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GLOBALIZATION OF A LOCAL MARKET DYNAMICS ONTO AN INFINITE CHAIN OF LOCAL MARKETS

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ABSTRACT. The purpose of this paper is to extend and globalize the Walrasian evolutionary cobweb model in an independent single local market of Brock and Hommes ([3]), to the case of the global market evolution over an infinite chain of many local markets interacting each other through a diffusion of prices between them. In the case of decreasing demands and increasing supplies with a weighted average of rational and naive predictors, we investigate, via the methods of Lattice Dynamical System, the spatial–temporal behaviors of global market dynamics and show that some kind of bounded dynamics of global market do exist and can be controlled by using the parameters in the model.

1. INTRODUCTION-THE COBWEB MODEL

The Cobweb model for the local market dynamics has been well introduced and studied by many researchers (e.g., [3], [4], [5], [8]). The Cobweb model describes the dynamics of equilibrium prices in a single independent local market for a non–storable good, that takes one time period to produce, so that producers must form price expectations one period ahead using the past history of prices.

Let $p_n^e = H(\mathbf{P}_{n-1})$, where p_n^e is the expected price by the producers at time n and $\mathbf{P}_{n-1} = (p_{n-1}, p_{n-2}, \cdots, p_{n-L})$ is a vector of past prices of lag-length L and $H(\cdot) : \mathbf{R}^L \to \mathbf{R}$ is a real-valued function, so called predictor. Let p_n be the actual market price at time n by the consumers, and let $D(p_n)$ be the consumer demand and $S(p_n^e)$ be the producer supply for the goods. The supply $S(p_n^e)$ is derived from producer's maximizing expected profit with a cost function $c(q)$, i.e.,

$$S(p_n^e) = arg \max_{q_n}\{p_n^e q_n - c(q_n)\}. \tag{1.1}$$

The demand function $D(\cdot)$ depends on the current market price p_n and is assumed to be strictly decreasing in the price p_n to ensure that its inverse D^{-1} is well-defined.

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The supply function $S(\cdot)$ depends on the expected price p_n^e and will be assumed to be strictly increasing. Let p^* be the unique steady state equilibrium price corresponding to the intersection point of the demand and supply curve, i.e., $D(p^*) = S(p^*)$.

If the beliefs of producers are homogeneous, i.e., all producers use the same predictor H, then the market equilibrium price dynamics in the cobweb model is described by

$$D(p_n) = S(H(\mathbf{P}_{n-1})), \quad \text{or,} \quad p_n = D^{-1}(S(H(\mathbf{P}_{n-1}))). \tag{1.2}$$

Note that the Equation (1.2) is well defined due to the above assumption that the demand D is decreasing and the supply S is increasing in such a way that its curve intersects the demand curve at a unique point in the relevant interval. Thus, the actual equilibrium price dynamics depends on the demand D, the supply S, and the predictor H used by the producers.

Now, as a model for the global market dynamics, we take the Lattice Dynamical System(LDS) of the following form (c.f., [1], [2], [6], [7], [9], [10]):

$$p_j(n+1) = (1-\alpha)p_j(n) + \alpha f(p_j(n)) + \varepsilon\{p_{j-1}(n) - 2p_j(n) + p_{j+1}(n)\}, \tag{1.3}$$

where a solution $p_j(n)$, $j \in \mathbf{Z}$, $n \in \mathbf{Z}^+$ represents the price of a good at the site (or local market)j at the time n, and $f : \mathbf{R} \to \mathbf{R}$ is a Walrasian local market price dynamics at each site j, and $\alpha \in [0, 1]$ is a parameter denoting the weighted average between $p_j(n)$ and $f(p_j(n))$, and the parameter ε is a diffusion coefficient measuring the intensity of interaction between the neighboring local markets. Thus, in this global market model, the price $p_j(n+1)$ at site j and at time $n+1$ is determined by several factors, i.e., the previous price $p_j(n)$, the local market dynamics f, the weight $\alpha \in [0, 1]$ of the average between them, and the diffusion coefficient $\varepsilon > 0$ denoting the intensity of the interaction between neighboring local markets. Notice that if $\alpha = 1$, then the next price at the site j is determined only by the local market dynamics f and the diffusion between neighboring sites. On the other hand, if $\alpha = 0$, then the local market dynamics is completely suppressed and the next price at the site j is determined only by the present price and the diffusion between neighboring sites. Hence, the parameter $\alpha = 1$ may be called the *market control parameter.*

Remark. For a solution $p_j(n)$ of our model (1.3) to have a meaning in economic sense, we impose a boundary condition at infinity that $p_j(n)$ must be bounded, i.e., $|p_j(n)| \leq C \; \forall j \in \mathbf{Z}, n \in \mathbf{Z}^+$ for some $C > 0$. Also, we require that a solution $p_j(n)$

must have nonnegative value for all $j \in \mathbf{Z}, n \in \mathbf{Z}^+$. If a solution of (1.3) does not satisfy these conditions, then it would not be an admissible solution for our model.

