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ON SPECTRA OF 2-ISOMETRIC OPERATORS

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ABSTRACT. A Hilbert space operator T is a 2-isometry if $T^{*2}T^2 - 2T^*T + I = 0$. We shall study some properties of 2-isometries, in particular spectra of a non-unitary 2-isometry and give an example. Also we prove with alternate argument that the Weyl's theorem holds for 2-isometries.

1. INTRODUCTION

Let H be a nonzero complex Hilbert space. By a subspace M of H we mean a *closed linear manifold* of H, and by an operator T on H we mean a *bounded linear transformation* of H into itself. A subspace M is *invariant* for T if $T(M) \subseteq M$, *nontrivial* if $\{0\} \neq M \neq H$. Let $L(H)$ denote the algebra of all bounded linear operators on H. Let $\sigma(T)$, $\sigma_p(T)$, $\sigma_{ap}(T)$, $\pi_{00}(T)$, $\sigma_e(T)$ and $\omega(T)$, respectively, denote the *spectrum*, the *point spectrum*, the *approximate spectrum*, the *set of isolated points* of $\sigma(T)$ that are eigenvalues of finite multiplicity, the *essential spectrum* and the *Weyl spectrum* of an operator $T \in L(H)$. We write the symbol $\partial\sigma(T)$ for the *boundary* of $\sigma(T)$. It is obvious that for any operator T, $\sigma(T)$ and $\omega(T)$ are nonempty compact subsets of $\mathbb{C}$, and $\sigma_e(T) \subseteq \omega(T) \subseteq \sigma(T)$. If for an operator T, $\omega(T) = \sigma(T) \sim \pi_{00}(T)$, then we say the Weyl's theorem holds for T.

According to [1], an operator T is defined to be a *2-isometry* if $T^{*2}T^2 - 2T^*T + I = 0$. Equivalently, T is a 2-isometry if $2\|Tx\|^2 = \|T^2x\|^2 + \|x\|^2$ for every $x \in H$. Clearly every isometry is a 2 isometry and so the set of 2-isometries is a class of operators that properly includes isometry. According to [1, Proposition 1.23], an invertible 2-isometry turns out to be a unitary operator. It is obvious from the definition that every 2-isometry is left invertible. In particular if both T and T^* are 2-isometries then T is invertible and so must be unitary.

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In this paper we shall study some properties of 2-isometries, in particular spectra of a non-unitary 2-isometry and give an example. Also we prove with alternate argument that the Weyl's theorem holds for 2-isometries.

2. THE SPECTRUM OF 2-ISOMETRIC OPERATORS

For an operator T, it was shown that $\lambda \notin \sigma_{ap}(T)$ iff $\ker(T-\lambda) = \{0\}$ and ran $(T-\lambda)$ is closed iff ran $(T^* - \overline{\lambda}) = H$([2], [4]).

According to [1, Proposition 1.23], we obtain the following result.

Lemma 1. *Let T be a 2-isometry. Then the following statements are equivalent:*

(1) *T is normal.*
(2) *T is invertible*
(3) *T is unitary.*
(4) *T has its spectrum on the unit circle.*

Theorem 2. *Let T be a 2-isometry.*

(1) *If S is unitarily equivalent to T, then S is a 2-isometry.*
(2) *If $M \subseteq H$ is an invariant subspace for T, then $T|M$ is a 2-isometry.*
(3) *If T commutes with an isometry S, then TS is a 2-isometry.*

Proof. (1) Let $S = U^*TU$ where U is an unitary. Then

$$\begin{aligned} S^{*2}S^2 - 2S^*S + I &= U^*T^{*2}T^2U - 2U^*T^*TU + U^*U \\ &= U^*(T^{*2}T^2U - 2T^*T + I)U = 0 \end{aligned}$$

Hence S is a 2-isometry.

(2) If $u \in M$, then

$$2\|(T|M)u\|^2 = 2\|Tu\|^2 = \|T^2u\|^2 + \|u\|^2 = \|(T|M)^2u\|^2 + \|u\|^2.$$

Hence $T|M$ is a 2-isometry.

