J. Korean Soc. Math. Educ. Ser. B: Pure Appl. Math. ISSN 1226-0657
Volume 16, Number 4 (November 2009), Pages 327–344

POSITIVE SOLUTION FOR SYSTEMS OF NONLINEAR SINGULAR BOUNDARY VALUE PROBLEMS ON TIME SCALES

CHUNMEI MIAO[a,c,*], DEHONG JI[b], JUNFANG ZHAO[c], WEIGAO GE[c] AND JIANI ZHANG[c]

ABSTRACT. In this paper, we deal with the following system of nonlinear singular boundary value problems(BVPs) on time scale $\mathbb{T}$

$$\begin{cases} x^{\Delta\Delta}(t) + f(t, y(t)) = 0, & t \in (a, b]_{\mathbb{T}}, \\ y^{\Delta\Delta}(t) + g(t, x(t)) = 0, & t \in (a, b]_{\mathbb{T}}, \\ \alpha_1 x(a) - \beta_1 x^{\Delta}(a) = \gamma_1 x(\sigma(b)) + \delta_1 x^{\Delta}(\sigma(b)) = 0, \\ \alpha_2 y(a) - \beta_2 y^{\Delta}(a) = \gamma_2 y(\sigma(b)) + \delta_2 y^{\Delta}(\sigma(b)) = 0, \end{cases}$$

where $\alpha_i,\ \beta_i,\ \gamma_i,\ \delta_i \geq 0$ and $\rho_i = \alpha_i\gamma_i(\sigma(b) - a) + \alpha_i\delta_i + \gamma_i\beta_i > 0 (i = 1, 2)$, $f(t, y)$ may be singular at $t = a$, $y = 0$, and $g(t, x)$ may be singular at $t = a$. The arguments are based upon a fixed-point theorem for mappings that are decreasing with respect to a cone. We also obtain the analogous existence results for the related nonlinear systems $x^{\nabla\nabla}(t) + f(t, y(t)) = 0$, $y^{\nabla\nabla}(t) + g(t, x(t)) = 0$, $x^{\Delta\nabla}(t) + f(t, y(t)) = 0$, $y^{\Delta\nabla}(t) + g(t, x(t)) = 0$, and $x^{\nabla\Delta}(t) + f(t, y(t)) = 0$, $y^{\nabla\Delta}(t) + g(t, x(t)) = 0$ satisfying similar boundary conditions.

1. INTRODUCTION

The singular boundary value problem (BVP) arises in a variety of differential applied mathematics and physics such as gas dynamics, nuclear physics, chemical reactions, studies of atomic structures, and atomic calculations [21]. It also arises in the study of positive radial solutions of a nonlinear elliptic equations. Therefore, it has been studied extensively in recent years, see, for instance, [3-6, 9, 12-14, 16, 22, 24-25] and the references therein. More recently, several authors begin to pay attention to boundary value problems for dynamic equations on time scale and many excellent results have been obtained, see [11, 17-18, 23, 26, 27].

Received by the editors December 11, 2008 and, in revised form, November 19, 2009.
2000 *Mathematics Subject Classification.* 34B10, 34B16, 34B18.
Key words and phrases. singular boundary value problem, nonlinear system, time scale, fixed point theorem.
Supported by NNSF of China (10671012) and SRFDP of China (20050007011).
*Corresponding author.

In [18], Hao, Xiao and Liang researched the following singular boundary value problems

$$\begin{cases} x^{\Delta\Delta}(t)+m(t)f(t,x(\sigma(t)))=0,\ t\in[a,b], \\ \alpha x(a)-\beta x^{\Delta}(a)=0,\ \gamma x(\sigma(b))+\delta x^{\Delta}(\sigma(b))=0 \end{cases} \tag{1.1}$$

and

$$\begin{cases} x^{\Delta\Delta}(t)+m(t)f(t,x(\sigma(t)),x^{\Delta}(\sigma(t)))=0,\ t\in[a,b], \\ \alpha x(a)-\beta x^{\Delta}(a)=0,\ \gamma x(\sigma(b))+\delta x^{\Delta}(\sigma(b))=0, \end{cases} \tag{1.2}$$

