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ON THE DEBREU INTEGRAL OF FUZZY MAPPINGS IN BANACH SPACES

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Abstract. In this paper, we introduce Debreu integral of fuzzy mappings in Banach spaces in terms of the Debreu integral of set-valued mappings, investigate properties of Debreu integral of fuzzy mappings in Banach spaces and obtain the convergence theorem for Debreu integral of fuzzy mappings in Banach spaces.

1. Introduction

The notion of integral of set-valued mappings is very useful in many branches of mathematics like mathematical economics, control theory, convex analysis, etc. It has been introduced by several mathematicians and in different ways. In particular, the Debreu integral of set-valued mappings was studied by Byme [1], Cascales and Rodriguez [2], Debreu [4], Hiai and Umegaki [6], Klein and Thompson [7] and others. Another mathematicians also introduced the integrals of fuzzy mappings in Banach spaces in terms of the integrals of set-valued mappings. In particular, Kaleva [9] introduced the integral of fuzzy mappings in $\mathbb{R}^n$ in terms of the integral of set-valued mappings in $\mathbb{R}^n$. Xue, Ha and Ma [10] and Xue, Wang and Wu [11] also introduced the integrals of fuzzy mappings in Banach spaces in terms of Aumann-Pettis and Aumann-Bochner integrals of set-valued mappings.

In this paper, we introduce Debreu integral of fuzzy mappings in Banach spaces in terms of the Debreu integral of set-valued mappings, investigate properties of Debreu integral of fuzzy mappings in Banach spaces and obtain the convergence theorem for Debreu integral of fuzzy mappings in Banach spaces.

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2. PRELIMINARIES

Throughout this paper, (Ω, Σ, μ) denotes a complete finite measure space and X a Banach space with dual X^*. The closed unit ball of X^* is denoted by B_{X^*}. $CL(X)$ denotes the family of all nonempty closed subsets of X and $CWK(X)$ the family of all nonempty convex weakly compact subsets of X. For $A \subseteq X$ and $x^* \in X^*$, let $s(x^*, A) = \sup\{x^*(x) : x \in A\}$, the support function of A. For $A, B \in CL(X)$, let $H(A, B)$ denote the Hausdorff metric of A and B defined by

$$H(A,B) = \max\left(\sup_{a\in A} d(a,B),\ \sup_{b\in B} d(b,A)\right),$$

where $d(a, B) = \inf_{b\in B} \|a - b\|$ and $d(b, A) = \inf_{a\in A} \|a - b\|$. Especially, it is well-known fact that

$$H(A,B) = \sup_{\|x^*\|\le 1} |s(x^*,A) - s(x^*,B)|$$

whenever A, B are convex sets. Note that $(CWK(X), H)$ is a complete metric space. The number $\|A\|$ is defined by

$$\|A\| = H(A, \{0\}) = \sup_{x\in A} \|x\|.$$

Let $u : X \to [0, 1]$. We denote $[u]^r = \{x \in X : u(x) \ge r\}$ for $r \in (0, 1]$ and $[u]^0 = cl\{x \in X : u(x) > 0\}$. The function u is called a *generalized fuzzy number* if for each $r \in (0, 1]$, $[u]^r \in CWK(X)$. Let $\mathcal{F}(X)$ denote the set of all generalized fuzzy numbers on X. For $u, v \in \mathcal{F}(X)$ and $\lambda \in \mathbb{R}$, we define $u + v$ and λu as follows:

$$(u+v)(x) = \sup_{x=y+z} \min(u(y), v(z)),$$

$$(\lambda u)(x) = \begin{cases} u\left(\dfrac{1}{\lambda}x\right), & \lambda \ne 0 \\ \tilde{0},\ \lambda = 0, \text{ where } \tilde{0} = \chi_{\{0\}}. \end{cases}$$

For $u, v \in \mathcal{F}(X)$ and $\lambda \in \mathbb{R}$, $[u+v]^r = [u]^r + [v]^r$ and $[\lambda u]^r = \lambda[u]^r$ for each $r \in (0, 1]$, so $u + v, \lambda u \in \mathcal{F}(X)$. For $u, v \in \mathcal{F}(X)$, we define $u \le v$ as follows:

$$u \le v \quad \text{if} \quad u(x) \le v(x) \quad \text{for all } x \in X.$$

Clearly, we have the following fact: for $u, v \in \mathcal{F}(X)$, $u \le v$ if and only if $[u]^r \subseteq [v]^r$ for each $r \in (0, 1]$. Define $D : \mathcal{F}(X) \times \mathcal{F}(X) \to [0, +\infty]$ by the equation

$$D(u,v) = \sup_{r\in(0,1]} H([u]^r, [v]^r).$$

Then D is a metric on $\mathcal{F}(X)$. The norm $\|u\|$ of $u \in \mathcal{F}(X)$ is defined by

$$\|u\| = D(u, \tilde{0}) = \sup_{r \in (0,1]} H([u]^r, \{0\}) = \sup_{r \in (0,1]} \|[u]^r\|.$$