(3) Let $A = TS$. By hypothesis we have $S^*S = I$, $ST = TS$, $S^*T^* = T^*S^*$. Thus

$$\begin{aligned} A^{*2}A^2 - 2A^*A + I &= S^*T^*S^*T^*TSTS - 2S^*T^*TS + I \\ &= T^{*2}T^2 - 2T^*T + I = 0 \end{aligned}$$

Hence TS is a 2-isometry. □

An operator $T \in L(H)$ is called to be *paranormal* if $\|Tx\|^2 \leq \|T^2x\|\|x\|$ for every $x \in H$, and *normaloid* if $r(T) = \|T\|$, where $r(T)$ denotes the spectral radius of T.

Theorem 3. *Let T be a 2-isometry. Then*

(1) *If T is invertible, then T^{-1} is also a 2-isometry.*
(2) *If T^2 is an isometry, then T is a paranormal operator.*
(3) *αT is a 2-isometry if and only if $|\alpha| = 1$ or αT^2 is an isometry.*
(4) *If αT is a 2-isometry, then $|\alpha| \leq 1$.*

Proof. (1) Since every invertible 2-isometry T is a unitary, T^{-1} is a unitary and so T^{-1} is a 2-isometry.

(2) Take any x in H and note that T is a 2-isometry if and only if $2\|Tx\|^2 = (\|T^2x\| - \|x\|)^2 + 2\|T^2x\|\|x\|$. By hypothesis $\|Tx\|^2 = \|T^2x\|\|x\|$ for every $x \in H$, i.e., T is a paranormal.

(3) If T is a 2-isometry, then $2|\alpha|^2 T^*T = |\alpha|^2 T^{*2}T^2 + |\alpha|^2 I$ for every $\alpha \in \mathbb{C}$. So we have that for every $\alpha \in \mathbb{C}$,

$$|\alpha|^4 T^{*2}T^2 - 2|\alpha|^2 T^*T + I = (|\alpha|^2 - 1)(|\alpha|^2 T^{*2}T^2 - I),$$

which implies the result.

(4) By [6], T^2 is a 2-isometry and so $1 \leq \|T^2\|$. Thus if $|\alpha| \neq 1$, then αT^2 is an isometry i.e., $\|\alpha T^2\| = 1$ i.e., $|\alpha|\|T^2\| = 1$, and so this yields $|\alpha| < 1$. □

We denote $\sigma_l(T)$ ($\sigma_r(T)$ and $\sigma_{le}(T)$ ($\sigma_{re}(T)$) for the left(right) spectrum and the left(right) essential spectrum of an operator T respectively.

Lemma 4 ([1]). *If T is a 2-isometry, then the approximate point spectrum lies in the unit circle. Thus either $\sigma(T) \subseteq \partial\mathbb{D}$ or $\sigma(T) = \overline{\mathbb{D}}$ where $\mathbb{D}$ denotes the open unit disc. In particular, T is injective and* $\operatorname{ran} T$ *is closed.*

Theorem 5. *Let T be a non-unitary 2-isometry. Then*

(1) *$\sigma(T) = \overline{\mathbb{D}}$, $\sigma_{ap}(T) = \partial\mathbb{D}$, and $\pi_{00}(T) = \emptyset$.*
(2) *$\sigma_{le}(T) \cap \sigma_{re}(T) = \partial\mathbb{D}$ and $\sigma_{le}(T) = \partial\mathbb{D}$.*
(3) *$\sigma(T) = \omega(T)$.*
(4) *$T - \lambda$ is semi-Fredholm and ind $(T - \lambda) \leq 0$ if $|\lambda| < 1$.*
(5) *T is not a Weyl operator.*

Proof. (1) By [1] T is not invertible and so $\sigma(T) = \overline{\mathbb{D}}$. Since $\partial\mathbb{D} = \partial\sigma(T) \subseteq \sigma_{ap}(T)$ for any operator T and $\sigma_{ap}(T) \subseteq \partial\mathbb{D}$ by [1], $\sigma_{ap}(T) = \partial\mathbb{D}$. Since every point of $\overline{\mathbb{D}} = \sigma(T)$ is not an isolated point, $\pi_{00}(T) = \emptyset$.