where f is continuous, $m(\cdot):(a,\sigma(b))\to[0,\infty)$ is rd-continuous and may be singular at $t=a$ and/or $t=\sigma(b)$, $\alpha,\ \beta,\ \gamma,\ \delta\geq 0$ such that $d:=\gamma\beta+\alpha\delta+\alpha\gamma(\sigma(b)-a)>0$ and $\delta\geq\gamma[\sigma^2(b)-\sigma(b)]$. Applying the Krasnosel'skii fixed point theorem, they proved the existence of positive solutions to the problems. But, the nonlinear term f being nonsingular in its dependent variable.

In [26], Su, Li and Sun researched the following singular boundary value problem

$$\begin{cases} (\varphi_p(u^{\Delta}(t)))^{\nabla}+q(t)f(t,u(t))=0,\ t\in(0,T)_{\mathbb{T}}, \\ u(0)=0,\quad u(T)-\sum\limits_{i=1}^{m-2}\psi_i(u(\xi_i))=0 \end{cases} \tag{1.3}$$

where $\varphi_p(u)=|u|^{p-2}u$, $p>1$, $\psi_i:\mathbb{R}\to\mathbb{R}$ is continuous and nondecreasing, $0<\xi_1<\xi_2<\cdots<\xi_{m-2}<\rho(T)$. The nonlinearity f is allowed to change sign and may be singular at $u=0$. In addition, ψ_i may be nonlinear. By using the Schauder fixed point theorem and upper and lower solutions method, existence criteria for positive solutions of the boundary value problem are presented.

Naturally, we hope there are the same excellent results on systems of nonlinear singular boundary value problems on time scale. To the best of our knowledge, systems of nonlinear singular boundary value problems on time scale are seldom investigated. Therefore, in this paper we consider the following system of nonlinear singular boundary value problem on time scale $\mathbb{T}$

$$\begin{cases} x^{\Delta\Delta}(t)+f(t,y(t))=0,\quad t\in(a,b]_{\mathbb{T}}, \\ y^{\Delta\Delta}(t)+g(t,x(t))=0,\quad t\in(a,b]_{\mathbb{T}}, \\ \alpha_1x(a)-\beta_1x^{\Delta}(a)=\gamma_1x(\sigma(b))+\delta_1x^{\Delta}(\sigma(b))=0, \\ \alpha_2y(a)-\beta_2y^{\Delta}(a)=\gamma_2y(\sigma(b))+\delta_2y^{\Delta}(\sigma(b))=0, \end{cases} \tag{1.4}$$

where $\alpha_i,\ \beta_i,\ \gamma_i,\ \delta_i\geq 0$ and $\rho_i=\alpha_i\gamma_i\sigma(b)+\alpha_i\delta_i+\gamma_i\beta_i>0 (i=1,2)$, $f(t,y)$ and $g(t,x)$ may be singular at $t=a$, and $f(t,y)$ may be singular at $y=0$.

2. A Fixed Point Theorem

We begin by giving definitions and some properties of cones in a Banach space. For references, see Krasnosel'skii [20] and Amann [1].

Let B be a real Banach space. A nonempty set $K \subset B$ is called a cone if the following conditions are satisfied:

(a) the set K is closed;

(b) if $u,\ v \in K$ then $\alpha u + \beta v \in K$ for all real $\alpha, \beta \geq 0$;

(c) $u,\ -u \in K$ imply $u = 0$.

For $x, y \in K$ a cone of B, recall that $x \leq y$ if $y - x \in K$. If $x, y \in B$ with $x \leq y$, let $< x, y >= \{z \in B | x \leq z \leq y\}$. A cone K is normal in B provided there exists $\delta > 0$ such that $||e_1 + e_2|| \geq \delta$, for all $e_1, e_2 \in K$ with $||e_1|| = ||e_2|| = 1$.

Remark 2.1. If K is a normal cone in B, then closed order intervals are norm bounded (see [20]).

Next we state the fixed point theorem due to Gatica, Oliker, and Waltman [12] which is instrumental in proving our existence results.

