

APPROXIMATELY CONVEX SCHWARTZ DISTRIBUTIONS

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ABSTRACT. Generalizing the approximately convex function which is introduced by D.H. Hyers and S.M. Ulam we establish an approximately convex Schwartz distribution and prove that every approximately convex Schwartz distribution is an approximately convex function.

1. INTRODUCTION

The main purpose of this article is to establish a concept of *approximately convex Schwartz distributions*. In 1950, Laurent Schwartz introduced the theory of distributions in his monograph *Théorie des distributions*. In this book Schwartz systematizes the theory of generalized functions, basing it on the theory of linear topological spaces, relates all the earlier approaches, and obtains many important results. After his elegant theory appeared, many important concepts and results on the classical spaces of functions have been generalized to the space of distributions. For example, *positive functions and positive-definite functions* have been generalized to *positive distributions and positive-definite distributions*, respectively, and it is shown that every positive distribution is a positive measure [7, p. 38] and every positive-definite distribution is the Fourier transform of positive measure μ such that $\int (1 + |x|)^{-p} d\mu < \infty$ for some $p \geq 0$ [5, p. 157], which is called *Bochner-Schwartz theorem* and is a natural generalization of the famous *Bochner theorem* stating that every positive-definite function is the Fourier transform of a positive finite measure. For other examples, *the Paley-Wiener theorem* has been generalized to *the Paley-Wiener-Schwartz theorem* which characterizes the distributions with

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bounded supports [7, p. 181]. Also, Lars Hörmander characterized the distributions u such that $u'' \geq 0$ (it was treated for n -dimensional case) in his famous book [7, p. 91]. On the other hand, generalizing the convex functions, D. H. Hyers and S. M. Ulam introduced *approximately convex functions* [8]. In this article, generalizing the distributions u such that $u'' \geq 0$ and the approximately convex functions, we establish *an approximately convex distributions* by means of a functional inequality in the space of distributions and prove that every approximately convex distribution u can be written in the form $u = g + r$ where g is a convex function and r is a bounded function.

2. SOME OPERATIONS ON DISTRIBUTIONS

In this section we briefly introduce the space of Schwartz distributions and some operations on it. Here we use the notations: $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$, for $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$, where \mathbb{N}_0 is the set of non-negative integers and $\partial_j = \frac{\partial}{\partial x_j}$.

Let Ω be an open subset of \mathbb{R}^n . We denote by $C_c^\infty(\Omega)$ the space of all infinitely differentiable functions on Ω with compact supports. A distribution $u : C_c^\infty(\Omega) \rightarrow \mathbb{C}$ is a linear functional on $C_c^\infty(\Omega)$ such that for every compact set $K \subset \Omega$ there exist constants C and k satisfying

$$(2.1) \quad |\langle u, \varphi \rangle| \leq C \sum_{|\alpha| \leq k} \sup |\partial^\alpha \varphi|$$

for all $\varphi \in C_c^\infty(\Omega)$ with supports contained in K . We denote by $\mathcal{D}'(\Omega)$ the space of Schwartz distributions on Ω .

Now we briefly introduce some basic operations in $\mathcal{D}'(\Omega)$. Let $u \in \mathcal{D}'(\Omega)$. Then the k -th partial derivative $\partial_k u$ of u is defined by

$$(2.2) \quad \langle \partial_k u, \varphi \rangle = -\langle u, \partial_k \varphi \rangle$$

for $k = 1, \dots, n$. Let $f \in C^\infty(\Omega)$. Then the multiplication fu is defined by

$$(2.3) \quad \langle fu, \varphi \rangle = \langle u, f\varphi \rangle.$$

Also we denote by $\tau_h u$ the translation of u by $h \in \mathbb{R}^n$ which is defined by

$$(2.4) \quad \langle \tau_h u, \varphi \rangle = \langle u, \varphi(\cdot - h) \rangle.$$

Let $u \in \mathcal{D}'(\mathbb{R}^n)$ and $\varphi \in C_c^\infty(\mathbb{R}^n)$. Then the convolution $u * \varphi$ is defined by

$$(2.5) \quad (u * \varphi)(x) = \langle u, \varphi(x - \cdot) \rangle.$$

Finally, let $f : \Omega_1 \rightarrow \Omega_2$ be a smooth function such that for each $x \in \Omega_1$ the derivative $f'(x)$ is surjective. Then there exists a unique continuous linear map $f^* : \mathcal{D}'(\Omega_2) \rightarrow \mathcal{D}'(\Omega_1)$ such that $f^*u = u \circ f$ when u is a continuous function. We call f^*u the pullback of u by f and often denoted by $u \circ f$. For more details of distributions we refer the reader to [7, 10].