Now, it is clear that in order to apply the model (1.3), we first need to determine the local market dynamics f. In this paper, for simplicity and applicability, we will restrict our attention to the case where D is decreasing, S is increasing, and H is a weighted average between rational and naive predictors. And then for such case of local markets, we will investigate the resulting global market dynamics of price distributions, say, spatially homogeneous solutions, static solutions, and traveling wave solutions along an infinite chain of local markets.

2. Local Market Dynamics

We assume that H is a predictor which is a weighted average between rational and naive predictors, that is, given by

$$p_n^e = H(\mathbf{P}_{n-1}) = \tau p_n + (1-\tau)p_{n-1}, \quad 0 < \tau < 1, \tag{2.1}$$

where the parameter τ will be called a "rationality", since it represents the amount of rationality in the expectation H in such a way that if $\tau = 1$, then H is perfectly rational, and if $\tau = 0$, then H is naive. Hence, we assume that $0 < \tau < 1$ from now on.

For the demand and supply, we again assume that D is linear decreasing, and S is linear increasing, i.e., we suppose that

$$D(p_n) = A - Bp_n, \quad S(p_n^e) = bp_n^e \quad (A, B, b > 0)$$

as in Brock and Hommes (1997). Then, the local market equilibrium price dynamics, $D(p_n) = S(p_n^e)$, is given by

$$\begin{aligned} A - Bp_n &= b\{\tau p_n + (1-\tau)p_{n-1}\}, \quad \text{or} \\ p_n &= \frac{A/b}{\tau + 1/\gamma} + \frac{\tau - 1}{\tau + 1/\gamma} p_{n-1}, \end{aligned} \tag{2.2}$$

where $\gamma = b/B$ and will be regarded as a fixed parameter hereafter. Hence, the local market dynamics f is given by

$$f(x) = \frac{A/b}{\tau + 1/\gamma} + \frac{\tau - 1}{\tau + 1/\gamma} x. \tag{2.3}$$

Here, f is an affine map and has a fixed point $p^* = \frac{A}{B+b}$. Let us first examine the dynamics of f. Letting $\tilde{p}_n = p_n - p^*$, (2.2) becomes

$$\tilde{p}_n = \frac{\tau - 1}{\tau + 1/\gamma} \tilde{p}_{n-1} = C_\gamma(\tau) \tilde{p}_{n-1}, \tag{2.4}$$

where $C_\gamma(\tau) = \frac{\tau-1}{\tau+1/\gamma}$ is a function of τ for each fixed γ. Note that $-\gamma < C_\gamma(\tau) < 0 \ \forall 0 < \tau < 1, \gamma > 0$, i.e., for each fixed $\gamma > 0$, $C_\gamma(\tau)$ is increasing from $-\gamma$ to 0 in the interval $0 < \tau < 1$. Now, according to the values of the parameters γ and τ, the local market dynamics f shows the following behaviors. The proof is trivial and so will be omitted.

Lemma 2.1. (i) *if* $0 < \gamma \leq 1$ *and* $0 < \tau < 1$, *then* $-1 < C_\gamma(\tau) < 0$, *and so* $\tilde{p}_n \to 0$ $(p_n \to p^*)(oscl)$ *as* $n \to +\infty$.
(ii) *if* $\gamma > 1$ *and* $\frac{\gamma-1}{2\gamma} < \tau < 1$, *then* $-1 < C_\gamma(\tau) < 0$, *and so* $\tilde{p}_n \to 0(oscl)$ *as* $n \to +\infty$.
(iii) *if* $\gamma > 1$ *and* $0 < \tau < \frac{\gamma-1}{2\gamma}$, *then* $C_\gamma(\tau) < -1$, *and so* $\tilde{p}_n \to \pm\infty(oscl)$ *as* $n \to +\infty$.
(iv) *if* $\gamma > 1$ *and* $\tau = \frac{\gamma-1}{2\gamma}$, *then* $C_\gamma(\tau) = -1$, *and so the orbit of any initial condition* $\tilde{p}_0 \neq 0$ *under* f *is a 2-cycle* $\{\tilde{p}_0, -\tilde{p}_0\}$.

