

POSITIVE SOLUTIONS OF MULTI-POINT BOUNDARY VALUE PROBLEMS OF NONLINEAR FRACTIONAL DIFFERENTIAL EQUATION AT RESONANCE

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ABSTRACT. This paper deals with the existence of positive solutions for a kind of multi-point nonlinear fractional differential boundary value problem at resonance. Our main approach is different from the ones existed and our main ingredient is the Leggett-Williams norm-type theorem for coincidences due to O'Regan and Zima. The most interesting point is the acquisition of positive solutions for fractional differential boundary value problem at resonance. And an example is constructed to show that our result here is valid.

1. INTRODUCTION

In this paper, we are concerned with positive solutions to the following fractional differential equation:

$$(1.1) \quad {}^c D_{0+}^{\alpha} u(t) + f(t, u(t)) = 0, \quad 0 < t < 1$$

with the boundary conditions

$$(1.2) \quad u'(0) = u'(1), \quad u(0) = \sum_{i=1}^{m-2} \mu_i u(\xi_i),$$

where $m > 2$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$, $\mu_i \geq 0$, $i = 1, 2, \dots, m-2$ and $\sum_{i=1}^{m-2} \mu_i = 1$, ${}^c D_{0+}^{\alpha}$ is the Caputo's fractional derivative of order α , $1 < \alpha \leq 2$ is a real number, and $f : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function.

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Due to $\sum_{i=1}^{m-2} \mu_i = 1$, the fractional differential operator ${}^c D_{0+}^\alpha$ is not invertible under Caputo's derivative, boundary value problems (in short:BVPs) of this type are referred to as problems at resonance.

Recently, fractional differential equations (in short:FDE) have been studied extensively. The motivation for those works stem from both the development of the theory of fractional calculus itself and the applications of such constructions in various sciences such as physics, mechanics, chemistry, engineering, etc. For an extensive collection of such results, we refer the readers to the monographs [1-4].

Some basic theory for the initial value problems of FDE involving Riemann-Liouville differential operator has been discussed by Lakshmikantham [5, 6, 7], A. M. A. El-Sayed et al [8, 9], Kai Diethelm and Neville J. Ford [10], M. Benchohra et al [11] and C. Bai [12], etc. Also, there are some papers which deal with the existence of positive solutions for BVPs of nonlinear FDE by using techniques of topological degree theory [13-16,19-20]. For example, the existence and multiplicity of positive solutions for the equation

$$(1.3) \quad D_{0+}^\alpha u(t) = f(t, u(t)), \quad 0 < t < 1, \quad 1 < \alpha \leq 2,$$

subject to the Dirichlet boundary condition

$$(1.4) \quad u(0) = u(1) = 0$$

have been studied by Bai and Lü [13] by means of the well-known Krasnosel'skii fixed point theorem and Leggett-Williams fixed point theorem. D_{0+}^α is the standard Riemann-Liouville fractional derivative there.

In [14] and [15], Zhang also studied the existence of positive solutions of Eq.(1.3) under the boundary conditions

$$(1.5) \quad u(0) = \nu \neq 0, \quad u(1) = \rho \neq 0$$

and

$$(1.6) \quad u(0) + u'(0) = 0, \quad u(1) + u'(1) = 0,$$

respectively. Due to the fact that conditions (1.5) and (1.6) are not zero boundary value, the Riemann-Liouville fractional derivative D_{0+}^α is not suitable. Therefore, the author investigated the BVPs (1.3)-(1.5) and (1.3)-(1.6) by involving the Caputo's fractional derivative.

M. El-Shahed [16] established the existence of positive solutions to BVP

$$(1.7) \quad D_{0+}^\alpha u(t) + \lambda a(t)f(u(t)) = 0, \quad 0 < t < 1, \quad 2 < \alpha \leq 3,$$

$$(1.8) \quad u(0) = u'(0) = u'(1) = 0$$

by applying Krasnosel'skii fixed point theorem.

From above works, we can see a fact, although the BVPs of nonlinear FDE have been studied by some authors, to the best of our knowledge, all of existing works are limited to non-resonance boundary conditions. For the resonance case, as far as we know, no contributions exist. The aim of this paper is to fill the gap in the relevant literature. Our main tool is the recent Leggett-Williams norm-type theorem for coincidences due to O'Regan and Zima [17].

2. PRELIMINARIES

For the convenience of the reader, we present the definitions and some fundamental facts of Caputo's derivatives of fractional order which can be found in the recent literatures [1-4].