3. APPROXIMATELY CONVEX DISTRIBUTIONS

Recall that a continuous function $g : \mathbb{R}^n \rightarrow \mathbb{R}$ is called *convex* if

$$(3.1) \quad g((1 - \theta)x + \theta y) \leq (1 - \theta)g(x) + \theta g(y)$$

for all $0 < \theta < 1$, $x, y \in \mathbb{R}^n$ and *midconvex* if the inequality (3.1) holds for $\theta = \frac{1}{2}$.

Generalizing the convex functions, D.H. Hyers and S.M. Ulam introduced ϵ -convex functions.

Definition 3.1 ([8]). Let $\epsilon \geq 0$. Then a function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is called ϵ -convex if

$$(3.2) \quad f((1 - \theta)x + \theta y) \leq (1 - \theta)f(x) + \theta f(y) + \epsilon$$

for all $0 < \theta < 1$, $x, y \in \mathbb{R}^n$.

Also they proved the following.

Theorem 3.2 ([8]). *Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be an ϵ -convex function. Then there corresponds a convex function $g : \mathbb{R}^n \rightarrow \mathbb{R}$ such that*

$$(3.3) \quad |f(x) - g(x)| \leq \left(\frac{n^2 + 3n}{4n + 4} \right) \epsilon$$

for all $x \in \mathbb{R}^n$.

Later, P.W. Cholewa [2] improved the above result and proved that the inequality (3.3) can be replaced by the better inequalities

$$(3.4) \quad |f(x) - g(x)| \leq \frac{1}{2} q_n \epsilon$$

where q_n is given by the recursion formula

$$(3.5) \quad q_1 = 1, q_2 = \frac{5}{3}, q_{2k-1} = q_{2k} = 1 + q_k, k \geq 2.$$

On the other hand, it is well known in elementary calculus that if g is twice differentiable, the inequality (3.1) is equivalent to

$$(3.6) \quad \sum_{j=1}^n \sum_{k=1}^n x_j x_k \partial_j \partial_k g(x) \geq 0, \quad x = (x_1, \dots, x_n) \in \mathbb{R}^n.$$

Generalizing the inequality (3.6), L. Hörmander characterized the distributions $u \in \mathcal{D}'(\mathbb{R}^n)$ satisfying

$$(3.7) \quad \sum_{j=1}^n \sum_{k=1}^n x_j x_k \partial_j \partial_k u \geq 0, \quad x = (x_1, \dots, x_n) \in \mathbb{R}^n.$$

Here a distribution $u \geq 0$ means that $\langle u, \varphi \rangle \geq 0$ for all test functions $\varphi \in C_c^\infty(\mathbb{R}^n)$ with $\varphi \geq 0$.

Theorem 3.3 ([7]). *Let $u \in \mathcal{D}'(\mathbb{R}^n)$ satisfy the inequality (3.7). Then there exists a convex function $g : \mathbb{R}^n \rightarrow \mathbb{R}$ such that $u = g(x)$ in the sense that*

$$(3.8) \quad \langle u, \varphi \rangle = \int g(x) \varphi(x) dx$$

for all $\varphi \in C_c^\infty(\mathbb{R}^n)$.

Generalizing the inequalities (3.1) and (3.2) to the space of Schwartz distributions we introduce *convex distributions* and ϵ -*convex distributions*. For distributions $u_1, u_2, u_1 \geq u_2$ means that $u_1 - u_2 \geq 0$, that is, $\langle u_1, \varphi \rangle \geq \langle u_2, \varphi \rangle$ for all test functions $\varphi \in C_c^\infty(\mathbb{R}^n)$ with $\varphi \geq 0$.

Definition 3.4. We call $u \in \mathcal{D}'(\mathbb{R}^n)$ *convex* if

$$(3.9) \quad \tau_{\theta h} u \leq (1 - \theta)u + \theta \tau_h u, \quad 0 < \theta < 1, \quad h \in \mathbb{R}^n,$$

and *midconvex* if the inequality (3.9) holds for $\theta = \frac{1}{2}$. Also we call $u \in \mathcal{D}'(\mathbb{R}^n)$ ϵ -*convex* if

$$(3.10) \quad \tau_{\theta h} u \leq (1 - \theta)u + \theta \tau_h u + \epsilon, \quad 0 < \theta < 1, \quad h \in \mathbb{R}^n,$$

and ϵ -*midconvex* if the inequality (3.10) holds for $\theta = \frac{1}{2}$.