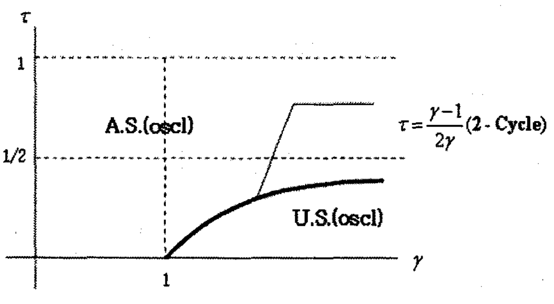

Figure 1. The stability regions of the fixed point p^* of f in the (γ, τ)-parameter space.

Hence, the steady state equilibrium price $p^* = \frac{A}{B+b}$ is asymptotically stable in the case (i) and (ii), and is neutrally stable in the case (iv), and is unstable in the case (iii). In the Figure 1, we show the motions of the local market dynamics f for the values of parameters in the (γ, τ)-space.

From Figure 1, we can make a following observation about the local market dynamics f in the cobweb model with linear supply and demand, and the expectations of the form given in (2.1).

1. If $0 < \gamma = \frac{b}{B} \le 1$, i.e., the supply rate b is less than or equal to the demand rate B, then for any rationality $\tau \in (0,1)$, the price fluctuations always converge to the steady state equilibrium price $p^* = \frac{A}{B+b}$.

2. If $\gamma > 1$, i.e., the supply rate b is greater than the demand rate B, then for each $\gamma_0 > 1$, there exists some critical value of the rationality $\tau_0 \in (0, \frac{1}{2})$ given by $\tau_0 = \frac{\gamma_0 - 1}{2\gamma_0}$ such that for $\tau > \tau_0$, the prices converge to p^* and for $\tau < \tau_0$, the prices diverge from p^*. Such critical values τ_0 are increasing to $\frac{1}{2}$ as γ_0 increases, but always $\tau_0 < \frac{1}{2}$. In other words, the curve $\tau = \frac{\gamma-1}{2\gamma}$ in the (γ, τ)-parameter space can be regarded as a curve of bifurcation values for the local market dynamics f to make a change of stability.

3. If $\frac{1}{2} \le \tau < 1$, i.e., the rationality τ is greater than or equal to $\frac{1}{2}$, then for any $\gamma > 0$, the prices always converge to the steady state equilibrium price p^*.

3. Global Market Dynamics

Now, let us return to considering the global market dynamics given by (1.3), where the local market dynamics f is defined by (2.3), i.e.,

$$p_j(n+1) = (1-\alpha)p_j(n) + \alpha\left\{\frac{A/b}{\tau + 1/\gamma} + \frac{\tau - 1}{\tau + 1/\gamma}p_j(n)\right\} + \varepsilon\{p_{j-1}(n) - 2p_j(n) + p_{j+1}(n)\}, \tag{3.1}$$

where $0 < \tau < 1$, $\gamma = \frac{b}{B} > 0$.

Before starting this section, let us first consider several kinds of basic motions (or solutions) in the LDS (1.3) or (3.1).

Definition 3.1. (i) A state (or solution) $p(n) = \{p_j(n)\}$ for the LDS (1.3) is *spatially homogeneous* if $p_j(n) = \psi(n)\, \forall j \in \mathbf{Z}$, i.e., a spatially homogeneous solution $\{\psi(n)\}$ does not depend on the space coordinates j and so has the same value at each site j.

(ii)A solution $p(n) = \{p_j(n)\}$ is *static* (or *stationary, steady state,standing wave*) if

$p_j(n) = \phi_j \, \forall n \in \mathbf{Z}^+$, i.e., a static solution $\{\phi_j\}$ does not depend on time n, and is standing there along the space coordinates j at all times n, so they are sometimes called "standing waves" in contrast to the traveling waves defined below.
(iii) A solution $p(n) = \{p_j(n)\}$ is a *traveling wave with wave velocity* m/l if $p_j(n) = \xi(lj+mn)$, where $l > 0$, $m \in \mathbf{Z}$ and $(l, m) = 1$ (i.e., relatively prime). Here, the ratio m/l is called the *wave velocity* of the traveling wave. At time $n = 0$, the value of $p_j(0)$ at site j is given by the value of some function ξ at the site lj, i.e., $\xi(lj)$, where l is a positive integer representing the unit of distance, and then at time $n = 1$, the value of $p_j(1)$ at site j is given by shifting the value of $\xi(lj)$ to the left by m, i.e., $\xi(lj + m)$ if $m > 0$. In this way, a traveling wave of the form $p_j(n) = \xi(lj + mn)$ is moving to the left by the distance m over the l unit of distance, i.e., with the velocity m/l at every increment of time by one.