The mapping $F : \Omega \to CL(X)$ is called a *set-valued mapping*. F is said to be *scalarly measurable* if for every $x^* \in X^*$, the real-valued function $s(x^*, F)$ is measurable. F is said to be *measurable* if $F^{-1}(A) = \{\omega \in \Omega : F(\omega) \cap A \neq \phi\} \in \Sigma$ for every $A \in CL(X)$. Note that measurability is stronger than scalar measurability.

Let $F : \Omega \to CL(X)$ be a set-valued mapping. Then the following statements are equivalent:

(1) $F : \Omega \to CL(X)$ is measurable;

(2) $F^{-1}(U) = \{\omega \in \Omega : F(\omega) \cap U \neq \phi\} \in \Sigma$ for every open subset U of X;

(3) (Castaing representation) there exists a sequence $\{f_n\}$ of measurable functions $f_n : \Omega \to X$ such that $F(\omega) = cl\{f_n(\omega)\}$ for all $\omega \in \Omega$.

A set-valued mapping $F : \Omega \to CL(X)$ is said to be *weakly integrably bounded* if the real-valued function $|x^*F| : \Omega \to \mathbb{R}, |x^*F|(\omega) = \sup\{|x^*(x)| : x \in F(\omega)\}$ is integrable for every $x^* \in X^*$. A set-valued mapping $F : \Omega \to CL(X)$ is said to be *integrably bounded* if there exists an integrable real-valued function h such that for each $\omega \in \Omega$, $\|x\| \leq h(\omega)$ for all $x \in F(\omega)$. A set-valued mapping $F : \Omega \to CL(X)$ is said to be *scalarly integrable* if for every $x^* \in X^*$, $s(x^*, F)$ is integrable. A set-valued mapping $F : \Omega \to CL(X)$ is said to be *scalarly uniformly integrable* if the set $\{s(x^*, F) : x^* \in B_{X^*}\}$ is uniformly integrable. A function $f : \Omega \to X$ is called a *measurable selector* of $F : \Omega \to CL(X)$ if f is measurable and $f(\omega) \in F(\omega)$ for all $\omega \in \Omega$. A measurable selector f of F is called a *Bochner integrable selector* of F if f is Bochner integrable. We denote by S_F the set of all Bochner integrable selectors of F.

Definition 2.1 ([8]). A set-valued mapping $F : \Omega \to CL(X)$ is said to be *Aumann-Bochner integrable* if $S_F \neq \phi$. In this case, the *Aumann-Bochner integral* of F on $A \in \Sigma$ is defined by

$$\int_A F d\mu = \left\{ \int_A f d\mu : f \in S_F \right\}.$$

Theorem 2.2 ([3]). *Let $\ell_\infty(B_{X^*})$ be the Banach space of bounded real-valued functions defined on B_{X^*} endowed with the supremum norm $\|\cdot\|_\infty$. Then the map $j : CWK(X) \longrightarrow \ell_\infty(B_{X^*})$ given by $j(A) := s(\cdot, A)$ satisfies the following properties:*

(1) $j(A+B) = j(A) + j(B)$ *for every* $A, B \in CWK(X)$;
(2) $j(\lambda A) = \lambda j(A)$ *for every* $\lambda \geq 0$ *and* $A \in CWK(X)$;
(3) $H(A, B) = \|j(A) - j(B)\|_\infty$ *for every* $A, B \in CWK(X)$;
(4) $j(CWK(X))$ *is closed in* $\ell_\infty(B_{X^*})$.

Definition 2.3 ([1,7]). A set-valued mapping $F : \Omega \to CWK(X)$ is said to be *Debreu integrable* if the composition $j \circ F : \Omega \to \ell_\infty(B_{X^*})$ is Bochner integrable. In this case, the *Debreu integral* of F on Ω is the unique element $(D)\int_\Omega F d\mu \in CWK(X)$ such that $j((D)\int_\Omega F d\mu) = \int_\Omega j \circ F d\mu$, where $\int$ denotes the Bochner integral.

If $F : \Omega \to CWK(X)$ is Debreu integrable, then for each $A \in \Sigma$ there exists a unique element $(D)\int_A F d\mu \in CWK(X)$, that is called the *Debreu integral* of F on A, such that $j((D)\int_A F d\mu) = \int_A j \circ F d\mu$.