Definition 2.1. The Riemann-Liouville fractional integral of order α is defined by

$$(2.1) \quad (I_{0+}^\alpha y)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{y(s)}{(t-s)^{1-\alpha}} ds, \quad (t > 0, \alpha > 0)$$

where $\Gamma(\alpha)$ is the Euler gamma function defined by

$$(2.2) \quad \Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt, \quad (z > 0)$$

for which, the reduction formula

$$(2.3) \quad \Gamma(z + 1) = z\Gamma(z), \quad (z > 0), \quad \Gamma(1) = 1, \quad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

and formula

$$(2.4) \quad \int_0^1 t^{z-1} (1-t)^{\omega-1} dt = \frac{\Gamma(z)\Gamma(\omega)}{\Gamma(z+\omega)}, \quad (z, \omega \notin \mathbb{Z}_0^-)$$

hold.

Definition 2.2. Caputo's derivative of order α for a function $y \in AC^n[0, 1]$ can be represented by

$$(2.5) \quad ({}^cD_{0+}^\alpha y)(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{y^{(n)}(s)}{(t-s)^{\alpha+1-n}} ds =: (I_{0+}^{n-\alpha} D^n y)(t), \quad (t > 0, \alpha > 0)$$

where $D^n = \frac{d^n}{dt^n}$ and $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of α , and $AC^n[0, 1] = \{f : [0, 1] \rightarrow \mathbb{R} \mid D^{n-1}f \in AC[0, 1]\}$.

Remark 2.1. Under natural conditions on the function $y(t)$, Caputo's derivative becomes a conventional m -th derivative of the function $y(t)$ as $\alpha \rightarrow m$ (see [2]).

From definitions 2.1 and 2.2, we can deduce the following statement.

Lemma 2.1. *The fractional differential equation*

$${}^c D_{0+}^\alpha y(t) = 0$$

has solutions $y(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1}$, $c_i \in \mathbb{R}$, $i = 0, 1, \dots, n-1$, $n = [\alpha] + 1$. Furthermore, for $y \in AC^n[0, 1]$,

$$(2.6) \quad (I_{0+}^\alpha {}^c D_{0+}^\alpha y)(t) = y(t) - \sum_{k=0}^{n-1} \frac{y^{(k)}(0)}{k!} t^k$$

and

$$(2.7) \quad ({}^c D_{0+}^\alpha I_{0+}^\alpha y)(t) = y(t).$$

In the following, we review some standard facts on Fredholm operators and cones in Banach spaces. Let X, Y be real Banach spaces. Consider a linear mapping $L : \text{dom}L \subset X \rightarrow Y$ and a nonlinear mapping $N : X \rightarrow Y$. Throughout we assume

1° L is a Fredholm operator of index zero, that is, $\text{Im}L$ is closed and $\dim \text{Ker}L = \text{codim} \text{Im}L < \infty$.

The assumption 1° implies that there exist continuous projections $P : X \rightarrow X$ and $Q : Y \rightarrow Y$ such that $\text{Im}P = \text{Ker}L$ and $\text{Ker}Q = \text{Im}L$. Moreover, since $\dim \text{Im}Q = \text{codim} \text{Im}L$, there exists an isomorphism $J : \text{Im}Q \rightarrow \text{Ker}L$. Denote by L_p the restriction of L to $\text{Ker}P \cap \text{dom}L$. Clearly, L_p is an isomorphism from $\text{Ker}P \cap \text{dom}L$ to $\text{Im}L$, we denote its inverse by $K_p : \text{Im}L \rightarrow \text{Ker}P \cap \text{dom}L$. It is known (see [18]) that the coincidence equation $Lx = Nx$ is equivalent to

$$x = (P + JQN)x + K_p(I - Q)Nx.$$

Let C be a cone in X such that

- (i) $\mu x \in C$ for all $x \in C$ and $\mu \geq 0$,
- (ii) $x, -x \in C$ implies $x = \theta$.

It is well known that C induces a partial order in X by

$$x \preceq y \text{ if and only if } y - x \in C.$$

We will write $x \not\leq y$ for $y - x \notin C$. Moreover, for every $u \in C \setminus \{0\}$ there exists a positive number $\sigma(u)$ such that

$$\|x + u\| \geq \sigma(u)\|x\|$$

for all $x \in C$. It is clearly that if $\sigma(u) > 0$ is such that $\|x + u\| \geq \sigma(u)\|x\|$ for all $x \in C$, then for every $\lambda > 0$,

$$\|x + \lambda u\| \geq \sigma(u)\|x\| \text{ for all } x \in C.$$

Let $\gamma : X \rightarrow C$ be a retraction, that is, a continuous mapping such that $\gamma(x) = x$ for all $x \in C$. Set

$$\Psi := P + JQN + K_p(I - Q)N \text{ and } \Psi_\gamma := \Psi \circ \gamma.$$

We make use of the following result due to O'Regan and Zima [17].

