J. Korean Soc. Math. Educ. Ser. B: Pure Appl. Math. ISSN 1226-0657
Volume 16, Number 3 (August 2009), Pages 271–276

SCREEN CONFORMAL LIGHTLIKE HYPERSURFACES OF A SEMI-RIEMANNIAN SPACE FORM

DAE HO JIN

ABSTRACT. We study the geometry of screen conformal lightlike hypersurfaces M of a semi-Riemannian manifold $\bar{M}$. The main result is a characterization theorem for screen conformal lightlike hypersurfaces of a semi-Riemannian space form.

1. INTRODUCTION

It is well known that the normal bundle $TM^{\perp}$ of the lightlike hypersurfaces M is a subbundle of TM, of rank 1. A complementary vector bundle $S(TM)$ of $TM^{\perp}$ in TM is non-degenerate distribution on M, called a *screen distribution* on M, and

$$TM = TM^{\perp} \oplus_{orth} S(TM), \tag{1.1}$$

where $\oplus_{orth}$ denotes the orthogonal direct sum. Although, in general, the screen distribution $S(TM)$ is not unique, but it has been proved in [2] that screen conformal hypersurfaces (our working space in this paper) do admit canonical screens (also see [3] on existence of canonical screens for lightlike hypersurfaces). We denote such a lightlike hypersurface by $(M, g, S(TM))$. Denote by $F(M)$ the algebra of smooth functions on M and by $\Gamma(E)$ the $F(M)$ module of smooth sections of a vector bundle E over M. We know [4] that, for any null section ξ of $TM^{\perp}$ on a coordinate neighborhood $\mathcal{U} \subset M$, there exists a unique null section N of a unique vector bundle $tr(TM)$ in $S(TM)^{\perp}$ satisfying

$$\bar{g}(\xi, N) = 1, \qquad \bar{g}(N, N) = \bar{g}(N, X) = 0, \ {}^{\forall}X \in \Gamma(S(TM)|_{\mathcal{U}}). \tag{1.2}$$

Then the tangent space $T\bar{M}$ of $\bar{M}$ is decomposed as follows:

$$T\bar{M} = TM \oplus tr(TM) = \{TM^{\perp} \oplus tr(TM)\} \oplus_{orth} S(TM). \tag{1.3}$$

Received by the editors September 5, 2008. Revised April 9, 2009. Accepted July 9, 2009.
2000 *Mathematics Subject Classification.* 53B25, 53C40, 53C50.
Key words and phrases. screen conformal, semi-Riemannian space forms.

We call $tr(TM)$ and N the *transversal vector bundle* and the *null transversal vector field* of M with respect to $S(TM)$ respectively.

The purpose of this paper is to study the geometry of screen conformal lightlike hypersurfaces of a semi-Riemannian space form. We prove a characterization theorem for screen conformal lightlike hypersurfaces M of a semi-Riemannian space form $(\bar{M}(c), \bar{g})$ with a constant curvature c: If $\dim M > 3$, then $c = 0$ (Theorem 2.1). Using this theorem, we prove an annexed theorem for screen conformal lightlike hypersurfaces M of a semi-Riemannian space form $(\bar{M}(c), \bar{g})$: If $c \neq 0$, then M is totally umbilical and locally a product manifold $M = L \times M^*$, where L is a lightlike curve and M^* is a totally umbilical semi-Riemannian 2-surface or a non-lightlike curve (Theorem 2.3). Recall the following structure equations:

Let $\bar{\nabla}$ be the Levi-Civita connection of $\bar{M}$ and P the projection morphism of $\Gamma(TM)$ on $\Gamma(S(TM))$ with respect to the decomposition (1.1). Then, for any vector fields $X, Y \in \Gamma(TM)$, the local Gauss and Weingartan formulas are given by

$$\bar{\nabla}_X Y = \nabla_X Y + B(X, Y)N, \tag{1.4}$$

$$\bar{\nabla}_X N = -A_N X + \tau(X)N, \tag{1.5}$$

$$\nabla_X PY = \nabla^*_X PY + C(X, PY)\xi, \tag{1.6}$$

$$\nabla_X \xi = -A^*_\xi X - \tau(X)\xi, \tag{1.7}$$

where ∇ and ∇^* are the linear connections on TM and $S(TM)$ respectively, B and C are the local second fundamental forms on TM and $S(TM)$ respectively, A_N and A^*_ξ are the shape operators on TM and $S(TM)$ respectively and τ is a 1-form on TM. Since $\bar{\nabla}$ is torsion-free, ∇ is also torsion-free and B is symmetric. From the fact that $B(X, Y) = \bar{g}(\bar{\nabla}_X Y, \xi)$, we know that B is independent of the choice of a screen distribution and satisfies