In fact, Debreu integrability does not depend on the particular embedding j considered and in order to define the Debreu integral we can use any map $i : CWK(X) \to Y$, in a Banach space Y, as long as properties (1) – (4) in Theorem 2.2 are fulfilled [7]. In particular, if X is separable and $F : \Omega \to CWK(X)$ is Debreu integrable, then $F : \Omega \to CWK(X)$ is Aumann-Bochner integrable and $(D)\int_A F d\mu = \int_A F d\mu$ for each $A \in \Sigma$ [1,7].

3. Results

A mapping $\tilde{F} : \Omega \to \mathcal{F}(X)$ is called a *fuzzy mapping* in a Banach space X. In this case, $\tilde{F}^r : \Omega \to CWK(X)$ defined by $\tilde{F}^r(\omega) = [\tilde{F}(\omega)]^r$ is a set-valued mapping for each $r \in (0,1]$. A fuzzy mapping $\tilde{F} : \Omega \to \mathcal{F}(X)$ is said to be *measurable* (resp., *scalarly measurable*) if $\tilde{F}^r : \Omega \to CWK(X)$ is measurable (resp., scalarly measurable) for each $r \in (0,1]$.

Definition 3.1 ([10]). A fuzzy mapping $\tilde{F} : \Omega \to \mathcal{F}(X)$ is said to be *integrable* if for each $A \in \Sigma$ there exists $u_A \in \mathcal{F}(X)$ such that $[u_A]^r = \int_A \tilde{F}^r d\mu$ for each $r \in (0,1]$. In this case, $u_A = \int_A \tilde{F} d\mu$ is called the *integral* of $\tilde{F}$ on A.

Definition 3.2. A fuzzy mapping $\tilde{F} : \Omega \to \mathcal{F}(X)$ is said to be *Debreu integrable* if for each $A \in \Sigma$ there exists $u_A \in \mathcal{F}(X)$ such that $[u_A]^r = (D)\int_A \tilde{F}^r d\mu$ for each

$r \in (0,1]$. In this case, $u_A = (D)\int_A \tilde{F} d\mu$ is called the *Debreu integral* of $\tilde{F}$ on A.

Theorem 3.3. *Let $\tilde{F} : \Omega \to \mathcal{F}(X)$ and $\tilde{G} : \Omega \to \mathcal{F}(X)$ be Debreu integrable and $\lambda \geq 0$. Then*

(1) *$\tilde{F} + \tilde{G}$ is Debreu integrable and for each $A \in \Sigma$*

$$(D)\int_A (\tilde{F} + \tilde{G}) d\mu = (D)\int_A \tilde{F} d\mu + (D)\int_A \tilde{G} d\mu,$$

(2) *$\lambda\tilde{F}$ is Debreu integrable and for each $A \in \Sigma$*

$$(D)\int_A \lambda\tilde{F} d\mu = \lambda(D)\int_A \tilde{F} d\mu.$$

Proof. The proof is straightforward. □

Lemma 3.4. *Let $F : \Omega \to CWK(X)$ and $G : \Omega \to CWK(X)$ be Debreu integrable set-valued mappings. Then $F(\omega) = G(\omega)$ μ-a.e. if and only if $(D)\int_A F d\mu = (D)\int_A G d\mu$ for each $A \in \Sigma$.*

Proof. Since $F : \Omega \to CWK(X)$ and $G : \Omega \to CWK(X)$ are Debreu integrable, $j \circ F$ and $j \circ G$ are Bochner integrable and there exist $(D)\int_A F d\mu, (D)\int_A G d\mu \in CWK(X)$ such that $j((D)\int_A F d\mu) = \int_A j \circ F d\mu$, $j((D)\int_A G d\mu) = \int_A j \circ G d\mu$ for each $A \in \Sigma$.

If $F(\omega) = G(\omega)$ μ-a.e., then $(j \circ F)(\omega) = (j \circ G)(\omega)$ μ-a.e. Hence

$$j((D)\int_A F d\mu) = \int_A j \circ F d\mu = \int_A j \circ G d\mu = j((D)\int_A G d\mu)$$

for each $A \in \Sigma$. Thus

$$\begin{aligned} s(x^*, (D)\int_A F d\mu) &= < x^*, j((D)\int_A F d\mu) > \\ &= < x^*, j((D)\int_A G d\mu) > \\ &= s(x^*, (D)\int_A G d\mu) \end{aligned}$$

for each $x^* \in B_{X^*}$ and $A \in \Sigma$. Since $(D)\int_A F d\mu, (D)\int_A G d\mu \in CWK(X)$ for each $A \in \Sigma$, by the separation theorem $(D)\int_A F d\mu = (D)\int_A G d\mu$ for each $A \in \Sigma$.

