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ON $s\gamma$-GENERALIZED SETS

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Abstract. In this paper, we introduce the notions of $s\gamma$-generalized closed sets and $s\gamma$-generalized sets, and investigate some properties for such notions.

1. Introduction

Generalized closed sets in a topological space were introduced by Levine in [2] and he investigated many of the extended properties of closed sets. In [3], Mashhourr et.al. introduced the notion of supratopological spaces which are generalized topological spaces. In [4], the author introduced the notion of $s\gamma$-sets in a supratopological space. In this paper, we introduce the notion of $s\gamma$-generalized closed (shortly $s\gamma$-g-closed) sets which are generalized $s\gamma$-closed sets in a supratopological space and study their properties. $s\gamma$-generalized sets (shortly $s\gamma$-g-sets) are also introduced and their properties are investigated.

2. Preliminaries

Let X be a nonempty set. A family $\tau \subset 2^X$ is called a supratopology on X [3] if $X \in \tau$ and τ is closed under arbitrary union. The pair (X, τ) is called a supratopological space. The members of τ are called supraopen sets. The complement of supraopen sets are called supraclosed sets. Let (X, τ) be a supratopological space and $S \subset X$. The supra-closure of S, denoted by $scl(S)$, is the intersection of supraclosed sets including S. And the interior of S, denoted by $sint(S)$, the union of supraopen sets included in S.

Definition 2.1 ([4]). Let (X, τ) be a supratopological space and let $S(x) = \{A \in \tau : x \in A\}$ for each $x \in X$. Then we call $\mathbf{S}_x = \{A \subset X$: there exists $\mu \subset S(x)$ such

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that μ is finite and $\cap\mu \subset A\}$ the supra-neighborhood filter at x.

A filter $\mathbf{F}$ on X supra-converges to x if $\mathbf{F}$ is finer than the supra-neighborhood filter $\mathbf{S}_x$.

Definition 2.2 ([4]). Let (X, τ) be a supratopological space. A subset U of X is called an *$s\gamma$-set* in X if whenever a filter $\mathbf{F}$ on X supra-converges to x and $x \in U$, then $U \in \mathbf{F}$.

The class of all $s\gamma$-sets in X will be denoted by $s\gamma(X)$. In particular, The class of all $s\gamma$-sets induced by the supratopology τ will be denoted by $s\gamma_\tau$.

Definition 2.3 ([4]). Let (X, τ) be a supratopological space and $A \subset X$.

$sI_\gamma(A) = \cup\{U : U \subset A,$ U is an $s\gamma$-set $\}$ is the $s\gamma$-interior of A.

$scl_\gamma(A) = \{x \in X : A \cap U \neq \emptyset$ for all $U \in \mathbf{S}_x\}$ is the $s\gamma$-closure of A.

Theorem 2.4 ([4]). *Let (X, τ) be a supratopological space and $A \subset X$.*

(1) *$sI_\gamma(A) \subset A$ and $A \subset scl_\gamma(A)$;*
(2) *A is $s\gamma$-set if and only if $A = sI_\gamma(A)$;*
(3) *A is $s\gamma$-closed if and only if $A = scl_\gamma A$;*
(4) *$sI_\gamma(A) = X - scl_\gamma(X - A)$ and $scl_\gamma(A) = X - sI_\gamma(X - A)$.*

Definition 2.5 ([4]). Let (X, τ) and (Y, μ) be supratopological spaces. A function $f : X \to Y$ is called $s\gamma^*$ if the inverse image of each $s\gamma$-set of Y is an $s\gamma$-set in X.

For $Y \subset X$, let $s\gamma_X(Y) = \{Y \cap U : U \in s\gamma(X)\}$; then we call $(Y, s\gamma_X(Y))$ a *$s\gamma$-subspace* of (X, τ). An element of $s\gamma_X(Y)$ is called an *$s\gamma$-set relative to* Y. Let $A \subset Y$; then the $s\gamma$-interior of A in an $s\gamma$-subspace Y is denoted by $sint_{\gamma Y}(A)$ and the $s\gamma$-closure of A in an $s\gamma$-subspace Y is denoted by $scl_{\gamma Y}(A)$.

3. $s\gamma$-GENERALIZED CLOSED SETS

Definition 3.1. Let (X, τ) be a supratopological space. A subset A of X is called an *$s\gamma$-generalized closed set* (shortly *$s\gamma$-g-closed set*) in X if $scl_\gamma A \subset U$ whenever $A \subset U$ and U is an $s\gamma$-set.