Theorem 2.1. *Let C be a cone in X and let Ω_1, Ω_2 be open bounded subsets of X with $\bar{\Omega}_1 \subset \Omega_2$ and $C \cap (\bar{\Omega}_2 \setminus \Omega_1) \neq \emptyset$. Assume that the following conditions hold.*

- 2° $QN : X \rightarrow Y$ is continuous and bounded and $K_p(I - Q)N : X \rightarrow X$ is compact on every bounded subset of X ,
- 3° $Lx \neq \lambda Nx$ for all $x \in C \cap \partial\Omega_2 \cap \text{Im}L$ and $\lambda \in (0, 1)$,
- 4° γ maps subsets of $\bar{\Omega}_2$ into bounded subsets of C ,
- 5° $\text{deg}_B\{[I - (P + JQN)\gamma]|_{\text{Ker}L}, \text{Ker}L \cap \Omega_2, 0\} \neq 0$, where deg_B stands for the Brouwer degree,
- 6° there exists $u_0 \in C \setminus \{0\}$ such that $\|x\| \leq \sigma(u_0)\|\Psi x\|$ for $x \in C(u_0) \cap \partial\Omega_1$, where $C(u_0) = \{x \in C : \mu u_0 \leq x \text{ for some } \mu > 0\}$ and $\sigma(u_0)$ such that $\|x + u_0\| \geq \sigma(u_0)\|x\|$ for every $x \in C$,
- 7° $(P + JQN)\gamma(\partial\Omega_2) \subset C$,
- 8° $\Psi_\gamma(\bar{\Omega}_2 \setminus \Omega_1) \subset C$.

Then the equation $Lx = Nx$ has a solution in the set $C \cap (\bar{\Omega}_2 \setminus \Omega_1)$.

For simplicity of notation, we set

$$G(t, s) = \begin{cases} 1 - \frac{\alpha t - 1}{\alpha^2 \Gamma(\alpha)} \cdot \frac{\sum_{i=1}^{m-2} \mu_i \xi_i^\alpha}{\sum_{i=1}^{m-2} \mu_i \xi_i} + \frac{t^\alpha}{\alpha \Gamma(\alpha)} - \frac{\Gamma(\alpha)}{\Gamma(2\alpha)} + \frac{(1-s)^{2-\alpha}}{\Gamma(\alpha)} \int_s^1 \frac{(\tau - s)^{\alpha-1}}{(1-\tau)^{2-\alpha}} d\tau \\ + \frac{(\alpha t - 1)(1-s)^{2-\alpha}}{(\alpha-1)\Gamma(\alpha+1)} \cdot \frac{\sum_{i=1}^{m-2} \mu_i (\xi_i - s)^{\alpha-1}}{\sum_{i=1}^{m-2} \mu_i \xi_i} - \frac{(1-s)^{2-\alpha}(t-s)^{\alpha-1}}{(\alpha-1)\Gamma(\alpha)}, \quad 0 \leq s \leq \min\{\xi_1, t\}, \end{cases}$$