Remark 3.1. Besides the solutions given in Definition 3.1, of course, there can be many other solutions, e.g., spatially and/or temporally periodic solutions, spatially and/or temporally chaotic bounded solutions, and so on. In this paper, we restrict our attention only to those periodic solutions or bounded chaotic solutions which are the basic solutions mentioned in Definition 3.1, e.g., spatially periodic static solutions, temporally periodic spatially homogeneous solutions, spatially and temporally periodic traveling wave solutions, etc.

3.1. SPATIALLY HOMOGENEOUS SOLUTIONS

Setting $p_j(n) = \psi(n) \, \forall j \in \mathbf{Z}$ in (3.1), then we have

$$\psi(n+1) = \alpha \frac{A/b}{\tau + 1/\gamma} + [1 + \alpha\{C_\gamma(\tau) - 1\}]\, \psi(n), \tag{3.2}$$

where $C_\gamma(\tau) = \frac{\tau - 1}{\tau + 1/\gamma}$ as before. Note that for any $\tau \in (0, 1)$, Equation (3.2) has a fixed point given by $\psi^* = \frac{A}{B+b}$ if $\alpha \in (0, 1]$ and $\psi^* = \psi(0)$ if $\alpha = 0$, which is the static spatially homogeneous solution. Now the solutions of (3.2) and their dynamics are given in the following theorem. The proof will be given in the Appendix.
Theorem 3.1. *The spatially homogeneous solutions* $p_j(n) = \psi(n)$ *are given by*

$$\psi(n) = [1 + \alpha\{C_\gamma(\tau) - 1\}]^n \left(\psi(0) - \psi^*\right) + \psi^*, \tag{3.3}$$

where $\psi(0) > 0$ *is an arbitrary initial condition. The dynamics of* $\psi(n)$ *depend on the values of the parameter* γ, α *and* τ *as follows:*

(i) *If* $\alpha(\gamma + 1) = 0$ *(i.e.,* $\alpha = 0$*), then* $\psi(n) = \psi(0) \, \forall n$.
(ii) *If* $0 < \alpha(\gamma+1) \le 1$*, then for any* $\tau \in (0, 1)$, $\psi(n) \to \psi^*$ *(mono) as* $n \to +\infty$.

(iii) *If* $1 < \alpha(\gamma+1) \le 2$, *then for* $0 < \tau < \frac{\alpha(\gamma+1)-1}{\gamma}$, $\psi(n) \to \psi^*$ *(oscl) as* $n \to +\infty$ *and for* $\tau = \frac{\alpha(\gamma+1)-1}{\gamma}$, $\psi(n) = \psi^*$ *for all time* n, *and for* $\frac{\alpha(\gamma+1)-1}{\gamma} < \tau < 1$, $\psi(n) \to \psi^*$ *(mono) as* $n \to +\infty$.

(iv) *If* $\alpha(\gamma+1) > 2$, *then for* $0 < \tau < \frac{\alpha(\gamma+1)-2}{2\gamma}$, $\psi(n) \to \pm\infty$ *(oscl) as* $n \to +\infty$ *and for* $\tau = \frac{\alpha(\gamma+1)-2}{2\gamma}$, *every orbit of* $\psi(0)$ *is a 2-cycle* $\{\psi(0), -\psi(0) + 2\psi^*\}$, *and for* $\frac{\alpha(\gamma+1)-2}{2\gamma} < \tau < \frac{\alpha(\gamma+1)-1}{\gamma}$, $\psi(n) \to \psi^*$*(oscl) as* $n \to +\infty$, *and for* $\tau = \frac{\alpha(\gamma+1)-1}{\gamma}$, $\psi(n) = \psi^*$ *for all* $n \ge 0$, *and for* $\frac{\alpha(\gamma+1)-1}{\gamma} < \tau < 1$, $\psi(n) \to \psi^*$ *(mono) as* $n \to +\infty$.