Theorem 2.1. *Let B be a Banach space, $K \subset B$ be a normal cone, and $D \subset K$ be such that if $x, y \in D$ with $x \leq y$, then $< x, y > \subset D$. Let $T : D \to K$ be a continuous, decreasing mapping which is compact on any closed order interval contained in D, and suppose there exists an $x_0 \in D$ such that $T^2 x_0$ is defined (where $T^2 x_0 = T(Tx_0)$) and Tx_0, $T^2 x_0$ are order comparable to x_0. Then T has a fixed point in D provided that either:*

(i) *$Tx_0 \leq x_0$ and $T^2 x_0 \leq x_0$ or $Tx_0 \geq x_0$ and $T^2 x_0 \geq x_0$; or*

(ii) *The complete sequence of iterates $\{T^n x_0\}_{n=0}^{\infty}$ is defined and there exists $y_0 \in D$ such that $Ty_0 \in D$ and $y_0 \leq T^n x_0$ for all n.*

3. The Delta-delta Problem

Consider the following Banach space $B = \mathbb{C}[a, \sigma^2(b)]_{\mathbb{T}}$, with the norm

$$\|u\| = \sup_{t \in [a, \sigma^2(b)]_{\mathbb{T}}} |u(t)|.$$

Define the normal cone, $K \subset B$, as

$$K := \{u \in B |\ u(t) \geq 0,\ t \in [a, \sigma^2(b)]_{\mathbb{T}}\}.$$

Moreover, define the function $h_1 : [a, \sigma^2(b)]_{\mathbb{T}} \to \mathbb{R}^+ (\mathbb{R}^+ = [0, \infty))$ by

$$h_1(t) = \begin{cases} t-a, & \text{if } t \in \left[a, \dfrac{a+\sigma^2(b)}{2}\right]_{\mathbb{T}}, \\ \sigma^2(b)-t, & \text{if } t \in \left[\sigma\left(\dfrac{a+\sigma^2(b)}{2}\right),\ \sigma^2(b)\right]_{\mathbb{T}}. \end{cases}$$

Finally, for $\theta > 0$, let

$$h_\theta = \theta \cdot h_1. \tag{3.1}$$

Let $\mathbb{R}_0^+ = (0,\infty)$, $\mathbb{R}^+ = [0,\infty)$. We make the following assumptions:

(H_1) $\alpha_i,\ \beta_i,\ \gamma_i,\ \delta_i \geq 0,\ \rho_i = \alpha_i\gamma_i(\sigma(b)-a)+\beta_i\gamma_i+\alpha_i\delta_i > 0,\ a, b \in \mathbb{T}$ with a right dense;

(H_2) $f \in \mathbb{C}((a,\sigma(b)]_{\mathbb{T}} \times \mathbb{R}_0^+, \mathbb{R}_0^+)$ and $f(t,\cdot)$ is decreasing for every $t \in (a,\sigma(b)]_{\mathbb{T}}$, $g \in \mathbb{C}((a,\sigma(b)]_{\mathbb{T}} \times \mathbb{R}^+, \mathbb{R}_0^+)$ and $g(t,\cdot)$ is increasing for every $t \in (a,\sigma(b)]_{\mathbb{T}}$;

(H_3) $\lim\limits_{y\to 0^+} f(t,y) = \infty,\ \lim\limits_{y\to\infty} f(t,y) = 0,\ \lim\limits_{x\to 0^+} g(t,x) = 0,\ \lim\limits_{x\to\infty} g(t,x) = \infty$ uniformly on compact subsets of $(a,\sigma(b)]_{\mathbb{T}}$;

(H_4) $0 < \int_a^{\sigma(b)} f(t,h_\theta(t))\Delta t < \infty$ for all $\theta > 0$ and $0 < \int_a^{\sigma(b)} g(t,x)\Delta t < \infty$ for all $x \in \mathbb{R}^+$.