$$\left\{ \begin{aligned}
 & 1 - \frac{\alpha t - 1}{\alpha^2 \Gamma(\alpha)} \cdot \frac{\sum_{i=1}^{m-2} \mu_i \xi_i^\alpha}{\sum_{i=1}^{m-2} \mu_i \xi_i} + \frac{t^\alpha}{\alpha \Gamma(\alpha)} - \frac{\Gamma(\alpha)}{\Gamma(2\alpha)} + \frac{(1-s)^{2-\alpha}}{\Gamma(\alpha)} \int_s^1 \frac{(\tau-s)^{\alpha-1}}{(1-\tau)^{2-\alpha}} d\tau \\
 & + \frac{(\alpha t - 1)(1-s)^{2-\alpha}}{(\alpha-1)\Gamma(\alpha+1)} \cdot \frac{\sum_{i=1}^{m-2} \mu_i (\xi_i - s)^{\alpha-1}}{\sum_{i=1}^{m-2} \mu_i \xi_i}, \quad 0 \leq t \leq s \leq \xi_1, \\
 & 1 - \frac{\alpha t - 1}{\alpha^2 \Gamma(\alpha)} \cdot \frac{\sum_{i=1}^{m-2} \mu_i \xi_i^\alpha}{\sum_{i=1}^{m-2} \mu_i \xi_i} + \frac{t^\alpha}{\alpha \Gamma(\alpha)} - \frac{\Gamma(\alpha)}{\Gamma(2\alpha)} + \frac{(1-s)^{2-\alpha}}{\Gamma(\alpha)} \int_s^1 \frac{(\tau-s)^{\alpha-1}}{(1-\tau)^{2-\alpha}} d\tau \\
 & + \frac{(\alpha t - 1)(1-s)^{2-\alpha}}{(\alpha-1)\Gamma(\alpha+1)} \cdot \frac{\sum_{i=j+1}^{m-2} \mu_i (\xi_i - s)^{\alpha-1}}{\sum_{i=1}^{m-2} \mu_i \xi_i} - \frac{(1-s)^{2-\alpha}(t-s)^{\alpha-1}}{(\alpha-1)\Gamma(\alpha)}, \quad \xi_j \leq s \leq \xi_{j+1}, s \leq t, \\
 & 1 - \frac{\alpha t - 1}{\alpha^2 \Gamma(\alpha)} \cdot \frac{\sum_{i=1}^{m-2} \mu_i \xi_i^\alpha}{\sum_{i=1}^{m-2} \mu_i \xi_i} + \frac{t^\alpha}{\alpha \Gamma(\alpha)} - \frac{\Gamma(\alpha)}{\Gamma(2\alpha)} + \frac{(1-s)^{2-\alpha}}{\Gamma(\alpha)} \int_s^1 \frac{(\tau-s)^{\alpha-1}}{(1-\tau)^{2-\alpha}} d\tau \\
 & + \frac{(\alpha t - 1)(1-s)^{2-\alpha}}{(\alpha-1)\Gamma(\alpha+1)} \cdot \frac{\sum_{i=j+1}^{m-2} \mu_i (\xi_i - s)^{\alpha-1}}{\sum_{i=1}^{m-2} \mu_i \xi_i}, \quad \xi_j \leq s \leq \xi_{j+1}, s \geq t, \\
 & 1 - \frac{\alpha t - 1}{\alpha^2 \Gamma(\alpha)} \cdot \frac{\sum_{i=1}^{m-2} \mu_i \xi_i^\alpha}{\sum_{i=1}^{m-2} \mu_i \xi_i} + \frac{t^\alpha}{\alpha \Gamma(\alpha)} - \frac{\Gamma(\alpha)}{\Gamma(2\alpha)} + \frac{(1-s)^{2-\alpha}}{\Gamma(\alpha)} \int_s^1 \frac{(\tau-s)^{\alpha-1}}{(1-\tau)^{2-\alpha}} d\tau \\
 & - \frac{(1-s)^{2-\alpha}(t-s)^{\alpha-1}}{(\alpha-1)\Gamma(\alpha)}, \quad \xi_{m-2} \leq s \leq t \leq 1, \\
 & 1 - \frac{\alpha t - 1}{\alpha^2 \Gamma(\alpha)} \cdot \frac{\sum_{i=1}^{m-2} \mu_i \xi_i^\alpha}{\sum_{i=1}^{m-2} \mu_i \xi_i} + \frac{t^\alpha}{\alpha \Gamma(\alpha)} - \frac{\Gamma(\alpha)}{\Gamma(2\alpha)} + \frac{(1-s)^{2-\alpha}}{\Gamma(\alpha)} \int_s^1 \frac{(\tau-s)^{\alpha-1}}{(1-\tau)^{2-\alpha}} d\tau, \\
 & \quad \max\{\xi_{m-2}, t\} \leq s \leq 1,
 \end{aligned} \right.$$

where $j = 1, 2, \dots, m - 3$.

Note that $G(t, s) \geq 0$ for $t, s \in [0, 1]$. Set $0 < \kappa \leq \min \left\{ 1, \frac{1}{\max_{t,s \in [0,1]} G(t,s)} \right\}$.

3. MAIN RESULTS

Now we state our main result on the existence of a positive solution for BVP (1.1)-(1.2).

Theorem 3.1. *Assume that*

- (H1) *there exist positive constants $b_1, b_2, b_3, c_1, c_2, B$ with $B > \frac{c_2}{c_1} + \frac{3b_2c_2 + 3b_3c_1}{b_1c_1(\alpha-1)}$ such that*

$$\begin{aligned} -\kappa x &\leq f(t, x), \\ f(t, x) &\leq -c_1x + c_2, \\ f(t, x) &\leq -b_1|f(t, x)| + b_2x + b_3 \end{aligned}$$

for $t \in [0, 1], x \in [0, B],$

(H2) there exist $b \in (0, B), t_0 \in [0, 1], \rho \in (0, 1), \delta \in (0, 1)$ and $q \in C[0, 1], q(t) \geq 0$ on $[0, 1], h \in C((0, b], \mathbb{R}^+)$ such that $f(t, x) \geq q(t)h(x)$ for $t \in [0, 1]$ and $x \in (0, b]. \frac{h(x)}{x^\rho}$ is non-increasing on $x \in (0, b]$ with

$$(3.1) \quad h(b) \int_0^1 G(t_0, s) \frac{q(s)}{(1-s)^{2-\alpha}} ds \geq \frac{b(1-\delta)}{(\alpha-1)\delta^\rho}.$$