Remark 3.2. Every $s\gamma$-closed set is $s\gamma$-g-closed but the converse is not true as the next example.

Example 3.3. Let $X = \{a, b, c, d\}$ and let $\tau = \{\emptyset, \{a, b, c\}, \{c, d\}, X\}$; then $S\gamma(X) = \{\emptyset, \{a, b, c\}, \{c\}, \{c, d\}, X\}$. Consider a set $A = \{a, d\}$. Since X is the only $s\gamma$-set containing A, A is $s\gamma$-g-closed but not $s\gamma$-closed.

The following implications are obtained

supraclosed $\Rightarrow$ $s\gamma$-closed $\Rightarrow$ $s\gamma$-g-closed

Theorem 3.4. *Let (X, τ) be a supratopological space. A set A is $s\gamma$-g-closed iff $scl_\gamma A - A$ contains no nonempty $s\gamma$-closed sets.*

Proof. Suppose A is an $s\gamma$-g-set and F is an $s\gamma$-closed set such that $F \subset scl_\gamma A - A$. Since F^c is an $s\gamma$-set and $A \subset F^c$, $scl_\gamma A \subset F^c$. Thus $F = \emptyset$.

For the converse, let $A \subset U$ for an $s\gamma$-set U in X. If $scl_\gamma A$ is not contained in U, then $scl_\gamma A \cap U^c \neq \emptyset$. It is a contradiction since $scl_\gamma A \cap U^c \subset scl_\gamma A - A$. □

Corollary 3.5. *Let (X, τ) be a supratopological space and let A be an $s\gamma$-g-closed set. Then a set A is $s\gamma$-closed iff $scl_\gamma A - A$ is $s\gamma$-closed.*

Proof. Suppose A is an $s\gamma$-closed set, then $scl_\gamma(A) = A$, so $scl_\gamma(A) - A = \emptyset$ is $s\gamma$-closed.

Suppose that $scl_\gamma(A) - A$ is $s\gamma$-closed. Since A is $s\gamma$-g-closed, by Theorem 3.4, $scl_\gamma(A) - A = \emptyset$, so we get $scl_\gamma(A) = A$. □

Theorem 3.6. *Let (X, τ) be a supratopological space. If both A and B are $s\gamma$-g-closed sets, then $A \cup B$ is $s\gamma$-g-closed.*

Proof. Let $A \cup B \subset U$ for $U \in S\gamma(X)$; then since $scl_\gamma(A \cup B) = scl_\gamma(A) \cup scl_\gamma(B) \subset U$, so $A \cup B$ is an $s\gamma$-g-closed set. □

The intersection of two $s\gamma$-g-closed sets is generally not an $s\gamma$-g-closed set as the next example.

Example 3.7. As Example 3.3, let $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, \{a, b, c\}, \{c, d\}, X\}$; then $S\gamma(X) = \{\emptyset, \{a, b, c\}, \{c\}, \{c, d\}, X\}$. Consider two $s\gamma$-g-closed sets $A = \{a, c, d\}$ and $B = \{b, c, d\}$. Since $scl_\gamma\{c, d\} = X$, $A \cap B$ is not $s\gamma$-g-closed.

Theorem 3.8. *Let (X, τ) be a supratopological space, $B \subset Y \subset X$. If B is an $s\gamma$-g-closed set relative to Y and if Y is $s\gamma$-g-closed in X, then B is $s\gamma$-g-closed in X.*

Proof. Let $B \subset U$ and let U be an $s\gamma$-set in X; then $scl_{\gamma Y}(B) \subset Y \cap U$, so $scl_{\gamma Y}(B) = Y \cap scl_\gamma(B) \subset Y \cap U$ and $Y \subset (scl_\gamma(B))^c \cup U$. Since Y is an $s\gamma$-g-closed set in X, $scl_\gamma(B) \subset scl_\gamma(Y) \subset U \cup (scl_\gamma(B))^c$, and so $scl_\gamma(B) \subset U$. □

From Theorem 3.8 we get the next corollary.