Conversely, if $(D)\int_A F d\mu = (D)\int_A G d\mu$ for each $A \in \Sigma$, then

$$\int_A j \circ F d\mu = j((D)\int_A F d\mu) = j((D)\int_A G d\mu) = \int_A j \circ G d\mu$$

for each $A \in \Sigma$. By [5, Corollary II.5], $(j \circ F)(\omega) = (j \circ G)(\omega)$ μ-a.e. and so $H(F(\omega), G(\omega)) = \|(j \circ F)(\omega) - (j \circ G)(\omega)\|_\infty = 0$ μ-a.e. Hence $F(\omega) = G(\omega)$ μ-a.e. □

Lemma 3.5. *Let X be a separable Banach space and let $F : \Omega \to CWK(X)$ and $G : \Omega \to CWK(X)$ be Debreu integrable set-valued mappings. If $F(\omega) \subseteq G(\omega)$ on Ω, then $(D)\int_A F d\mu \subseteq (D)\int_A G d\mu$ for each $A \in \Sigma$.*

Proof. Since $F : \Omega \to CWK(X)$ and $G : \Omega \to CWK(X)$ are Debreu integrable, $F : \Omega \to CWK(X)$ and $G : \Omega \to CWK(X)$ are Aumann-Bochner integrable and $(D)\int_A F d\mu = \int_A F d\mu$, $(D)\int_A G d\mu = \int_A G d\mu$ for each $A \in \Sigma$. If $F(\omega) \subseteq G(\omega)$ on Ω, then $S_F \subseteq S_G$. Hence $\int_A F d\mu \subseteq \int_A G d\mu$ and so $(D)\int_A F d\mu \subseteq (D)\int_A G d\mu$ for each $A \in \Sigma$. □

Theorem 3.6. *Let $\tilde{F} : \Omega \to \mathcal{F}(X)$ and $\tilde{G} : \Omega \to \mathcal{F}(X)$ be Debreu integrable fuzzy mappings. If $\tilde{F}(\omega) = \tilde{G}(\omega)$ μ-a.e., then $(D)\int_A \tilde{F} d\mu = (D)\int_A \tilde{G} d\mu$ for each $A \in \Sigma$.*

Proof. Since $\tilde{F} : \Omega \to \mathcal{F}(X)$ and $\tilde{G} : \Omega \to \mathcal{F}(X)$ are Debreu integrable, for each $A \in \Sigma$ there exist $u_A, v_A \in \mathcal{F}(X)$ such that $[u_A]^r = (D)\int_A \tilde{F}^r d\mu$, $[v_A]^r = (D)\int_A \tilde{G}^r d\mu$ for each $r \in (0,1]$. If $\tilde{F}(\omega) = \tilde{G}(\omega)$ μ-a.e., then $\tilde{F}^r(\omega) = \tilde{G}^r(\omega)$ μ-a.e. for each $r \in (0,1]$. By Lemma 3.4, $[u_A]^r = (D)\int_A \tilde{F}^r d\mu = (D)\int_A \tilde{G}^r d\mu = [v_A]^r$ for each $r \in (0,1]$ and $A \in \Sigma$ and so $(D)\int_A \tilde{F} d\mu = u_A = v_A = (D)\int_A \tilde{G} d\mu$ for each $A \in \Sigma$. □

Theorem 3.7. *Let X be a separable Banach space and let $\tilde{F} : \Omega \to \mathcal{F}(X)$ and $\tilde{G} : \Omega \to \mathcal{F}(X)$ be Debreu integrable fuzzy mappings. If $\tilde{F}(\omega) \subseteq \tilde{G}(\omega)$ on Ω, then $(D)\int_A \tilde{F} d\mu \subseteq (D)\int_A \tilde{G} d\mu$ for each $A \in \Sigma$.*

Proof. The proof is similar to Theorem 3.6. □

Theorem 3.8. *Let X be a separable Banach space. If a fuzzy mapping $\tilde{F} : \Omega \to \mathcal{F}(X)$ is Debreu integrable, then $\tilde{F} : \Omega \to \mathcal{F}(X)$ is integrable and $(D)\int_A \tilde{F} d\mu = \int_A \tilde{F} d\mu$ for each $A \in \Sigma$.*