Thus, when $0 \le \alpha(1+\gamma) \le 2$ *for* $0 < \tau < 1$ *or when* $\alpha(1+\gamma) > 2$ *for* $\frac{\alpha(\gamma+1)-2}{2\gamma} \le \tau < 1$, *we do have a bounded spatially homogeneous global market dynamics.*

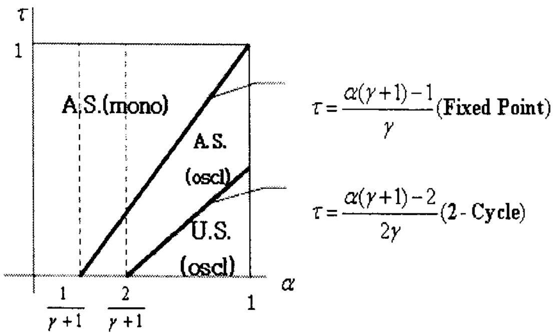

Figure 2. Stability regions of the fixed point $\psi^* = \frac{A}{B+b}$ *in the* (α, τ)*-parameter space for the spatially homogeneous solutions* $\psi(n)$.

In the (α, τ)-parameter space, we can show the various stability regions as in the Figure 2. The curve $\tau = \frac{\alpha(\gamma+1)-2}{2\gamma}$ can be regarded as a curve of bifurcation values of parameters at which change of stability occurs.

From the Figure 2, we can observe the following facts:

1. If $0 < \alpha(\gamma+1) \le 2$, then for any $\tau \in (0,1)$, the spatially homogeneous solutions always converge to the unique static spatially homogeneous solution $\psi^* = \frac{A}{B+b}$. Hence, even if the local market equilibrium price p^* is unstable when $\gamma > 1, \tau < \frac{\gamma-1}{2\gamma}$ (cf. Lemma 2.1), by taking the global market control parameter α sufficiently small so that $0 < \alpha(\gamma+1) \le 2$, we can make global market spatially homogeneous solutions to converge to ψ^*.
2. Even in the case $\alpha(\gamma+1) > 2$, if we take τ sufficiently large so that $\tau > \frac{\alpha(\gamma+1)-2}{2\gamma}$,

then we can make the spatially homogeneous solutions to converge to ψ^*. This is the key role of the parameter τ.

3. If $\frac{1}{2} \le \tau < 1$, then for any $\alpha \in (0,1), \gamma > 0$, the spatially homogeneous solutions always converge to ψ^*.

Hence, Theorem 2.1 tells us about the roles of the parameters α, γ and τ in the dynamics of the spatially homogeneous solutions. That is, bigger values of γ make the spatially homogeneous solutions to be unstable, while smaller values of α or bigger values of τ stabilize them.

3.2. STATIC SOLUTIONS

Setting $p_j(n) = \phi_j$ in (3.1), then we have

$$\phi_{j+1} = -\beta \frac{A/b}{\tau + 1/\gamma} + [2 - \beta\{C_\gamma(\tau) - 1\}]\phi_j - \phi_{j-1}, \tag{3.4}$$

where $\beta = \alpha/\varepsilon$ and $C_\gamma(\tau) = \frac{\tau-1}{\tau+1/\gamma}$. For any $\tau \in (0,1), \gamma > 0$, Equation (3.4) has a fixed point given by $\phi^* = \frac{A}{B+b}$ if $0 < \alpha \le 1$ and $\phi^* = \phi_0$ $\forall \phi_0 \ge 0$ if $\alpha = 0$, which is identical with the static spatially homogeneous solution ψ^* given in (3.2) as we expected. Now we give the results about the static solutions of (3.4) in the following theorem. The proof is given in the Appendix.

Theorem 3.2. *The static solutions $p_j(n) = \phi_j$ of (3.1) are given by*

$$\phi_j = c_1\lambda_1^j + c_2\lambda_2^j + \phi^*, \tag{3.5}$$

where c_1 and c_2 are arbitrary constants that can be determined by initial conditions ϕ_0 and ϕ_1, and λ_1 and λ_2 are characteristic roots given by

$$\lambda_{1,2} = \frac{1}{2}[2 - \beta\{C_\gamma(\tau) - 1\} \pm \sqrt{\Delta}]$$

with $\beta = \alpha/\varepsilon$ and $\Delta = [2 - \beta\{C_\gamma(\tau) - 1\}]^2 - 4$. Among the formal solutions of (3.5), there are only two kinds of bounded static solutions for all $\tau \in (0,1), \varepsilon > 0, \gamma > 0$:

(i) *the static spatially homogeneous solution $\phi_j = \frac{A}{B+b}$ for $\alpha \in (0,1]$*
(ii) *the static spatially homogeneous solution $\phi_j = \phi_0$ for $\alpha = 0$.*

All other static solutions are unbounded as the space coordinates $j \to \pm\infty$.