As a main result we prove the following.

Theorem 3.5. *Every ϵ -convex distribution $u \in \mathcal{D}'(\mathbb{R}^n)$ can be written in the form*

$$(3.11) \quad u = g(x) + r(x)$$

where $g(x)$ is a convex function and $r(x)$ is a bounded measurable function satisfying $\|r\|_{L^\infty} \leq \frac{1}{2} q_n \epsilon$ where q_n is given by the recursion formula (3.5).

The following elegant result of S. Banach is an essential tool in the proof of our theorem.

Lemma 3.6 ([1]). *Let $f_k, j = 1, 2, \dots$, be a sequence of bounded measurable functions such that $\|f_k\|_{L^\infty} \leq M, j = 1, 2, \dots$. Then there exists a subsequence $f_{k_j}, j = 1, 2, \dots$, and a bounded measurable function f with $\|f\|_{L^\infty} \leq M$ such that*

$$\lim_{j \rightarrow \infty} \int f_{k_j}(x)\varphi(x)dx = \int f(x)\varphi(x)dx$$

for all $\varphi \in L^1(\mathbb{R}^n)$.

Note that $C_c^\infty(\mathbb{R}^n) \subset L^1(\mathbb{R}^n)$ and the subsequence f_{k_j} in the above lemma converges to the f in the sense of distributions.

Lemma 3.7. *Let $u \in \mathcal{D}'(\mathbb{R}^n)$. Then the following conditions are equivalent.*

- (i) u is a convex distribution.
- (ii) u is a midconvex distribution.
- (iii) u satisfies (3.7).
- (iv) u is a convex function.

Proof. The implications (i) \Rightarrow (ii), (iv) \Rightarrow (i) are obvious. The implication (iii) \Rightarrow (iv) is just the Theorem 3.3. Thus it suffices to show that (ii) \Rightarrow (iii). We denote by $\delta(x)$ the function on \mathbb{R}^n such that

$$\delta(x) = \begin{cases} A \exp(-(1 - |x|^2)^{-1}), & |x| < 1 \\ 0, & |x| \geq 1, \end{cases}$$

where

$$A = \left(\int_{|x| < 1} \exp(-(1 - |x|^2)^{-1}) dx \right)^{-1}.$$

It is easy to see that $\delta(x)$ is an infinitely differentiable function with support $\{x : |x| \leq 1\}$. Now we employ the function $\delta_t(x) := t^{-n}\delta(x/t)$, $t > 0$. Let $u \in \mathcal{D}'(\mathbb{R}^n)$. Then for each $t > 0$, it is well known that $(u * \delta_t)(x) = \langle u_y, \delta_t(x - y) \rangle$ is a smooth function in \mathbb{R}^n and $(u * \delta_t)(x) \rightarrow u$ as $t \rightarrow 0^+$ in the sense of distributions [7, Chapter IV], that is, for every $\varphi \in C_c^\infty(\mathbb{R}^n)$

$$\langle u, \varphi \rangle = \lim_{t \rightarrow 0^+} \int (u * \delta_t)(x)\varphi(x) dx.$$

Convolving $\delta_t(x)$ in (3.9) with $\theta = \frac{1}{2}$, we have for each $t > 0$

$$(3.12) \quad 2U_t \left(x + \frac{h}{2} \right) \leq U_t(x) + U_t(x + h), \quad x, h \in \mathbb{R}^n,$$

where $U_t(x) = (u * \delta_t)(x)$. Thus for each $t > 0$, U_t is midconvex function, which implies U_t is convex function since U_t is a smooth function. Thus we have

$$\sum_{j=1}^n \sum_{k=1}^n x_j x_k \partial_{x_j} \partial_{x_k} U_t(x) \geq 0, \quad x = (x_1, \dots, x_n) \in \mathbb{R}^n$$

for each $t > 0$. Thus let $\varphi \in C_c^\infty(\mathbb{R}^n)$ such that $\varphi \geq 0$. Then integration by parts gives

$$(3.13) \quad \sum_{j=1}^n \sum_{k=1}^n \int_{\mathbb{R}^n} [x_j x_k \partial_{x_j} \partial_{x_k} U_t(x)] \varphi(x) dx = \sum_{j=1}^n \sum_{k=1}^n \int_{\mathbb{R}^n} U_t(x) x_j x_k [\partial_{x_j} \partial_{x_k} \varphi(x)] dx \geq 0.$$