Lemma 3.1. *Suppose that* (H_1) *holds and* $e \in B$, *then linear boundary value problems*

$$\begin{cases} u^{\Delta\Delta}(t)+e(t)=0, \quad t \in (a,b]_{\mathbb{T}}, \\ \alpha_i u(a)-\beta_i u^{\Delta}(a)=0,\ \gamma_i u(\sigma(b))+\delta_i u^{\Delta}(\sigma(b))=0,\ i=1,2 \end{cases} \tag{3.2$_i$}$$

has a unique solution

$$u_i(t) = \int_a^{\sigma(b)} G_i(t,s)e(s)\Delta s, \tag{3.3}$$

where $G_i(t,s) : [a,\sigma(b)]_{\mathbb{T}} \times [a,b]_{\mathbb{T}} \to [0,\infty)$ *is defined by*

$$G_i(t,s) = \frac{1}{\rho_i}\begin{cases} (\beta_i+\alpha_i s-\alpha a)(\gamma_i\sigma(b)+\delta_i-\gamma_i t), & a \leq s \leq \sigma(s) \leq t \leq \sigma(b), \\ (\beta_i+\alpha_i t-\alpha a)(\gamma_i\sigma(b)+\delta_i-\gamma_i s), & a \leq t \leq s \leq \sigma(s) \leq \sigma(b). \end{cases} \tag{3.4}$$

Remark 3.1. If (H_1) holds, then $G_i(t,s) > 0$ for $(t,s) \in (a,\sigma(b))_{\mathbb{T}} \times (a,b)_{\mathbb{T}}$.

Lemma 3.2. *Suppose that* (H_1) *holds and* $e \in B$, $e(t) > 0$ *on* $t \in (a,b]_{\mathbb{T}}$. *If* $u_i (i=1,2)$ *is a solution of* $BVP(3.2)_i$, *then*

(i) u_i *is concave;*

(ii) *there exists some* $\theta > 0$ *such that* $u_i(t) \geq h_\theta(t)$ *for all* $t \in [a,\sigma(b)]_{\mathbb{T}}$.

As we know, $(x,y) \in \mathbb{C}^2(a,\sigma^2(b))_{\mathbb{T}} \times \mathbb{C}^2(a,\sigma^2(b))_{\mathbb{T}}$ is a solution of BVP(1.4) if and only if $(x,y) \in \mathbb{C}[a,\sigma^2(b)]_{\mathbb{T}} \times \mathbb{C}[a,\sigma^2(b)]_{\mathbb{T}}$ is a solution of the following nonlinear integral equation system

$$\tag{3.5} \begin{cases} x(t) = \int_a^{\sigma(b)} G_1(t,s) f(s, y(s)) \Delta s, \\ y(t) = \int_a^{\sigma(b)} G_2(t,s) g(s, x(s)) \Delta s, \ t \in [a, \sigma(b)]_{\mathbb{T}}, \end{cases}$$

where $G_i(t,s)(i=1,2)$ are as defined in Lemma 3.1. Obviously, the above nonlinear system of integral equation can be regarded as the following nonlinear integral equation

$$\tag{3.6} x(t) = \int_a^{\sigma(b)} G_1(t,s) f(s, \int_a^{\sigma(b)} G_2(s,\tau) g(\tau, x(\tau)) \Delta\tau) \Delta s.$$

Define $D \subset K$ by

$$D := \{x \in K |\text{ there exists } \theta(x) > 0 \text{ such that } x(t) \geq h_\theta(t),\ t \in [a,\sigma(b)]_{\mathbb{T}}\},$$

and the operator $T : K \to K$ by

$$\tag{3.7} (Tx)(t) := \int_a^{\sigma(b)} G_1(t,s) f(s, \int_a^{\sigma(b)} G_2(s,\tau) g(\tau, x(\tau)) \Delta\tau) \Delta s.$$