$$B(X, \xi) = 0, \quad {}^\forall X \in \Gamma(TM). \tag{1.8}$$

The induced connection ∇ of M is not metric and satisfies

$$(\nabla_X g)(Y, Z) = B(X, Y)\,\eta(Z) + B(X, Z)\,\eta(Y), \tag{1.9}$$

for any $X, Y, Z \in \Gamma(TM)$, where η is a 1-form such that

$$\eta(X) = \bar{g}(X, N), \quad {}^\forall X \in \Gamma(TM). \tag{1.10}$$

But the connection ∇^* on $S(TM)$ is metric. The above two local second fundamental forms of M and on $S(TM)$ are related to their shape operators by

$$B(X, Y) = g(A^*_\xi X, Y), \qquad \bar{g}(A^*_\xi X, N) = 0, \tag{1.11}$$

$$C(X, PY) = g(A_N X, PY), \quad \bar{g}(A_N X, N) = 0. \tag{1.12}$$

From (1.11), the operator A^*_ξ is $S(TM)$-valued and self-adjoint on TM such that

$$A^*_\xi \xi = 0. \tag{1.13}$$

We denote by $\bar{R}$, R and R^* the curvature tensors of the Levi-Civita connection $\bar{\nabla}$ of $\bar{M}$, the induced connection ∇ of M and the connection ∇^* on $S(TM)$, respectively. Using the Gauss-Weingarten equations for M and $S(TM)$, we obtain the Gauss-Codazzi equations for M and $S(TM)$ such that, for any $X, Y, Z, W \in \Gamma(TM)$,

$$\begin{aligned}
\bar{g}(\bar{R}(X, Y)Z, PW) &= g(R(X, Y)Z, PW) \\
&\quad + B(X, Z)C(Y, PW) - B(Y, Z)C(X, PW), \qquad (1.14)\\
\bar{g}(\bar{R}(X, Y)Z, \xi) &= g(R(X, Y)Z, \xi) \\
&= (\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z) \\
&\quad + B(Y, Z)\tau(X) - B(X, Z)\tau(Y), \qquad (1.15)\\
\bar{g}(\bar{R}(X, Y)Z, N) &= g(R(X, Y)Z, N), \qquad (1.16)\\
g(R(X, Y)PZ, PW) &= g(R^*(X, Y)PZ, PW) \\
&\quad + C(X,PZ)B(Y,PW) - C(Y,PZ)B(X,PW), \qquad (1.17)\\
g(R(X, Y)PZ, N) &= (\nabla_X C)(Y, PZ) - (\nabla_Y C)(X, PZ) \\
&\quad + C(X, PZ)\tau(Y) - C(Y, PZ)\tau(X). \qquad (1.18)
\end{aligned}$$

2. Screen Conformal Hypersurfaces

Definition. A lightlike hypersurface $(M, g, S(TM))$ of a semi-Riemannian manifold $(\bar{M}, \bar{g})$ is *screen conformal* [2] if there exist a non-vanishing smooth function φ on a neighborhood $\mathcal{U}$ in M such that $A_N = \varphi A^*_\xi$, or equivalently,

$$C(X, PY) = \varphi B(X, Y), \quad \forall X, Y \in \Gamma(TM). \tag{2.1}$$

Note 1. For a screen conformal hypersurface M, since the second fundamental form C is symmetric, $S(TM)$ is integrable distribution. Thus M is locally a product manifold $L \times M^*$ where L is a lightlike curve and M^* is a leaf of $S(TM)$ [4].