Note that the conclusion of Theorem 3.2 says that the only bounded static solutions are the static spatially homogeneous solutions which are derived from the steady state equilibrium solutions of the local market dynamics. This is due to the effect of diffusion between neighboring sites and the linearity of the supply function that cause the resulting local market dynamics (and hence the global market dynamics) to be linear.

3.3. Traveling Waves with Wave Velocity 1

Setting $p_j(n) = \xi(j+n)$ in (3.1), and then letting $k = j+n$, we have

$$\xi(k+1) = (1-\alpha)\xi(k) + \alpha\left\{\frac{A/b}{\tau+1/\gamma} + \frac{\tau-1}{\tau+1/\gamma}\xi(k)\right\} + \varepsilon\{\xi(k-1) - 2\xi(k) + \xi(k+1)\}, \tag{3.6}$$

Equation (3.6) has a fixed point $\xi^* = \frac{A}{B+b} = \phi^* = \psi^*$ if $0 < \alpha \le 1$ and $\xi^* = \xi(0)$ if $\alpha = 0$, for all $\varepsilon, \gamma > 0$ and $0 < \tau < 1$. The existence and behavior of the solutions of (3.6) are given in the following theorem: The proof is given in the Appendix.

Theorem 3.3. *The traveling waves with wave velocity* 1, $p_j(n) = \xi(j+n) = \xi(k)$, *are given by*

$$\xi(k) = a_1\lambda_1^k + a_2\lambda_2^k + \xi^* \quad (\textit{when } \varepsilon \neq 1), \tag{3.7}$$

$$\xi(k) = \left\{\frac{1}{1-\alpha\{C_\gamma(\tau)-1\}}\right\}^k (\xi(0)-\xi^*) + \xi^* \quad (\textit{when } \varepsilon = 1),$$

where a_1,a_2 *and* $\xi(0)$ *are arbitrary constants and initial condition respectively and*

$$\lambda_{1,2} = \frac{1}{2}\left(c_1 \pm \sqrt{c_1^2 + 4c_2}\right)$$

with $c_1 = 1 + \frac{\alpha\{C_\gamma(\tau)-1\}-\varepsilon}{1-\varepsilon}$ *and* $c_2 = \frac{\varepsilon}{1-\varepsilon}$.
Among those formal solutions in (3.7), *there are only three kinds of bounded traveling wave solutions for all* $\tau \in (0,1), \gamma > 0$:

(i) *the static spatially homogeneous solution* $\xi(k) = \xi^* = \frac{A}{B+b}$ *for* $0 < \alpha \le 1, \varepsilon > 0$.
(ii) *the static spatially homogeneous solution* $\xi(k) = \xi(0)$ *for* $\alpha = 0$, $\varepsilon > 0$.
(iii) *the spatially and temporally* 2*-periodic solution of the form*

$$\xi(2k) = \xi(0),\ \xi(2k+1) = \xi(1) \quad \textit{for } \alpha = 0,\ \varepsilon = \frac{1}{2}, \tag{3.8}$$

where $\xi(0) \neq \xi(1)$ *are arbitrary initial conditions.*

Note that the conclusion of Theorem 3.3 implies that the only bounded traveling wave solutions are the static spatially homogeneous solutions which are obtained from the steady state equilibrium solutions of the local market dynamics. This is also due to the effect of diffusion between neighboring sites and the linearity of the supply function that cause the resulting local market dynamics (and hence the global market dynamics) to be linear. Recall that in the case of spatially homogeneous solutions, the bigger values of γ made them be unstable, while smaller values of α or bigger values of τ stabilized them (Theorem 3.1). But, note that such roles of the parameters do not hold any longer in the case of static or traveling wave solutions since Theorem 3.2 and 3.3 hold for any $\tau \in (0,1)$, $\gamma > 0$. In fact, as we can see in Theorem 3.2 and 3.3, in the case of static or traveling wave solutions, the presence of the diffusion parameter $\varepsilon > 0$ seems to make them be unstable.

4. Concluding Remarks

In Section 3, we have studied various global dynamics and have noticed that bounded spatially homogeneous solutions do exist and are directly affected by the local market dynamics because of the non-presence of the diffusion. Furthermore, even if the local market dynamics is unstable, the spatially homogeneous solutions of the global market can be controlled via market parameter, so that it can converge to the static spatially homogeneous solutions corresponding to the fixed points of the local market dynamics. However, the bounded static solutions and the bounded traveling wave solutions with wave velocity 1 do not exist except the trivial static spatially homogeneous solutions and the trivial 2-periodic solutions. This is mainly due to the monotonicity of the supply function which results in the linearity of the local market dynamics and hence of the global market dynamics.