In the following, we will prove T is well-defined. Note that for $x \in K$, there exists $\theta(x) > 0$ such that $y(t) = \int_a^b G_2(t,s) g(\tau, x(\tau)) \Delta\tau \geq h_\theta(t)$ for all $t \in [a,b]_{\mathbb{T}}$. Since $f(t,y)$ is decreasing with respect to y, we see $f(t,y) \leq f(t, h_\theta(t))$ for $t \in [a,b]_{\mathbb{T}}$. We obtain

$$\begin{aligned} 0 &< \int_a^{\sigma(b)} G_1(t,s) f(s, \int_a^{\sigma(b)} G_2(s,\tau) g(\tau, x(\tau)) \Delta\tau) \Delta s \\ &\leq \int_a^{\sigma(b)} G_1(t,s) f(s, h_\theta(s)) \Delta s < \infty. \end{aligned}$$

It can be easily verified that, T is decreasing with respect to $x \in K$ and $T : K \to D$.

Remark 3.2. It is easy to prove that the existence of solutions to system (3.5) is equivalent to the existence of solutions to integral equation (3.6). So, we have $(x,y) \in D \times D$ is a solution of system (1.4) if and only if $Tx = x$.

We now present the main result of this part.

Theorem 3.3. *Suppose f, g satisfy* (H_1)-(H_4), *then system* (1.4) *has at least one positive solution.*

Proof. For each $n \in \mathbb{N}$, let $\psi_n(t) = \int_a^{\sigma(b)} G_2(t,s) g\Big(s, \frac{1}{n}\Big)\Delta s$. Since g is increasing in its second component, then

$$\psi_{n+1}(t) \le \psi_n(t), \quad \psi_n(t) > 0, \ t \in [a, \sigma(b)]_{\mathbb{T}}. \tag{3.8}$$

By (H$_3$), $\lim_{n\to\infty} \psi_n(t) = 0$ uniformly on $[a,\sigma(b)]_{\mathbb{T}}$. Define

$$f_n(t,y) = f(t, \max\{y, \psi_n(t)\}).$$

Note that f_n has effectively "removed the singularity" in f at $y = 0$. Moreover, for $(t,y) \in (a,\sigma(b)]_{\mathbb{T}} \times (0,\infty)$, we see

$$f_n(t,y) \le f(t,y),$$

and

$$f_n(t,y) = f(t, \max\{y, \psi_n(t)\}) \le f(t, \psi_n(t)).$$

Next, define a sequence of operators $T_n : K \to K$ by

$$(T_n x)(t) = \int_a^{\sigma(b)} G_1(t,s) f_n\Big(s, \int_a^{\sigma(b)} G_2(s,\tau) g(\tau, x(\tau)\Big)\Delta\tau)\Delta s, \quad t \in [a,\sigma(b)]_{\mathbb{T}}.$$

Arzelà-Ascoli Theorem guarantees that T_n is a compact mapping on K. Furthermore, $T_n(0) \ge 0$, $T_n^2(0) \ge 0$. By Theorem 2.1, for each $n \in \mathbb{N}$, there exists $x_n \in K$ such that

$$T_n x_n(t) = x_n(t), \quad \text{for } t \in [a,\sigma(b)]_{\mathbb{T}}.$$

Hence, for each $n \in \mathbb{N}$, x_n satisfies the boundary conditions of problem (1.4).

First we claim that there exists $R > 0$ such that $||x_n|| \le R$ for all n. If this were not true we would find, by going to a subsequence if necessary, that there exists a sequence of functions $\{x_n\}_{n=1}^{\infty}$ from $[a,b]$ into $[0,\infty]$ such that $x_n(t) > 0$ for $t \in (a,b)$,

$$T_n x_n = x_n,$$
$$||x_n|| \le ||x_{n+1}||,$$

and $\lim_{n\to\infty} ||x_n|| = \infty$.