Then the BVP (1.1)-(1.2) has at least one positive solution on $[0, 1].$

Proof. Consider the Banach spaces $X = Y = C[0, 1]$ with the sup norm $\|x\| = \max_{t \in [0,1]} |x(t)|.$ Define $L : \text{dom}L \rightarrow Y$ and $N : X \rightarrow Y$ with

$$\text{dom}L = \left\{ x \in X : x \in AC^n[0, 1], x(0) = \sum_{i=1}^{m-2} \mu_i x(\xi_i), x'(0) = x'(1), {}^cD_{0+}^\alpha x \in C[0, 1] \right\}$$

by $Lx(t) = -{}^cD_{0+}^\alpha x(t)$ and $Nx(t) = f(t, x(t)).$ Then

$$\text{Ker}L = \{x \in \text{dom}L : x(t) \equiv c \in \mathbb{R} \text{ on } [0, 1]\}$$

and

$$(3.2) \quad \text{Im}L = \left\{ y \in Y : \int_0^1 \frac{y(s)}{(1-s)^{2-\alpha}} ds = 0 \right\}.$$

Next, define the projections $P : X \rightarrow X$ by $(Px)(t) = (\alpha - 1) \int_0^1 \frac{x(s)}{(1-s)^{2-\alpha}} ds$ and $Q : Y \rightarrow Y$ by

$$(Qy)(t) = (\alpha - 1) \int_0^1 \frac{y(s)}{(1-s)^{2-\alpha}} ds, \quad t \in [0, 1].$$

Clearly, $\text{Im}P = \text{Ker}L$ and $\text{Ker}Q = \text{Im}L.$ So $\dim\text{Ker}L = 1 = \dim\text{Im}Q = \text{codimIm}L.$ Notice that $\text{Im}L$ is closed, L is Fredholm operator of index zero, i.e. 1° holds.

Note that for $y \in \text{Im}L$ the inverse $K_p : \text{Im}L \rightarrow \text{dom}L \cap \text{Ker}P$ of L_p is given by

$$(K_p y)(t) = \int_0^1 k(t, s) \frac{y(s)}{(1-s)^{2-\alpha}} ds$$

where

$$k(t, s) = \frac{(1-s)^{2-\alpha}}{\Gamma(\alpha)}$$

$$\left\{ \begin{array}{ll} \frac{\alpha t-1}{\alpha \sum_{i=1}^{m-2} \mu_i \xi_i} \sum_{i=1}^{m-2} \mu_i (\xi_i - s)^{\alpha-1} + (\alpha-1) \int_s^1 \frac{(\tau-s)^{\alpha-1}}{(1-\tau)^{2-\alpha}} d\tau & 0 \leq s \leq \min\{\xi_1, t\}, \\ - (t-s)^{\alpha-1}, & \\ \frac{\alpha t-1}{\alpha \sum_{i=1}^{m-2} \mu_i \xi_i} \sum_{i=1}^{m-2} \mu_i (\xi_i - s)^{\alpha-1} + (\alpha-1) \int_s^1 \frac{(\tau-s)^{\alpha-1}}{(1-\tau)^{2-\alpha}} d\tau, & 0 \leq t \leq s \leq \xi_1, \\ \frac{\alpha t-1}{\alpha \sum_{i=1}^{m-2} \mu_i \xi_i} \sum_{i=j+1}^{m-2} \mu_i (\xi_i - s)^{\alpha-1} + (\alpha-1) \int_s^1 \frac{(\tau-s)^{\alpha-1}}{(1-\tau)^{2-\alpha}} d\tau & \\ - (t-s)^{\alpha-1}, & \xi_j \leq s \leq \xi_{j+1}, s \leq t, \\ \frac{\alpha t-1}{\alpha \sum_{i=1}^{m-2} \mu_i \xi_i} \sum_{i=j+1}^{m-2} \mu_i (\xi_i - s)^{\alpha-1} + (\alpha-1) \int_s^1 \frac{(\tau-s)^{\alpha-1}}{(1-\tau)^{2-\alpha}} d\tau, & \\ & \xi_j \leq s \leq \xi_{j+1}, s \geq t, \\ (\alpha-1) \int_s^1 \frac{(\tau-s)^{\alpha-1}}{(1-\tau)^{2-\alpha}} d\tau - (t-s)^{\alpha-1}, & \xi_{m-2} \leq s \leq t \leq 1, \\ (\alpha-1) \int_s^1 \frac{(\tau-s)^{\alpha-1}}{(1-\tau)^{2-\alpha}} d\tau, & \max\{\xi_{m-2}, t\} \leq s \leq 1. \end{array} \right.$$

where $j = 1, 2, \dots, m-3$. It is easy to see that $|k(t, s)| \leq 3$. Since f is continuous, 2° holds.

Consider the cone

$$C = \{x \in X : x(t) \geq 0 \text{ on } [0, 1]\}.$$

Let

$$\Omega_1 = \{x \in X : \delta \|x\| < |x(t)| < b \text{ on } [0, 1]\}$$

and

$$\Omega_2 = \{x \in X : \|x\| < B\}.$$

Clearly, Ω_1 and Ω_2 are bounded and open sets and

$$\bar{\Omega}_1 = \{x \in X : \delta \|x\| \leq |x(t)| \leq b \text{ on } [0, 1]\} \subset \Omega_2$$

(see [17]). Moreover, $C \cap (\bar{\Omega}_2 \setminus \Omega_1) \neq \emptyset$. Let $J = I$ and $(\gamma x)(t) = |x(t)|$ for $x \in X$. Then γ is a retraction and maps subsets of $\bar{\Omega}_2$ into bounded subsets of C , which means that 4° holds.