Appendix

Proof of Theorem 3.1. If $\alpha = 0$, then (3.2) reduces to $\psi(n+1) = \psi(n)$ which gives the trivial solution $\psi(n) = \psi(0)$ $\forall n \in \mathbf{Z}$. If $\alpha = 1$, then (3.2) reduces to the same equation as (2.2) with p_n replaced by $\psi(n)$. That is, when $\alpha = 1$, the dynamics of $\psi(n)$ can be obtained from the local market dynamics of p_n given in the Lemma 2.1 by replacing p_n by $\psi(n)$. Hence, we assume that $0 < \alpha < 1$ hereafter. Letting $\tilde{\psi}(n) = \psi(n) - \psi^*$, then Equation (3.2) is reduced to a homogeneous linear equation for $\tilde{\psi}(n)$:

$$\tilde{\psi}(n+1) = [1 + \alpha\{C_\gamma(\tau) - 1\}]\tilde{\psi}(n), \tag{A.1}$$

The solution of (A.1) is given by

$$\tilde{\psi}(n) = [1 + \alpha\{C_\gamma(\tau) - 1\}]^n \tilde{\psi}(0). \tag{A.2}$$

Since $C_\gamma(\tau)$ is monotone increasing from $-\gamma$ to 0 in the interval $0 < \tau < 1$, the coefficient $1 + \alpha\{C_\gamma(\tau) - 1\}$ is also monotone increasing from $1 - \alpha(\gamma + 1)$ to $1 - \alpha$ in the interval $0 < \tau < 1$. Also, for $\alpha(\gamma + 1) > 1$, $1 + \alpha\{C_\gamma(\tau) - 1\} = 0$ when $\tau = \frac{\alpha(\gamma+1)-1}{\gamma}$ and for $\alpha(\gamma + 1) > 2$, $1 + \alpha\{C_\gamma(\tau) - 1\} = -1$ when $\tau = \frac{\alpha(\gamma+1)-2}{2\gamma}$. Hence, for fixed $\gamma > 0$, by examining the conditions for α and τ under which the values of $1 + \alpha\{C_\gamma(\tau) - 1\}$ lies in the interval $(0, 1)$ or $(-1, 0)$ or $(-\infty, -1)$, we can determine the dynamics of the solutions of (A.1). □

Proof of Theorem 3.2. Letting $\tilde{\phi}_j = \phi_j - \phi^*$, Equation (3.4) is reduced to the homogeneous linear difference equation in $\tilde{\phi}_j$:

$$\tilde{\phi}_{j+1} = [2 - \beta\{C_\gamma(\tau) - 1\}]\tilde{\phi}_j - \tilde{\phi}_{j-1}. \tag{A.3}$$

The characteristic equation of (A.3) is $\lambda^2 - [2 - \beta\{C_\gamma(\tau) - 1\}]\lambda + 1 = 0$, which has the roots $\lambda_{1,2} = \frac{1}{2}[2 - \beta\{C_\gamma(\tau) - 1\} \pm \sqrt{\Delta}]$, $\Delta = [2 - \beta\{C_\gamma(\tau) - 1\}]^2 - 4$. Since $-\gamma < C_\gamma(\tau) < 0$ for $0 < \tau < 1$, we have $\Delta > 0 \, \forall \beta > 0$, $\gamma > 0$, $\tau \in (0, 1)$. Also, $\lambda_1\lambda_2 = 1$ and $\lambda_1 + \lambda_2 = 2 - \beta\{C_\gamma(\tau) - 1\} > 0$ implies that $\lambda_1 > 1 > \lambda_2 > 0$. Therefore, the general solution of (A.3), which is given by

$$\tilde{\phi}_j = c_1\lambda_1^j + c_2\lambda_2^j \quad (c_1 \text{ and } c_2 \text{ are arbitrary constants}) \tag{A.4}$$

is bounded as $j \to \pm\infty$ if and only if $c_1 = c_2 = 0$. Hence, the only bounded static solution is $\tilde{\phi}_j = 0$ i.e.,$\phi_j = \phi^*$. If $\alpha = 0$, i.e., $\beta = 0$, then the characteristic roots of (A.3) becomes $\lambda_{1,2} = 1, 1$ and so the general solution (A.4) takes the form $\tilde{\phi}_j = c_1 + c_2 j$. Clearly, $\tilde{\phi}_j$ is bounded as $j \to \pm\infty$ if and only if $c_2 = 0$, i.e., $\tilde{\phi}_j = \text{constant} = \tilde{\phi}_0$, or $\phi_j = \phi_0 \, \forall j \in \mathbf{Z}$ for any initial value $\phi_0 \geq 0$. Clearly, this is the fixed point of (3.4) when $\alpha = 0$. □