Let $y_n(t) = \int_a^{\sigma(b)} G_2(t,s) g(s, x_n(s))\Delta s$, by Lemma 3.2, for any n, $y_n(t)$ have their graph concave and possess exactly one point of maximum in the interval $[a,\sigma(b)]_{\mathbb{T}}$. For each n, let t_n be the unique point of maximum of $y_n(t)$ on $[a,\sigma(b)]_{\mathbb{T}}$. The fact that the graph of $y_n(t)$ is concave down implies

$$y_n(t) \ge \frac{\sigma(b) - a}{4} y_n(t_n), \quad t \in [\frac{\sigma(b) + 3a}{4}, \frac{3\sigma(b) + a}{4}]_{\mathbb{T}}. \tag{3.9}$$

Let $M_i = \sup\{G_i(t,s)|\ (t,s) \in [a,\sigma(b)]_{\mathbb{T}} \times [a,b]_{\mathbb{T}}\}$, $i = 1,2$. Our assumptions (H_3), coupled with the inequality (3.9) imply there exists n_0 such that if $n \geq n_0$, then

$$f\Big(t, \int_a^{\sigma(b)} G_2(t,s)g(s,x_n(s))\Delta s\Big) \leq \frac{2}{M_1},\quad t \in \Big[\frac{\sigma(b)+3a}{4}, \frac{3\sigma(b)+a}{4}\Big]_{\mathbb{T}}.$$

The fact that $\|x_n\| \leq \|x_{n+1}\|$ implies $\|y_n\| \leq \|y_{n+1}\|$ since g is increasing in its second component, then for $n \geq n_0$,

$$y_n\Big(\frac{\sigma(b)+3a}{4}\Big) \geq \frac{\sigma(b)-a}{4} y_n(t_n) \geq \frac{\sigma(b)-a}{4} y_{n_0}(t_{n_0}),$$
$$y_n\Big(\frac{3\sigma(b)+a}{4}\Big) \geq \frac{\sigma(b)-a}{4} y_n(t_n) \geq \frac{\sigma(b)-a}{4} y_{n_0}(t_{n_0}),$$

since the graph of each solution is concave down, if we let $\theta = \dfrac{\sigma(b)-a}{4} y_{n_0}(t_{n_0})$, then the line segments joining $(a,0)$ with $\Big(\dfrac{\sigma(b)+3a}{4}, \theta\Big)$ and $\Big(\theta, \dfrac{3\sigma(b)+a}{4}\Big)$ with $(\sigma(b),0)$ must lie under the graph of y_n for $n \geq n_0$, that is,

$$y_n(t) \geq h_\theta(t),\quad t \in \Big[a, \frac{\sigma(b)+3a}{4}\Big]_{\mathbb{T}} \cup \Big[\frac{3\sigma(b)+a}{4}, \sigma(b)\Big]_{\mathbb{T}},\ n \geq n_0.$$

Thus, for $n \geq n_0$ one has

$$\begin{aligned}
x_n(t) &= (T_n x_n)(t) = \int_a^{\sigma(b)} G_1(t,s) f_n\Big(s, \int_a^{\sigma(b)} G_2(s,\tau) g(\tau, x_n(\tau))\Delta\tau\Big)\Delta s \\
&\leq \int_a^{\sigma(b)} G_1(t,s) f\Big(s, \int_a^{\sigma(b)} G_2(s,\tau) g(\tau, x_n(\tau))\Delta\tau\Big)\Delta s \\
&\leq \int_a^{\frac{\sigma(b)+3a}{4}} G_1(t,s) f(s, h_\theta(s))\Delta s + \int_{\frac{\sigma(b)+3a}{4}}^{\frac{3\sigma(b)+a}{4}} M_1 \frac{2}{M_1}\Delta s \\
&\quad + \int_{\frac{3\sigma(b)+a}{4}}^{\sigma(b)} G_1(t,s) f(s, h_\theta(s))\Delta s \\
&\leq \int_a^{\sigma(b)} G_1(t,s) f(s, h_\theta(s))\Delta s + \sigma(b) - a < \infty,
\end{aligned}$$

which contradicts the fact that $\lim_{n\to\infty} \|x_n\| = \infty$. Thus there exists $R > 0$ such that $\|x_n\| \leq R$ for all n.

Then we claim that there exists $r > 0$ such that

$$\|x_n\| \geq r \text{ and } \|y_n\| \geq r,$$

where $y_n(t) = \displaystyle\int_a^{\sigma(b)} G_2(t,s) g(s, x_n(s))\Delta s$.