In order to prove 3°, suppose that there exist $x_0 \in \partial\Omega_2 \cap C \cap \text{dom}L$ and $\lambda_0 \in (0, 1)$ such that $Lx_0 = \lambda_0 Nx_0$. Then ${}^cD_{0+}^\alpha x_0(t) + \lambda_0 f(t, x_0(t)) = 0$ for all $t \in [0, 1]$. In

view of (H1), we have

$$-\frac{1}{\lambda_0} {}^c D_{0+}^\alpha x_0(t) = f(t, x_0(t)) \leq -\frac{1}{\lambda_0} b_1 |{}^c D_{0+}^\alpha x_0(t)| + b_2 x_0(t) + b_3.$$

Hence,

$$\begin{aligned} 0 &= -(I_{0+}^{\alpha-1} {}^c D_{0+}^{\alpha-1} D x_0)(1) \\ &\leq -\frac{b_1}{\Gamma(\alpha-1)} \int_0^1 \frac{|{}^c D_{0+}^\alpha x_0(s)|}{(1-s)^{2-\alpha}} ds + \frac{\lambda_0 b_2}{\Gamma(\alpha-1)} \int_0^1 \frac{x_0(s)}{(1-s)^{2-\alpha}} ds \\ &\quad + \frac{\lambda_0 b_3}{\Gamma(\alpha-1)} \int_0^1 \frac{1}{(1-s)^{2-\alpha}} ds, \end{aligned}$$

which gives

$$(3.3) \quad \int_0^1 \frac{|{}^c D_{0+}^\alpha x_0(s)|}{(1-s)^{2-\alpha}} ds \leq \frac{b_2}{b_1} \int_0^1 \frac{x_0(s)}{(1-s)^{2-\alpha}} ds + \frac{b_3}{b_1(\alpha-1)}.$$

Similarly, from (H1), we also obtain

$$(3.4) \quad \int_0^1 \frac{x_0(s)}{(1-s)^{2-\alpha}} ds \leq \frac{c_2}{c_1(\alpha-1)}.$$

On the other hand,

$$\begin{aligned} (3.5) \quad x_0(t) &= (\alpha-1) \int_0^1 \frac{x_0(s)}{(1-s)^{2-\alpha}} ds + \int_0^1 k(t,s) \frac{{}^c D_{0+}^\alpha x_0(s)}{(1-s)^{2-\alpha}} ds \\ &\leq \frac{c_2}{c_1} + \int_0^1 |k(t,s)| \cdot \frac{|{}^c D_{0+}^\alpha x_0(s)|}{(1-s)^{2-\alpha}} ds \\ &\leq \frac{c_2}{c_1} + \frac{3b_2c_2 + 3b_3c_1}{b_1c_1(\alpha-1)}. \end{aligned}$$

(3.3), (3.4) and (3.5) yield

$$B = \|x_0\| \leq \frac{c_2}{c_1} + \frac{3b_2c_2 + 3b_3c_1}{b_1c_1(\alpha-1)},$$

which contradicts (H1).

To prove 5°, consider $x \in \text{Ker}L \cap \bar{\Omega}_2$. Then $x(t) \equiv c \in \mathbb{R}$ on $[0, 1]$. Let

$$H(c, \lambda) = c - \lambda|c| - \lambda(\alpha-1) \int_0^1 \frac{f(s, |c|)}{(1-s)^{\alpha-2}} ds$$

for $c \in [-B, B]$ and $\lambda \in [0, 1]$. It is easy to show that $0 = H(c, \lambda)$ implies $c \geq 0$.

Suppose $0 = H(B, \lambda)$ for some $\lambda \in (0, 1]$. Then, (H1) leads to

$$0 \leq B(1-\lambda) = \lambda(\alpha-1) \int_0^1 \frac{f(s, B)}{(1-s)^{\alpha-2}} ds \leq \lambda(-c_1B + c_2) < 0,$$

which is a contradiction. In addition, if $\lambda = 0$, then $B = 0$, which is impossible. Thus, $H(x, \lambda) \neq 0$ for $x \in \text{Ker}L \cap \partial\Omega_2$, $\lambda \in [0, 1]$. As a result,

$$\deg_B\{H(\cdot, 1), \text{Ker}L \cap \Omega_2, 0\} = \deg_B\{H(\cdot, 0), \text{Ker}L \cap \Omega_2, 0\}.$$

However,

$$\deg_B\{H(\cdot, 0), \text{Ker}L \cap \Omega_2, 0\} = \deg_B\{I, \text{Ker}L \cap \Omega_2, 0\} = 1.$$