Proof of Theorem 3.3. When $\varepsilon \neq 1$, solving (3.6) for $\xi(k + 1)$, we have

$$\xi(k+1) = \frac{\alpha}{1-\varepsilon}\frac{A/b}{\tau + 1/\gamma} + \left[1 + \frac{\alpha\{C_\gamma(\tau) - 1\} - \varepsilon}{1 - \varepsilon}\right]\xi(k)) + \frac{\varepsilon}{1-\varepsilon}\xi(k-1), \tag{A.5}$$

where $C_\gamma(\tau) = \frac{\tau - 1}{\tau + 1/\gamma}$, $0 < \tau < 1$, $\gamma = b/B > 0$. Letting $\tilde{\xi}(k) = \xi(k) - \xi^*$, Equation (A.5) is reduced to a homogeneous linear equation for $\tilde{\xi}(k)$:

$$\tilde{\xi}(k+1) - c_1\tilde{\xi}(k) - c_2\tilde{\xi}(k-1) = 0, \tag{A.6}$$

where

$$c_1 = 1 + \frac{\alpha\{C_\gamma(\tau) - 1\} - \varepsilon}{1 - \varepsilon} \quad \text{and} \quad c_2 = \frac{\varepsilon}{1 - \varepsilon}.$$

The general solution of (A.6) is given by

$$(A.7) \qquad \tilde{\xi}(k) = a_1\lambda_1^k + a_2\lambda_2^k, \ \forall k \in \mathbf{Z},$$

where a_1 and a_2 are arbitrary constants, and λ_1 and λ_2 are roots of the characteristic equation $\lambda^2 - c_1\lambda - c_2 = 0$ of (A.6) and are given by $\lambda_{1,2} = \frac{1}{2}(c_1 \pm \sqrt{c_1^2 + 4c_2})$. We claim that for $\varepsilon > 0$, $\varepsilon \neq 1$, $\Delta = c_1^2 + 4c_2 > 0 \ \forall \alpha \in [0, 1]$, $\gamma > 0$, $\tau \in (0, 1)$. For if $0 < \varepsilon < 1$, then $c_2 = \varepsilon/(1 - \varepsilon) > 0$ and so clearly $\Delta > 0$, and if $\varepsilon > 1$, then recall that $-\gamma < C_\gamma(\tau) < 0 \forall \tau \in (0, 1)$, and so

$$\begin{aligned}\Delta = c_1^2 + 4c_2 &= \left[1 + \frac{\alpha\{C_\gamma(\tau) - 1\} - \varepsilon}{1 - \varepsilon}\right]^2 + 4\frac{\varepsilon}{1 - \varepsilon} \\ &\geq \left(1 - \frac{\varepsilon}{1 - \varepsilon}\right)^2 + 4\frac{\varepsilon}{1 - \varepsilon} = \left(1 + \frac{\varepsilon}{1 - \varepsilon}\right)^2 > 0.\end{aligned}$$

Hence, by the claim, λ_1 and λ_2 are real distinct and $\lambda_1 > \lambda_2$.

Now, considering the cases $0 < \varepsilon < 1$ and $\varepsilon > 1$, and using the similar argument in the proof of Theorem 3.2, we can easily show that when $\varepsilon > 0$, $\varepsilon \neq 1$, except the trivial case $\xi(k) = \xi^*$ and the spatially 2-periodic and temporally 2-periodic solution $\xi(2k) = \xi(0)$, $\xi(2k + 1) = \xi(1)$, all other solutions $\tilde{\xi}(k) \to \pm\infty$ as $k \to \pm\infty$. When $\varepsilon = 1$, Equation (3.6) is reduced to a 1st order linear difference equation:

$$(A.8) \qquad [1 - \alpha\{C_\gamma(\tau) - 1\}]\xi(k) = \frac{A/b}{\tau + 1/\gamma} + \xi(k - 1).$$

Now using the similar arguments as above, we can get the remaining results. □

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