Let M be a screen conformal lightlike hypersurface of a semi-Riemannian space form $(\bar{M}(c),\bar{g})$. Then, by (1.15), we have

$$(2.2)\qquad (\nabla_X B)(Y,Z)-(\nabla_Y B)(X,Z)=B(X,Z)\tau(Y)-B(Y,Z)\tau(X),$$

for all X, Y, $Z\in\Gamma(TM)$. Using this, (1.16), (1.18) and (2.1), we obtain

$$\begin{aligned}&\{X[\varphi]-2\varphi\tau(X)\}B(Y,Z)-\{Y[\varphi]-2\varphi\tau(Y)\}B(X,Z)\\ (2.3)\qquad &=c\{g(Y,Z)\eta(X)-g(X,Z)\eta(Y)\}.\end{aligned}$$

Replacing Y by ξ in (2.3), we obtain

$$(2.4)\qquad \{\xi[\varphi]-2\varphi\tau(\xi)\}B(X,Z)=cg(X,Z).$$

Theorem 2.1. *Let $(M,g,S(TM))$ be a screen conformal lightlike hypersurface of a semi-Riemannian space form $(\bar{M}(c),\bar{g})$. If dim $M>3$, then $c=0$.*

Proof. Assume that $c\neq 0$. Then $\xi[\varphi]-2\varphi\tau(\xi)\neq 0$ and $B\neq 0$, that is, M is not a totally geodesic. From (2.1) and (2.4), we have

$$(2.5)\qquad B(X,Y)=\rho\, g(X,Y),\ C(X,PY)=\varphi\rho\, g(X,PY),\ {}^{\forall}X,\ Y\in\Gamma(TM),$$

where $\rho=c(\xi[\varphi]-2\varphi\tau(\xi))^{-1}\neq 0$. If $\varphi\rho=0$, then $C=0$ by (2.5). From (2.1), we have $B=0$. Therefore, we get $\varphi\rho\neq 0$. Thus M and $S(TM)$ are totally umbilical which are not totally geodesic. Since M is screen conformal, by Note 1, M is locally a product manifold $L\times M^*$ where L is a lightlike curve and M^* is a leaf of $S(TM)$. Since $\bar{M}$ is a space of constant curvature, from (1.14), (1.17) and (2.5), we have

$$(2.6)\quad R^*(X,Y)Z=(c+2\varphi\rho^2)\{g(Y,Z)X-g(X,Z)Y\},\ {}^{\forall}X,\ Y,\ Z\in\Gamma(S(TM)).$$

Let Ric^* be the induced symmetric Ricci tensor of M^*. From (2.6), we have

$$(2.7)\qquad Ric^*(X,Y)=(c+2\varphi\rho^2)(m-1)\,g(X,Y),\ {}^{\forall}X,\ Y\in\Gamma(S(TM)).$$

Thus M^* is Einstein. Since dim $M^*>2$, the function $(c+2\varphi\rho^2)$ is a constant and M^* is a space of constant curvature $(c+2\varphi\rho^2)$. Differentiating the first equation of (2.5) and using (1.9), (2.2) and the first equation of (2.5), we have

$$(2.8)\qquad \{X[\rho]+\rho\tau(X)-\rho^2\eta(X)\}g(Y,Z)=\{Y[\rho]+\rho\tau(Y)-\rho^2\eta(Y)\}g(X,Z).$$

Replacing Y by ξ in this equation, we have $\xi[\rho]=\rho^2-\rho\tau(\xi)$. Since $\varphi\rho^2$ is a constant, we have $0=\xi[\varphi\rho^2]=\rho(c+2\varphi\rho^2)$. Since $(c+2\varphi\rho^2)$ is a constant and $\rho\neq 0$, we have $c+2\varphi\rho^2=0$. Thus M^* is a semi-Euclidean space and the second fundamental form C of M^* satisfies $C=0$. Thus, from (2.1), we have $B=0$. It is a contradiction to $B\neq 0$. Consequently, we have $c=0$. □

Corollary 1. *There exist no screen conformal lightlike hypersurfaces M of semi-Riemannian space form $(\bar{M}(c),\ \bar{g})$ with $c \neq 0$ and $\dim M > 3$.*

Corollary 2. *There exist no screen conformal totally geodesic lightlike hypersurfaces M of semi-Riemannian space form $(\bar{M}(c),\ \bar{g})$ with $c \neq 0$.*

Theorem 2.2. *Let $(M, g, S(TM))$ be a screen conformal lightlike hypersurface of $(\bar{M}(c), \bar{g})$ with dim $M > 3$ and M^* be a leaf of the screen distribution $S(TM)$. Then the curvature tensors R and R^* of M and M^* respectively, are related by*

$$R(X,Y)Z = \frac{1}{2}R^*(PX,PY)PZ, \quad \forall X,\ Y,\ Z \in \Gamma(TM). \tag{2.9}$$