Proof. If $\tilde{F} : \Omega \to \mathcal{F}(X)$ is Debreu integrable, then for each $A \in \Sigma$ there exists $u_A \in \mathcal{F}(X)$ such that $u_A = (D)\int_A \tilde{F} d\mu$. Thus $[u_A]^r = (D)\int_A \tilde{F}^r d\mu$ for each $r \in (0,1]$. Since $\tilde{F}^r : \Omega \to CWK(X)$ is Debreu integrable for each $r \in (0,1]$, $\tilde{F}^r : \Omega \to CWK(X)$ is Aumann-Bochner integrable for each $r \in (0,1]$ and $[u_A]^r = (D)\int_A \tilde{F}^r d\mu = \int_A \tilde{F}^r d\mu$ for each $r \in (0,1]$. Thus $u_A = \int_A \tilde{F} d\mu$. Therefore $\tilde{F} : \Omega \to \mathcal{F}(X)$ is integrable and $(D)\int_A \tilde{F} d\mu = \int_A \tilde{F} d\mu$ for each $A \in \Sigma$. □

Lemma 3.9. *If $F : \Omega \to CWK(X)$ and $G : \Omega \to CWK(X)$ are measurable, integrably bounded and Debreu integrable set-valued mappings, then $H(F,G)$ is integrable and*

$$H\left((D)\int_\Omega F d\mu, (D)\int_\Omega G d\mu\right) \le \int_\Omega H(F,G) d\mu.$$

Proof. Since $F : \Omega \to CWK(X)$ and $G : \Omega \to CWK(X)$ are measurable, there exist Castaing representations $\{f_n\}$ and $\{g_n\}$ for F and G. Since f_n and g_n are measurable for all $n \in \mathbb{N}$,

$$H(F(\omega), G(\omega)) = \max\left(\sup_{n\ge 1}\inf_{k\ge 1} \|f_n(\omega) - g_k(\omega)\|, \sup_{n\ge 1}\inf_{k\ge 1} \|g_n(\omega) - f_k(\omega)\|\right)$$

is measurable. Since $F : \Omega \to CWK(X)$ and $G : \Omega \to CWK(X)$ are integrably bounded, there exist integrable real-valued functions h_1 and h_2 such that for each $\omega \in \Omega$, $\|x\| \le h_1(\omega)$ for all $x \in F(\omega)$ and $\|x\| \le h_2(\omega)$ for all $x \in G(\omega)$. Hence

$$H(F(\omega), G(\omega)) \le H(F(\omega), \{0\}) + H(G(\omega), \{0\}) \le h_1(\omega) + h_2(\omega)$$

for each $\omega \in \Omega$. Therefore $H(F,G)$ is integrable. Since F and G are Debreu integrable, there exist $(D)\int_\Omega F d\mu, (D)\int_\Omega G d\mu \in CWK(X)$ such that $j((D)\int_\Omega F d\mu) = \int_\Omega j \circ F d\mu$ and $j((D)\int_\Omega G d\mu) = \int_\Omega j \circ G d\mu$. Hence

$$\begin{aligned} H\left((D)\int_\Omega F d\mu, (D)\int_\Omega G d\mu\right) &= \left\| j((D)\int_\Omega F d\mu) - j((D)\int_\Omega G d\mu)\right\|_\infty \\ &= \left\|\int_\Omega j \circ F d\mu - \int_\Omega j \circ G d\mu\right\|_\infty \\ &\le \int_\Omega \|j \circ F - j \circ G\|_\infty d\mu \end{aligned}$$

$$= \int_\Omega \sup_{\|x^*\|\le 1} | < x^*, (j\circ F)(\omega) - (j \circ G)(\omega) > | \, d\mu$$
$$= \int_\Omega \sup_{\|x^*\|\le 1} |s(x^*, F(\omega)) - s(x^*, G(\omega))| \, d\mu$$
$$= \int_\Omega H(F,G) d\mu$$

□

A fuzzy mapping $\tilde{F} : \Omega \to \mathcal{F}(X)$ is said to be *integrably bounded* if there exists an integrable real-valued function h such that for each $\omega \in \Omega$, $\|x\| \le h(\omega)$ for all $x \in \tilde{F}^0(\omega)$, where $\tilde{F}^0(\omega) = cl\left(\cup_{0<r\le 1}\tilde{F}^r(\omega)\right)$.

Theorem 3.10. *If $\tilde{F} : \Omega \to \mathcal{F}(X)$ and $\tilde{G} : \Omega \to \mathcal{F}(X)$ are measurable, integrably bounded and Debreu integrable fuzzy mappings, then $D(\tilde{F}, \tilde{G})$ is integrable and*

$$D\left((D)\int_\Omega \tilde{F}d\mu, (D)\int_\Omega \tilde{G}d\mu\right) \le \int_\Omega D(\tilde{F},\tilde{G})d\mu.$$