Again the argument is by contradiction. If this were not true, then, by going to a subsequence if necessary, we may assume that $x_n(t) \to 0$ uniformly on $[a,\sigma(b)]_{\mathbb{T}}$

as $n \to \infty$, by (H_3), then $y_n(t) \to 0$ uniformly on $[a, \sigma(b)]_{\mathbb{T}}$ as $n \to \infty$. Let

$$m_i = \inf\{G_i(t,x)|\ (t,s) \in [a,\sigma(b)]_{\mathbb{T}} \times [a,b]_{\mathbb{T}}\} > 0,\ i = 1,2.$$

From (H_3), $\lim_{y \to 0^+} f(t,y) = \infty$ uniformly on compact subsets of $(a, \sigma(b)]_{\mathbb{T}}$. Hence, there exists some $\delta > 0$ such that for $t \in [\frac{3a+\sigma(b)}{4}, \frac{3\sigma(b)+a}{4}]_{\mathbb{T}}$ and $0 < y < \delta$, we obtain

$$f(t,y) > \frac{2}{m_1}.$$

By assumption, there exists n_0 such that $n \geq n_0$ implies that $0 < x_n(t) < \frac{\delta}{2}$, $t \in (a, \sigma(b))_{\mathbb{T}}$. Furthermore, $\psi_n(t) = \int_a^{\sigma(b)} G_2(t,s) g\Big(s, \frac{1}{n}\Big) \Delta s$, hence there exists $n_1 > n_0$ such that if $n \geq n_1$ then

$$\psi_n(t) < \frac{\delta}{2},\quad t \in \Big[\frac{3a+\sigma(b)}{4}, \frac{3\sigma(b)+a}{4}\Big]_{\mathbb{T}}.$$

The conclusion is that if $n \geq n_1$ and $t \in \Big[\frac{3a+\sigma(b)}{4}, \frac{3\sigma(b)+a}{4}\Big]_{\mathbb{T}}$, then

$$\begin{aligned}
x_n(t) = (T_n x_n)(t) &= \int_a^{\sigma(b)} G_1(t,s) f_n\Big(s, \int_a^{\sigma(b)} G_2(s,\tau) g(\tau, x_n(\tau)) \Delta\tau\Big) \Delta s \\
&\geq \int_{\frac{3a+\sigma(b)}{4}}^{\frac{3\sigma(b)+a}{4}} G_1(t,s) f_n\Big(s, \int_a^{\sigma(b)} G_2(s,\tau) g(\tau, x_n(\tau)) \Delta\tau\Big) \Delta s \\
&\geq m_1 \int_{\frac{3a+\sigma(b)}{4}}^{\frac{3\sigma(b)+a}{4}} f\Big(s, \max\Big\{\psi_n(s), \int_0^{\sigma(1)} G_2(s,\tau) g(\tau, x_n(\tau)) \Delta\tau\Big\}\Big) \Delta s \\
&\geq m_1 \int_{\frac{3a+\sigma(b)}{4}}^{\frac{3\sigma(b)+a}{4}} f\Big(s, \frac{\delta}{2}\Big) \Delta s \\
&\geq 1,
\end{aligned}$$

and

$$y_n(t) \geq \int_a^{\sigma(b)} G_2(t,s) g(s,1) \Delta s.$$

But this contradicts the assumption that

$$\lim_{n \to \infty} x_n(t) \to 0$$

uniformly on $[a, \sigma(b)]_{\mathbb{T}}$. Hence, there exists $r > 0$ such that $\|x\| \geq r$ and $\|y\| \geq r$.

Let $\theta = \dfrac{\sigma(b) - a}{4} r$, the concavity of $y_n(t) = \int_a^{\sigma(b)} G_2(t,s) g(s, x_n(s)) \Delta s$ and $x_n(t)$ for $t \in [a, \sigma(b)]_{\mathbb{T}}$ yields

$$\begin{aligned} &x_n(t) \geq h_\theta(t), \\ &y_n(t) \geq h_\theta(t),\quad t \in [a, \sigma(b)]_{\mathbb{T}}. \end{aligned} \tag{3.10}$$

Therefore, the sequence $\{x_n\}_{n=1}^{\infty}$ is contained in the order interval $< h_\theta, R >$; that is, $\{x_n\}_{n=1}^{\infty} \subset D$. Since $T : K \to D$ is a compact mapping, $Tx_n \to x^*$ as $n \to \infty$ for some $x^* \in D$.