Letting $t \rightarrow 0^+$ in (3.13) we have

$$\sum_{j=1}^n \sum_{k=1}^n \langle u, x_j x_k \partial_{x_j} \partial_{x_k} \varphi \rangle \geq 0$$

for all $\varphi \in C_c^\infty(\mathbb{R}^n)$ with $\varphi \geq 0$. In view of (2.2) and (2.3) we have,

$$\langle \sum_{j=1}^n \sum_{k=1}^n x_j x_k \partial_{x_j} \partial_{x_k} u, \varphi \rangle = \sum_{j=1}^n \sum_{k=1}^n \langle u, x_j x_k \partial_{x_j} \partial_{x_k} \varphi \rangle \geq 0$$

for all $\varphi \in C_c^\infty(\mathbb{R}^n)$ with $\varphi \geq 0$, which gives (3.7). This completes the proof. \square

Proof of Theorem 3.5. Convolving $\delta_t(x)$ in (3.10) we have for each $t > 0$,

$$(3.14) \quad U_t(x + \theta h) \leq (1 - \theta)U_t(x) + \theta U_t(x + h) + \epsilon, \quad x, h \in \mathbb{R}^n, 0 < \theta < 1,$$

where $U_t(x) = (u * \delta_t)(x)$. Thus for each $t > 0$, $U_t(x)$ is an ϵ -convex function of $x \in \mathbb{R}^n$. Thus, due to the result of Cholewa[2], for each $t > 0$, there is a convex function $V_t(x)$ such that

$$(3.15) \quad |U_t(x) - V_t(x)| \leq \frac{1}{2} q_n \epsilon.$$

Now, let $f_k(x) := U_{\frac{1}{k}}(x) - V_{\frac{1}{k}}(x)$, $k = 1, 2, 3, \dots$. Then by Lemma 3.6, there exists a subsequence $k_j, j = 1, 2, 3, \dots$, and a bounded measurable function $r(x)$ such that $f_{k_j}(x) \rightarrow r(x)$ in $\mathcal{D}'(\mathbb{R}^n)$ as $j \rightarrow \infty$. Thus we have

$$\lim_{j \rightarrow \infty} V_{\frac{1}{k_j}} = \lim_{j \rightarrow \infty} (U_{\frac{1}{k_j}} - f_{k_j}) = u - r(x) := v$$

in $\mathcal{D}'(\mathbb{R}^n)$.

Now we show that $v = u - r(x)$ is a convex function. Since for each $t > 0$, V_t is a convex function, we have

$$V_t(x + \theta h) \leq (1 - \theta)V_t(x) + \theta V_t(x + h), \quad x, h \in \mathbb{R}^n, 0 < \theta < 1.$$

Thus for all $\varphi \in C_c^\infty(\mathbb{R}^n)$ such that $\varphi \geq 0$ we have

$$(3.16) \quad \int_{\mathbb{R}^n} V_t(x + \theta h) \varphi(x) dx \leq (1 - \theta) \int_{\mathbb{R}^n} V_t(x) \varphi(x) dx + \theta \int_{\mathbb{R}^n} V_t(x + h) \varphi(x) dx, \quad 0 < \theta < 1.$$

Letting $t = \frac{1}{k_j}$ and $j \rightarrow \infty$ in (3.16), we have

$$\langle \tau_{\theta h} v, \varphi \rangle \leq (1 - \theta) \langle v, \varphi \rangle + \theta \langle \tau_h v, \varphi \rangle, \quad 0 < \theta < 1,$$

for all $\varphi \in C_c^\infty(\mathbb{R}^n)$ with $\varphi \geq 0$. Thus v is a convex distribution. By Lemma 3.7, v is a convex function. This completes the proof with $v = g(x)$. \square

Remark 1. Since every ϵ -midconvex function is 2ϵ -convex it is easy to see that every ϵ -midconvex distribution can be written in the form

$$u = g(x) + r(x),$$

where $g(x)$ is a convex function and r is a bounded measurable function satisfying $\|r\| \leq q_n \epsilon$.

Remark 2. The inequality (3.10) can be generalized to the space of distributions in a different way as follows

$$(3.17) \quad u \circ A_\theta \leq (1 - \theta)(u \circ P_1) + \theta(u \circ P_2) + \epsilon, \quad 0 < \theta < 1,$$

where $u \circ A_\theta$, $u \circ P_1$, $u \circ P_2$ denote the pullbacks of u by $A_\theta(x, y) = (1 - \theta)x + \theta y$, $P_1(x, y) = x$, $P_2(x, y) = y$, respectively. The author would like to know that if u satisfies the inequality (3.17), then u is an ϵ -convex distribution and can be written in the form (3.11).

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