_{d,0})}{\log\frac{1}{1-r}}=\mu. \tag{3.28}$$

By (3.18), we have

$$A_{d,0}(z)=\frac{\nu_d^{(k-d)}}{\nu_d}+A_{d,k-d-1}(z)\frac{\nu_d^{(k-d-1)}}{\nu_d}+\cdots+A_{d,1}(z)\frac{\nu_d'}{\nu_d}. \tag{3.29}$$

By (3.27) and (3.29), we get that

$$\begin{aligned}m(r,A_{d,0}) &\le \sum_{s=1}^{k-d} m\left(r,\frac{\nu_d^{(s)}}{\nu_d}\right)+\sum_{j=1}^{k-d-1} m(r,A_{d,j}) \qquad (3.30)\\ &= O\left(\log T(r,\nu_d)+\log\frac{1}{1-r}\right)+O\left\{\left(\frac{1}{1-r}\right)^{\delta+\varepsilon}\right\},\ (r\in(R,1)\backslash E_1).\end{aligned}$$

By (3.28) and Lemma 4, there is a point range $\{r_n\} \subset (R, 1)\backslash E_1), r_n \to 1^-$, such that

$$m(r_n, A_{d,0}) \geq \left(\frac{1}{1-r_n}\right)^{\mu-\varepsilon} \tag{3.31}$$

Since $3\varepsilon < \mu - \delta$, by (3.30) and (3.31), we deduce that

$$\left(\frac{1}{1-r_n}\right)^{\mu-2\varepsilon} \leq M\left(\log T(r_n, \nu_d) + \log\frac{1}{1-r_n}\right), \tag{3.32}$$

where M (> 0) is some constant. Since ε is arbitrary, by (3.32), we get that

$$\sigma_2(\nu_d) \geq \mu. \tag{3.33}$$

All solutions ν_d of the equation (3.18) satisfy (3.33). But $\nu_{d,d+1}$ is a solution of (3.18). Thus, (3.33) contradicts (3.20). □

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