Proof. Since $\tilde{F} : \Omega \to \mathcal{F}(X)$ and $\tilde{G} : \Omega \to \mathcal{F}(X)$ are measurable, there exist Castaing representations $\{f_n^r\}$ and $\{g_n^r\}$ for $\tilde{F}^r$ and $\tilde{G}^r$ for each $r \in (0,1]$. Since f_n^r and g_n^r are measurable for all $n \in \mathbb{N}$,

$$H(\tilde{F}^r(\omega), \tilde{G}^r(\omega)) = \max\left(\sup_{n\ge 1}\inf_{k\ge 1}\|f_n^r(\omega) - g_k^r(\omega)\|, \sup_{n\ge 1}\inf_{k\ge 1}\|g_n^r(\omega) - f_k^r(\omega)\|\right)$$

is measurable for each $r \in (0,1]$. Hence $D(\tilde{F}(\omega), \tilde{G}(\omega)) = \sup_{k\ge 1} H(\tilde{F}^{r_k}(\omega), \tilde{G}^{r_k}(\omega))$ is measurable, where $\{r_k : k \in \mathbb{N}\}$ is dense in $(0,1]$. Since $\tilde{F}$ and $\tilde{G}$ are integrably bounded, there exist integrable real-valued functions h_1 and h_2 such that for each $\omega \in \Omega$, $\|x\| \le h_1(\omega)$ for all $x \in \tilde{F}^0(\omega)$ and $\|x\| \le h_2(\omega)$ for all $x \in \tilde{G}^0(\omega)$. Hence we have

$$D(\tilde{F}(\omega), \tilde{G}(\omega)) \le D(\tilde{F}(\omega), \tilde{0}) + D(\tilde{G}(\omega), \tilde{0}) \le h_1(\omega) + h_2(\omega)$$

for each $\omega \in \Omega$. Therefore $D(\tilde{F}, \tilde{G})$ is integrable and by Lemma 3.9

$$H\left((D)\int_\Omega \tilde{F}^r d\mu, (D)\int_\Omega \tilde{G}^r d\mu\right) \le \int_\Omega H(\tilde{F}^r, \tilde{G}^r) d\mu$$

for each $r \in (0,1]$. Hence we have

$$D\left((D)\int_\Omega \tilde{F}d\mu, (D)\int_\Omega \tilde{G}d\mu\right) = \sup_{r\in(0,1]} H\left(\left[(D)\int_\Omega \tilde{F}d\mu\right]^r, \left[(D)\int_\Omega \tilde{G}d\mu\right]^r\right)$$
$$= \sup_{r\in(0,1]} H\left((D)\int_\Omega \tilde{F}^r d\mu, (D)\int_\Omega \tilde{G}^r d\mu\right)$$

$$\begin{aligned}
&\le \sup_{r\in(0,1]} \int_\Omega H(\tilde{F}^r, \tilde{G}^r) d\mu \\
&\le \int_\Omega \sup_{r\in(0,1]} H(\tilde{F}^r, \tilde{G}^r) d\mu \\
&= \int_\Omega D(\tilde{F}, \tilde{G}) d\mu.
\end{aligned}$$

□

The following lemma is the Convergence Theorem for the set-valued Debreu integral.

Lemma 3.11. *Let $F_n : \Omega \to CWK(X)$ be a Debreu integrable set-valued mapping for each $n \in \mathbb{N}$ and let $F : \Omega \to CWK(X)$ be a set-valued mapping such that $\lim_{n\to\infty} H(F_n(\omega), F(\omega)) = 0$ on Ω. If there exists an integrable real-valued function h such that $\|F_n(\omega)\| \le h(\omega)$ on Ω for each $n \in \mathbb{N}$, then $F : \Omega \to CWK(X)$ is Debreu integrable and*

$$\lim_{n\to\infty} H\left((D)\int_\Omega F_n d\mu, (D)\int_\Omega F d\mu\right) = 0.$$

Proof. Since $F_n : \Omega \to CWK(X)$ is Debreu integrable for each $n \in \mathbb{N}$, $j \circ F_n$ is Bochner integrable and there exists $(D)\int_\Omega F_n d\mu \in CWK(X)$ such that $j((D)\int_\Omega F_n d\mu) = \int_\Omega j \circ F_n d\mu$ for each $n \in \mathbb{N}$. Since $\lim_{n\to\infty} \|(j \circ F_n)(\omega) - (j \circ F)(\omega)\|_\infty = \lim_{n\to\infty} H(F_n(\omega), F(\omega)) = 0$ on Ω, $\lim_{n\to\infty} (j \circ F_n)(\omega) = (j \circ F)(\omega)$ on Ω. For each $n \in \mathbb{N}$