Then

$$\deg_B\{[I - (P + JQN)\gamma]_{\text{Ker}L}, \text{Ker}L \cap \Omega_2, 0\} = \deg_B\{H(\cdot, 1), \text{Ker}L \cap \Omega_2, 0\} \neq 0.$$

Next, we prove 8°. Let $x \in \bar{\Omega}_2 \setminus \Omega_1$ and $t \in [0, 1]$,

$$\begin{aligned} (\Psi_\gamma x)(t) &= (\alpha - 1) \int_0^1 \frac{|x(s)|}{(1-s)^{2-\alpha}} ds + (\alpha - 1) \int_0^1 \frac{f(s, |x(s)|)}{(1-s)^{2-\alpha}} ds \\ &\quad + \int_0^1 k(t, s) \frac{[f(s, |x(s)|) - (\alpha - 1) \int_0^1 \frac{f(\tau, |x(\tau)|)}{(1-\tau)^{2-\alpha}} d\tau]}{(1-s)^{2-\alpha}} ds \\ &= (\alpha - 1) \int_0^1 \frac{|x(s)|}{(1-s)^{2-\alpha}} ds + (\alpha - 1) \int_0^1 G(t, s) \frac{f(s, |x(s)|)}{(1-s)^{2-\alpha}} ds \\ &\geq (\alpha - 1) \int_0^1 (1 - \kappa G(t, s)) \frac{|x(s)|}{(1-s)^{2-\alpha}} ds \geq 0. \end{aligned}$$

Hence, $\Psi_\gamma(\bar{\Omega}_2 \setminus \Omega_1) \subset C$, i.e. 8° holds.

Since for $x \in \partial\Omega_2$,

$$\begin{aligned} (P + JQN)\gamma x &= (\alpha - 1) \int_0^1 \frac{|x(s)|}{(1-s)^{2-\alpha}} ds + (\alpha - 1) \int_0^1 \frac{f(s, |x(s)|)}{(1-s)^{2-\alpha}} ds \\ &\geq (\alpha - 1) \int_0^1 (1 - \kappa) \frac{|x(s)|}{(1-s)^{2-\alpha}} ds \geq 0. \end{aligned}$$

Thus, $(P + JQN)\gamma x \subset C$ for $x \in \partial\Omega_2$, 7° holds.

It remains to verify 6°. Let $u_0(t) \equiv 1$ on $[0, 1]$. Then $u_0 \in C \setminus \{0\}$, $C(u_0) = \{x \in C : x(t) > 0 \text{ on } [0, 1]\}$ and we can take $\sigma(u_0) = 1$. Let $x \in C(u_0) \cap \partial\Omega_1$. Then $x(t) > 0$ on $[0, 1]$, $0 < \|x\| \leq b$ and $x(t) \geq \delta \|x\|$ on $[0, 1]$. For every $x \in C(u_0) \cap \partial\Omega_1$, by (H2), we have

$$(\Psi x)(t_0) = (\alpha - 1) \int_0^1 \frac{x(s)}{(1-s)^{2-\alpha}} ds + (\alpha - 1) \int_0^1 G(t_0, s) \frac{f(s, x(s))}{(1-s)^{2-\alpha}} ds$$

$$\begin{aligned}
 &\geq \delta \|x\| + (\alpha - 1) \int_0^1 G(t_0, s) \frac{q(s)h(x(s))}{(1-s)^{2-\alpha}} ds \\
 &= \delta \|x\| + (\alpha - 1) \int_0^1 \frac{G(t_0, s)q(s)}{(1-s)^{2-\alpha}} \cdot \frac{h(x(s))}{x^\rho(s)} x^\rho(s) ds \\
 &\geq \delta \|x\| + (\alpha - 1) \delta^\rho \|x\|^\rho \int_0^1 \frac{G(t_0, s)q(s)}{(1-s)^{2-\alpha}} \cdot \frac{h(b)}{b^\rho} ds \\
 &= \delta \|x\| + (\alpha - 1) \delta^\rho \|x\| \cdot \frac{h(b)}{b} \cdot \frac{b^{1-\rho}}{\|x\|^{1-\rho}} \int_0^1 \frac{G(t_0, s)q(s)}{(1-s)^{2-\alpha}} ds \\
 &\geq \|x\|.
 \end{aligned}$$

Thus, $\|x\| \leq \sigma(u_0) \|\Psi x\|$ for all $x \in C(u_0) \cap \partial\Omega_1$.

By Theorem 2.1, the BVP (1.1)-(1.2) has a positive solution x^* on $[0, 1]$ with $b \leq \|x^*\| \leq B$. This completes the proof of Theorem 3.1. □

Remark 3.1. Note that with the projection $P(x) = x(0)$, condition 7° and 8° of Theorem 2.1 are no longer satisfied.

To illustrate how our main result can be used in practice, we present here an example.