(2) Since $\sigma_{le}(T) \subseteq \sigma_l(T) = \sigma_{ap}(T)$ for any operator T, $\sigma_{le}(T) \cap \sigma_{re}(T) \subseteq \sigma_{ap}(T) = \partial\mathbb{D}$ by (1). On the other hand, if $\lambda \in \partial\mathbb{D}$, then $\lambda \in \sigma_{ap}(T)$ by (1) and so ran $(T - \lambda)$

is not closed([6]). Thus $\lambda \in \sigma_{le}(T) \cap \sigma_{re}(T)$. Also $\sigma_{le}(T) = \partial\mathbb{D}$ follows from the above argument.

(3) Clearly $\omega(T) \subseteq \sigma(T)$ for every operator T. Let $\lambda \in \sigma(T)$. If $\lambda \in \partial\mathbb{D}$, then by (2), $\lambda \in \sigma_{le}(T) \cap \sigma_{re}(T)$ and so $T - \lambda$ is not Fredholm, i.e.,$\lambda \in \sigma_e(T)$. Thus $\lambda \in \omega(T)$. On the other hand, if $\lambda \in \mathbb{D}$ then $\lambda \notin \sigma_{ap}(T) = \partial\mathbb{D}$ by (1). By the equivalent condition of $\sigma_{ap}(T)$([6]), ran $(T-\lambda)$ is closed and $\dim\ker(T-\lambda) = 0$. Also ran $(T^* - \overline{\lambda}) = H$. Since $\overline{\lambda} \in \sigma(T^*)(= \sigma(T)^*)$ and ran $(T^* - \overline{\lambda}) = H$, we must have $\ker(T-\lambda)^* \neq \{0\}$ and so $\dim\ker(T-\lambda)^* \neq 0$. Thus $T - \lambda$ is Fredholm and ind $(T-\lambda) \neq 0$ and so $\lambda \in \omega(T)$. Therefore $\sigma(T) = \omega(T)$.

(4) & (5) : These follow from the proof of (3) and $0 \in \omega(T)$. □

Corollary 6. *If T is a 2-isometry, then T is isoloid, i.e., isolated points of $\sigma(T)$ are eigenvalues of T.*

Proof. By [1], either $\sigma(T) \subseteq \partial D$ or $\sigma(T) = \overline{D}$, and so $\sigma(T) \subseteq \partial D$ since every point of $\overline{D}$ is not an isolated point. Thus T is a unitary (so normal) and hence T is isoloid. □

We prove here with the alternate argument that the Weyl's theorem holds for 2-isometries.

Theorem 7. *Weyl's theorem holds for 2-isometries.*

Proof. Let T be a 2-isometry. If T is a unitary (so normal), the result is obvious by the fact that Weyl's theorem holds for normal operators. If T is a non-unitary, then $\pi_{00}(T) = \emptyset$ by Theorem 5(1). Thus $\sigma(T) - \pi_{00}(T) = \sigma(T) = \omega(T)$ by Theorem 5(3). □

Example Let $T \in L(H)$ be a unilateral weighted shift with weights

$$\alpha_n = \sqrt{1 + \frac{1}{n}} \quad (n = 1, 2, \cdots).$$

Then $|\alpha_n| \neq 1$ and

$$|\alpha_n|^2 |\alpha_{n+1}|^2 - 2|\alpha_n|^2 + 1 = 0$$

for each $n = 1, 2, \cdots$, and so T is a non-isometric 2-isometry. Clearly T is not unitary. By Theorem 5 and the direct computation, $\sigma(T) = \omega(T) = \overline{D}$, $\sigma_{ap}(T) = \partial\mathbb{D}$ and $\sigma_p(T) = \emptyset$. If $|\lambda| < 1$, then ran $(T - \lambda)$ is closed and $\dim[ran(T-\lambda)]^{\perp} = 1$. This implies that

$$\sigma_{le}(T) = \sigma_{re}(T) = \sigma_e(T) = \partial\mathbb{D},$$

and that $T-\lambda$ is Fredholm and ind $(T-\lambda) = -1$ if $|\lambda| < 1$. Furthermore, $\|T\| = \sqrt{2}$ and so T is not normaloid.

The above example shows that a 2-isometry T is not normaloid.

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