$$\begin{aligned}
\|(j \circ F_n)(\omega)\|_\infty &= \sup_{\|x^*\|\le 1} | < x^*, (j \circ F_n)(\omega) > | \\
&= \sup_{\|x^*\|\le 1} |s(x^*, F_n(\omega))| \\
&= \sup_{\|x^*\|\le 1} |s(x^*, F_n(\omega)) - s(x^*, \{0\})| \\
&= H(F_n(\omega), \{0\}) \\
&= \|F_n(\omega)\| \\
&\le h(\omega)
\end{aligned}$$

on Ω. By the Dominated Convergence Theorem for the Bochner integral, $j \circ F$ is Bochner integrable and $\lim_{n\to\infty} \int_\Omega j \circ F_n d\mu = \int_\Omega j \circ F d\mu$. Hence $F : \Omega \to CWK(X)$

is Debreu integrable and

$$\lim_{n\to\infty} H\left((D)\int_\Omega F_n d\mu, (D)\int_\Omega F d\mu\right) = \lim_{n\to\infty}\left\|j((D)\int_\Omega F_n d\mu) - j((D)\int_\Omega F d\mu)\right\|_\infty$$
$$= \lim_{n\to\infty}\left\|\int_\Omega j\circ F_n d\mu - \int_\Omega j\circ F d\mu\right\|_\infty = 0.$$

□

The following theorem is the Convergence Theorem for the Debreu integral of fuzzy mappings.

Theorem 3.12. *Let X be a separable Banach space and let $\tilde{F}_n : \Omega \to \mathcal{F}(X)$ be a measurable and Debreu integrable fuzzy mapping for each $n \in \mathbb{N}$ and let $\tilde{F} : \Omega \to \mathcal{F}(X)$ be a measurable fuzzy mapping such that $\lim_{n\to\infty} D(\tilde{F}_n(\omega), \tilde{F}(\omega)) = 0$ on Ω. If there exists an integrable real-valued function h such that $\|\tilde{F}_n^0(\omega)\| \le h(\omega)$ on Ω for each $n \in \mathbb{N}$, then $\tilde{F} : \Omega \to \mathcal{F}(X)$ is Debreu integrable and*

$$\lim_{n\to\infty} D\left((D)\int_\Omega \tilde{F}_n d\mu, (D)\int_\Omega \tilde{F} d\mu\right) = 0.$$

Proof. Since $\lim_{n\to\infty} D(\tilde{F}_n(\omega), \tilde{F}(\omega)) = 0$ on Ω, for each $\epsilon > 0$ and $\omega \in \Omega$ there exists $N \in \mathbb{N}$ such that $n \ge N \Rightarrow D(\tilde{F}_n(\omega), \tilde{F}(\omega)) < \epsilon$. For some $n \in \mathbb{N}$ with $n \ge N$,

$$\|\tilde{F}^0(\omega)\| = D(\tilde{F}(\omega), \tilde{0}) \le D(\tilde{F}(\omega), \tilde{F}_n(\omega)) + D(\tilde{F}_n(\omega), \tilde{0})$$
$$< \|\tilde{F}_n{}^0(\omega)\| + \epsilon \le h(\omega) + \epsilon$$

for each $\omega \in \Omega$. Since $\epsilon > 0$ is arbitrary, $\|\tilde{F}^0(\omega)\| \le h(\omega)$ on Ω. Thus $\tilde{F} : \Omega \to \mathcal{F}(X)$ is integrably bounded. Since $\tilde{F}_n : \Omega \to \mathcal{F}(X)$ is Debreu integrable for each $n \in \mathbb{N}$, there exists $u_n \in \mathcal{F}(X)$ such that $[u_n]^r = (D)\int_\Omega \tilde{F}_n{}^r d\mu$ for each $r \in (0,1]$ and $n \in \mathbb{N}$. Since $\lim_{n\to\infty} D(\tilde{F}_n(\omega), \tilde{F}(\omega)) = 0$ on Ω, $\lim_{n\to\infty} H(\tilde{F}_n^r(\omega), \tilde{F}^r(\omega)) = 0$ on Ω for each $r \in (0,1]$. Since $\|\tilde{F}_n{}^0(\omega)\| \le h(\omega)$ on Ω for each $n \in \mathbb{N}$, $\|\tilde{F}_n{}^r(\omega)\| \le h(\omega)$ on Ω for each $r \in (0,1]$ and $n \in \mathbb{N}$. By Lemma 3.11, $\tilde{F}^r : \Omega \to CWK(X)$ is Debreu integrable for each $r \in (0,1]$. Let $A \in \Sigma$. Then there exists $M_r \in CWK(X)$ such that $M_r = (D)\int_A \tilde{F}^r d\mu$ for each $r \in (0,1]$. For $r_1, r_2 \in (0,1]$ with $r_1 < r_2$, $\tilde{F}^{r_1}(\omega) \supseteq \tilde{F}^{r_2}(\omega)$ for each $\omega \in \Omega$. By Lemma 3.5 $M_{r_1} = (D)\int_A \tilde{F}^{r_1} d\mu \supseteq (D)\int_A \tilde{F}^{r_2} d\mu = M_{r_2}$. Let $r \in (0,1]$ and $\{r_n\}$ be a sequence in $(0,1]$ such that $r_1 \le r_2 \le r_3 \le \cdots$ and $\lim_{n\to\infty} r_n = r$. Then $\tilde{F}^r(\omega) = \cap_{n=1}^\infty \tilde{F}^{r_n}(\omega)$ for each $\omega \in \Omega$. By [10, Lemma 4.2],