Proof. From (1.16) with $c = 0$, we have $\bar{g}(R(X,Y)Z, N) = 0$. Thus we see that (2.9) is equivalent to

$$g(R(X,Y)Z, PW) = \frac{1}{2}g(R^*(PX,PY)PZ, PW), \ \forall X,\ Y,\ Z,\ W \in \Gamma(TM). \tag{2.10}$$

Due to (1.15) with $c = 0$, we have $g(R(X,Y)\xi, Z) = 0$. Thus we see that (2.10) is true for $Z = \xi$. Using (1.8), (1.14) and (1.17) with $c = 0$ and (2.1), we derive (2.10). □

Theorem 2.3. *Let $(M, g, S(TM))$ be a screen conformal lightlike hypersurface of a semi-Riemannian space form $(\bar{M}(c), \bar{g})$ with $c \neq 0$. Then M is totally umbilical and locally a product manifold $L \times M^*$, where L is a lightlike curve and M^* is a totally umbilical semi-Riemannian 2-surface or a non-lightlike curve.*

Proof. Since M is a screen conformal lightlike hypersurface of a semi-Riemannian space form $(\bar{M}(c), \bar{g})$, by Note 1 M is locally a product manifold $L \times M^*$ where L is a lightlike curve and M^* is a leaf of $S(TM)$. Since $c \neq 0$, by Theorem 2.1 we see that dim $M \leq 3$. As $c \neq 0$, by (2.4), we see that $\xi[\varphi] - 2\varphi\tau(\xi) \neq 0$ and $B \neq 0$. Thus, from (2.1) and (2.4), the second fundamental forms B and C of M and M^* respectively satisfy the following equations

$$B(X,Y) = \rho\, g(X,Y),\ C(X,PY) = \varphi\rho\, g(X,PY),\ \forall X,\ Y \in \Gamma(TM),$$

where $\rho = c(\xi[\varphi] - 2\varphi\tau(\xi))^{-1} \neq 0$. Thus M and M^* are totally umbilical which are not totally geodesic. Since dim $M \leq 3$ and the Riemannian curvature tensor R^* of M^* is given by $R^*(X,Y)Z = (c + 2\varphi\rho^2)\{g(Y,Z)X - g(X,Z)Y\}$ for all $X,\ Y,\ Z \in \Gamma(S(TM))$, the leaf M^* of $S(TM)$ is a proper totally umbilical semi-Riemannian 2-surface of sectional curvature $(c + 2\varphi\rho^2)$ or a non-lightlike curve. □

REFERENCES

1. Akivis, M. A. & Goldberg, V. V.: Lightlike hypersurfaces on manifolds endowed with a conformal structure of Lorentzian signature. *Acta Appli. Math.* **57** (1999), 255-285.
2. Atindogbe, C. & Duggal, K. L.: Conformal screen on lightlike hypersurfaces. *International J. of Pure and Applied Math.* **11** (2004), no. 4, 421-442.
3. Duggal, K. L.: A report on canonical null curves and screen distributions for lightlike geometry. *Acta Appl Math.* **95** (2007), 135-149.
4. Duggal, K. L. & Bejancu, A.: *Lightlike Submanifolds of Semi-Riemannian Manifolds and Applications.* Kluwer Acad. Publishers, Dordrecht, 1996.
5. Duggal, K. L. & Jin, D. H.: *Null curves and Hypersurfaces of Semi-Riemannian Manifolds.* World Scientific, 2007.
6. Duggal, K. L. & Jin, D. H.: A classification of Einstein lightlike hypersurfaces of a Lorentzian space form. to appear in *J. Geom. Phys.*
7. Jin, D. H.: Totally umbilical hypersurfaces of Lorentz manifolds. *Korean J. Math. Sciences* **8** (2001), no. 1, 41-55.
8. O'Neill, B.: *Semi-Riemannian Geometry with Applications to Relativity.* Academic Press, 1983.

DEPARTMENT OF MATHEMATICS, DONGGUK UNIVERSITY, GYEONGJU, GYEONGBUK 780-714, KOREA
Email address: jindh@dongguk.ac.kr