Corollary 3.9. *If A is an $s\gamma$-g-closed set and if F is an $s\gamma$-set, then $A \cap F$ is an $s\gamma$-g-closed set.*

Theorem 3.10. *Let (X, τ) be a supratopological space. If A is $s\gamma$-g-closed and $A \subset B \subset scl_\gamma(A)$, then B is $s\gamma$-g-closed.*

Proof. Since $scl_\gamma(B) - B \subset scl_\gamma(A) - A$, $scl_\gamma(B) - B$ has no a nonempty $s\gamma$-closed subset. Thus by Theorem 3.4, B is $s\gamma$-g-closed. □

Theorem 3.11. *Let (X, τ) be a supratopological space and $A \subset Y \subset X$. If A is $s\gamma$-g-closed in X, then A is an $s\gamma$-g-closed set relative to Y.*

Proof. Let $A \subset Y \cap U$ and let U be an $s\gamma$-closed set in X; then $scl_\gamma A \subset U$. It follows that $scl_{\gamma Y}(A) = Y \cap scl_\gamma(A) \subset Y \cap U$. □

4. $s\gamma$-GENERALIZED SETS

Definition 4.1. Let (X, τ) be a supratopological space. A subset U of X is called an *$s\gamma$-generalized set* (shortly *$s\gamma$-g-set*) in X if the complement of U (shortly U^c) is an $s\gamma$-g-closed set.

The class of all $s\gamma$-g-sets in X will be denoted by $s\gamma g(X)$.

Remark 4.2. In a supratopological space (X, τ), it is always true that

$$\tau \subset s\gamma(X) \subset s\gamma g(X).$$

From Definition 3.1, we get the following theorem.

Theorem 4.3. *A set A is an $s\gamma$-g-set iff $F \subset sI_\gamma A$ whenever F is $s\gamma$-closed and $F \subset A$.*

Theorem 4.4. *Let (X, τ) be a supratopological space. A set A is an $s\gamma$-g-set of X iff $U = X$ whenever U is an $s\gamma$-set and $sI_\gamma U \cup A^c \subset U$.*

Proof. Suppose that U is an $s\gamma$-set and $sI_\gamma U \cup A^c \subset U$. Then $U^c \subset scl_\gamma(A^c) \cap A = scl_\gamma(A^c) - A^c$. Since U^c is an $s\gamma$-closed set and A^c is $s\gamma$-g-closed, by Theorem 3.4, we get $U^c = \emptyset$, so $X = U$.

Suppose that F is an $s\gamma$-closed set and $F \subset A$. Then $sI_\gamma A \cup A^c \subset sI_\gamma A \cup F^c$ and so $sI_\gamma A \cup F^c = X$. It follows that $F \subset sI_\gamma A$. □

Let (X, τ) be a supratopological space and A, B be nonempty subsets of X. Then A and B are said to be $s\gamma$-separated if $A \cap scl_\gamma B = scl_\gamma A \cap B = \emptyset$.

Theorem 4.5. *Let (X, τ) be a supratopological space and let A, B be nonempty $s\gamma$-separated subsets of X. If both A and B are $s\gamma$-g-sets, then $A \cup B$ is an $s\gamma$-g-set.*

Proof. Let F be an $s\gamma$-closed set of $A \cup B$; then $F \cap scl_\gamma A \subset A$ and by Theorem 4.3, $F \cap scl_\gamma A \subset sI_\gamma A$. In the same manner, we have $F \cap scl_\gamma B \subset sI_\gamma B$. Thus

$F = F \cap (A \cup B) \subset sI_\gamma A \cup sI_\gamma B \subset sI_\gamma(A \cup B)$, so by Theorem 4.3, $A \cup B$ is an $s\gamma$-g-set. □

From Theorem 4.5 we get the following:

Corollary 4.6. *Let A and B be two $s\gamma$-g-closed sets. If both A^c and B^c are $s\gamma$-separated, then $A \cap B$ is $s\gamma$-g-closed.*

Theorem 4.7. *Let (X, τ) be a supratopological space. If $A \subset Y \subset X$ where A is an $s\gamma$-g-set relative to Y and if Y is an $s\gamma$-g-set in X, then A is an $s\gamma$-g-set in X.*

Proof. Let F be an $s\gamma$-closed set in X and $F \subset A$. Then F is $s\gamma$-closed relative to Y, so $F \subset sI_{\gamma Y}A$. Thus there exists an $s\gamma$-set U in X such that $F \subset U \cap Y \subset sI_{\gamma Y}A \subset A$ and $F \subset sI_\gamma Y \subset Y$ since Y is an $s\gamma$-g-set in X. Thus $F \subset sI_\gamma A$. From Theorem 4.3, A is an $s\gamma$-g-set in X. □