$\lim_{n\to\infty} s(x^*, \tilde{F}^{r_n}(\omega)) = s(x^*, \tilde{F}^r(\omega))$ for each $\omega \in \Omega$ and $x^* \in X^*$. For each $x^* \in B_{X^*}$ and $\omega \in \Omega$,

$$\begin{aligned}\lim_{n\to\infty} < x^*, (j \circ \tilde{F}^{r_n})(\omega) > &= \lim_{n\to\infty} s(x^*, \tilde{F}^{r_n}(\omega)) \\ &= s(x^*, \tilde{F}^r(\omega)) \\ &=< x^*, (j \circ \tilde{F}^r)(\omega) > .\end{aligned}$$

Thus $\lim_{n\to\infty} (j \circ \tilde{F}^{r_n})(\omega) = (j \circ \tilde{F}^r)(\omega)$ on Ω. For each $n \in \mathbb{N}$ and $\omega \in \Omega$,

$$\begin{aligned}\|(j \circ \tilde{F}^{r_n})(\omega)\|_\infty &= \sup_{\|x^*\|\le 1} |s(x^*, \tilde{F}^{r_n}(\omega))| \\ &= \sup_{\|x^*\|\le 1} |s(x^*, \tilde{F}^{r_n}(\omega)) - s(x^*, \{0\})| \\ &= H(\tilde{F}^{r_n}(\omega), \{0\}) \\ &= \|\tilde{F}^{r_n}(\omega)\| \\ &\le \|\tilde{F}^0(\omega)\| \\ &\le h(\omega).\end{aligned}$$

By the Dominated Convergence Theorem for the Bochner integral,

$$\lim_{n\to\infty} \int_A j \circ \tilde{F}^{r_n} d\mu = \int_A j \circ \tilde{F}^r d\mu.$$

For each $x^* \in B_{X^*}$,

$$\begin{aligned}|s(x^*, M_{r_n}) - s(x^*, M_r)| &= \left| s(x^*, (D)\int_A \tilde{F}^{r_n} d\mu) - s(x^*, (D)\int_A \tilde{F}^r d\mu) \right| \\ &= \left| < x^*, j((D)\int_A \tilde{F}^{r_n} d\mu) > - < x^*, j((D)\int_A \tilde{F}^r d\mu) > \right| \\ &= \left| < x^*, \int_A j \circ \tilde{F}^{r_n} d\mu > - < x^*, \int_A j \circ \tilde{F}^r d\mu > \right| \\ &\le \left\| \int_A j \circ \tilde{F}^{r_n} d\mu - \int_A j \circ \tilde{F}^r d\mu \right\|_\infty \to 0 \text{ as } n \to \infty.\end{aligned}$$

Thus for each $x^* \in B_{X^*}$, $\lim_{n\to\infty} s(x^*, M_{r_n}) = s(x^*, M_r)$. And so for each $x^* \in X^*$, $\lim_{n\to\infty} s(x^*, M_{r_n}) = s(x^*, M_r)$. By [10, Lemma 4.2], $M_r = \cap_{n=1}^\infty M_{r_n}$. Let $M_0 = X$. By [10, Lemma 4.1], there exists $u_A \in \mathcal{F}(X)$ such that $[u_A]^r = M_r = (D)\int \tilde{F}^r d\mu$ for each $r \in (0, 1]$. Hence $\tilde{F} : \Omega \to \mathcal{F}(X)$ is Debreu integrable. By Theorem 3.10 and the Dominated Convergence Theorem for the Bochner integral,

$$D\left((D)\int_\Omega \tilde{F}_n d\mu, (D)\int_\Omega \tilde{F} d\mu\right) \le \int_\Omega D(\tilde{F}_n, \tilde{F}) d\mu \to 0 \text{ as } n \to \infty.$$

Thus $\lim_{n\to\infty} D\left((D)\int_\Omega \tilde{F}_n d\mu, (D)\int_\Omega \tilde{F} d\mu\right) = 0.$ □

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