Example 3.1. Consider

$$(3.6) \quad \begin{cases} {}^c D_{0^+}^{1.5} x(t) + \frac{1}{500}(1+t-t^2)(x^2 - 8x + 12)x = 0, & t \in (0, 1), \\ x'(0) = x'(1), \quad x(0) = \frac{3}{4}x(\frac{2}{3}) + \frac{1}{4}x(\frac{5}{6}). \end{cases}$$

Corresponding to the BVP (1.1)-(1.2), $\alpha = 1.5$ and $f(t, x) = \frac{1}{500}(1+t-t^2)(x^2 - 8x + 12)x$. Let $\kappa = \frac{1}{5}$, $B = 6$ and $b = \frac{1}{2}$, we may choose $b_1 = 4$, $b_2 = \frac{3}{20}$, $b_3 = \frac{1}{6}$, $c_1 = \frac{1}{50}$, $c_2 = \frac{1}{25}$ such that (H1) holds, and take $\rho = 1$, $t_0 = \frac{2}{3}$, $\delta = 0.995$, $q(t) = 1 + t(1 - t)$ and $h(x) = \frac{1}{50}x$ for $t \in [0, 1]$, $x \in (0, \frac{1}{2}]$ such that (H2) holds. Therefore, the BVP (3.6) has at least one positive solution on $[0, 1]$ according to Theorem 3.1.

REFERENCES

1. S. G. Samko, A. A. Kilbas & O.I. Marichev: *Fractional Integrals and Derivatives. Theory and Applications*. Gordon and Breach, Yverdon, 1993.
2. I. Podlubny: *Fractional Differential Equations, Mathematics in Sciences and Engineering*. **198**, Academic Press, San Diego, 1999.
3. K.S. Miller & B. Ross: *An Introduction to the Fractional Calculus and Differential Equations*. John Wiley, New York, 1993.

4. A. A. Kilbas, H. M. Srivastava & J.J. Trujillo: *Theory and Applications of Fractional Differential Equations*. North-Holland Mathematics Studies, 204. Elsevier Science B. V., Amsterdam, 2006.
5. V. Lakshmikantham & A.S. Vatsala: Basic theory of fractional differential equations. *Nonlinear Anal.* **69** (2008), 2677-2682.
6. _____ : Theory of fractional functional differential equations. *Nonlinear Anal.* **69** (2008), 2677-2682.
7. _____ : General uniqueness and monotone iterative technique for fractional differential equations. *Appl. Math. Letters* **21** (2008), 828-834.
8. A.M.A. El-Sayed, A.E.M. El-Mesiry & H.A.A. El-Saka: On the fractional-order logistic equation. *Appl. Math. Letters* **20** (2007), 817-823.
9. A.M.A. El-Sayed & E. M. El-Maghrabi: Stability of a monotonic solution of a non-autonomous multidimensional delay differential equation of arbitrary (fractional) order. *Electronic Journal of Qualitative Theory of Differential Equations* **16** (2008), 1-9.
10. K. Diethelm & N.J. Ford: Analysis of fractional differential equations. *J. Math. Anal. Appl.* **265** (2002), 229-248.
11. M. Benchohra, J. Henderson, S. K. Ntouyas & A. Ouahab: Existence results for fractional order functional differential equations with infinite delay. *J. Math. Anal. Appl.* **338** (2008), 1340-1350.
12. C. Bai: Positive solutions for nonlinear fractional differential equations with coefficient that changes sign. *Nonlinear Anal.* **64** (2006), 677-685.
13. Z. Bai & H. Lü: Positive solutions for boundary value problem of nonlinear fractional differential equation. *J. Math. Anal. Appl.* **311** (2005), 495-505.
14. S. Zhang: Existence of solution for a boundary value problem of fractional order. *Acta Mathematica Scientia* **26** (2006), 220-228.
15. _____ : Positive solutions for boundary-value problems of nonlinear fractional differential equations. *Electronic Journal of Qualitative Theory of Differential Equations* **2006** (2006), 1-12.
16. M. El-Shahed: Positive solutions for boundary value problem of nonlinear fractional differential equation. *Abstract and Appl. Anal.* Article Number: 10368 Published: 2007, 8 pages.
17. D. O'Regan & M. Zima: Leggett-Williams norm-type theorems for coincidences. *Arch. Math.* **87** (2006), 233-244.
18. J. Mawhin: *Topological degree methods in nonlinear boundary value problems*. in: NS-FCBMS Regional Conference Series in Mathematics, American Mathematical Society, Providence, RI, 1979.
19. Aijun Yang & Weigao Ge: Positive solutions for boundary value problems of N-dimension nonlinear fractional differential system. *Boundary Value Problems*, In press.

20. _____ : Positive solutions of multi-point boundary value problems with multivalued operators at resonance. *J. Appl. Math. Comput.* On line: 10.1007/s12190-008-0217-2.

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