Theorem 4.8. *If A is an $s\gamma$-g-set and $sI_\gamma A \subset B \subset A$, then B is an $s\gamma$-g-set.*

Proof. $A^c \subset B^c \subset scl_\gamma(A^c)$ and since A^c is an $s\gamma$-g-closed set, it follows that B^c is an $s\gamma$-g-closed set. Thus B is an $s\gamma$-g-set. □

Theorem 4.9. *A set A is $s\gamma$-g-closed iff $scl_\gamma A - A$ is an $s\gamma$-g-set.*

Proof. Suppose that A is an $s\gamma$-g-closed subset and $F \subset scl_\gamma A - A$, where F is $s\gamma$-closed. Then by Theorem 3.4, $F = \emptyset$ and hence $F \subset sI_\gamma(scl_\gamma A - A)$. Thus we get $scl_\gamma A - A$ is an $s\gamma$-g-set.

Suppose $A \subset U$ where U is an $s\gamma$-set. Then $scl_\gamma A \cap U^c \subset scl_\gamma A \cap A^c = scl_\gamma A - A$ and since $scl_\gamma A \cap U^c$ is $s\gamma$-closed and $scl_\gamma A - A$ is an $s\gamma$-g-set, it follows that $scl_\gamma A \cap U^c \subset sI_\gamma(scl_\gamma A \cap A^c) = \emptyset$. Therefore $scl_\gamma A \subset U$. Thus A is $s\gamma$-g-closed. □

Definition 4.10. For two supratopological spaces (X, τ) and (Y, μ), a function $f : (X, \tau) \to (Y, \mu)$ is $s\gamma^*$-closed if for every $s\gamma$-closed set G in X, $f(G)$ is $s\gamma$-closed in Y.

Theorem 4.11. *Let $f : X \to Y$ be an $s\gamma^*$-continuous and $s\gamma^*$-closed function between supratopological spaces. If A is an $s\gamma$-g-closed set in X, then $f(A)$ is $s\gamma$-g-closed in Y.*

Proof. Let $f(A) \subset U$ where U is an $s\gamma$-set in Y; then $A \subset f^{-1}(U)$ and hence $scl_\gamma A \subset f^{-1}(U)$. Thus $f(scl_\gamma A) \subset U$ and $f(scl_\gamma A)$ is an $s\gamma$-closed set. It follows that $scl_\gamma f(A) \subset scl_\gamma f(scl_\gamma A) \subset f(scl_\gamma A) \subset U$. Then $f(A)$ is an $s\gamma$-g-closed set in Y. □

Let $(X, \tau), (Y, \mu)$ be supratopological spaces. We recall that a function $f : X \to Y$

is $s\gamma^*$-continuous iff $f(scl_{\gamma_\tau}(U)) \subset scl_{\gamma_\mu}(f(U))$, for every $U \subset X$ [4].

Theorem 4.12. *Let $f : X \to Y$ be a function between supratopological spaces and let f be $s\gamma^*$-continuous and $s\gamma^*$-closed. If B is an $s\gamma$-g-closed set in Y, then $f^{-1}(B)$ is $s\gamma$-g-closed in X.*

Proof. Let B be an $s\gamma - g$-closed set in Y and $f^{-1}(B) \subset U$ where U is an $s\gamma$-set in X. Then since U is an $s\gamma$-set, $scl_\gamma(f^{-1}(B)) \cap U^c$ is $s\gamma$-closed and $f(scl_\gamma(f^{-1}(B)) \cap U^c)$ is also $s\gamma$-closed by Definition 4.11. Since f is an $s\gamma^*$-continuous function, we get the following:

$$\begin{aligned} f(scl_\gamma(f^{-1}(B)) \cap U^c) &\subset f(scl_\gamma(f^{-1}(B))) \cap f(U^c) \\ &\subset scl_\gamma f(f^{-1}(B)) \cap f(U^c) \\ &\subset scl_\gamma(B) \cap f(U^c) \\ &\subset scl_\gamma B - B. \end{aligned}$$

Since $f(scl_\gamma(f^{-1}(B)) \cap U^c)$ is an $s\gamma$-closed set, from Theorem 3.4 it follows $f(scl_\gamma(f^{-1}(B)) \cap U^c) = \emptyset$, i.e., $scl_\gamma(f^{-1}(B)) \cap U^c = \emptyset$. Hence $f^{-1}(B)$ is an $s\gamma - g$-closed set. □

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