To conclude the proof of this theorem, we need to show that

$$\lim_{n\to\infty} (Tx_n(t) - x_n(t)) = 0 \text{ uniformly on } [a, \sigma(b)]_{\mathbb{T}}.$$

Fix $\theta = \dfrac{\sigma(b) - a}{4} r$, and let $\varepsilon > 0$ be given. The latter part of assumption (H$_1$) permits us to choose $\delta \in (a, \sigma(b))_{\mathbb{T}}$ such that

$$\int_a^{\delta} f(t, h_\theta(t))\Delta t < \frac{\varepsilon}{2M_1}.$$

By (3.8) and (3.10), there exists $n_0 \in \mathbb{N}$ such that for $n \geq n_0$,

$$\psi_n(t) \leq h_\theta(t) \leq \min\{x_n(t), y_n(t)\}, \quad t \in [\delta, \sigma(b)]_{\mathbb{T}}.$$

Thus, for $t \in [\delta, \sigma(b)]_{\mathbb{T}}$,

$$f_n(t, y_n(t)) = f(t, \max\{y_n(t), \psi_n(t)\}) = f(t, y_n(t)),$$

and for $t \in [a, \sigma(b)]_{\mathbb{T}}$,

$$\begin{aligned}
(Tx_n)(t) - x_n(t) &= (Tx_n)(t) - (T_n x_n)(t) \\
&= \int_a^{\sigma(b)} G_1(t,s) f(s, y_n(s))\Delta s - \int_a^{\sigma(b)} G_1(t,s) f_n(s, y_n(s))\Delta s \\
&= \int_a^{\delta} G_1(t,s) f(s, y_n(s))\Delta s - \int_a^{\delta} G_1(t,s) f_n(s, y_n(s))\Delta s \\
&\quad + \int_\delta^{\sigma(b)} G_1(t,s) f(s, y_n(s))\Delta s - \int_\delta^{\sigma(b)} G_1(t,s) f_n(s, y_n(s))\Delta s \\
&= \int_a^{\delta} G_1(t,s) f(s, y_n(s))\Delta s - \int_a^{\delta} G_1(t,s) f_n(s, y_n(s))\Delta s.
\end{aligned}$$

Therefore, for $t \in [a, \sigma(b)]_{\mathbb{T}}$,

$$\begin{aligned}
|(Tx_n)(t) - x_n(t)| &\leq M_1 \left[\int_a^{\delta} f(s, y_n(s))\Delta s + \int_a^{\delta} f_n(s, y_n(s))\Delta s\right] \\
&= M_1 \left[\int_a^{\delta} f(s, y_n(s))\Delta s + \int_a^{\delta} f(s, \max\{y_n(s), \psi_n(s)\})\Delta s\right] \\
&\leq 2M_1 \int_a^{\delta} f(s, y_n(s))\Delta s \\
&\leq 2M_1 \int_a^{\delta} f(s, h_\theta(s))\Delta s < \varepsilon.
\end{aligned}$$

Since $t \in [a, \sigma(b)]_{\mathbb{T}}$ is arbitrary, we conclude that $\|Tx_n - x_n\| < \varepsilon$ for all $n \geq n_0$. Hence, $x^* \in < h_\theta, R >$ and for $t \in [a, \sigma(b)]_{\mathbb{T}}$,

$$Tx^*(t) = T\Big(\lim_{n\to\infty} Tx_n(t)\Big) = T\Big(\lim_{n\to\infty} x_n(t)\Big) = \lim_{n\to\infty} Tx_n(t) = x^*(t).$$

The proof is completed. □

4. THE NABLA-NABLA PROBLEM

We now extend the existence results of the previous section to the following system of nonlinear singular boundary value problems on time scale

$$\left\{\begin{array}{l} x^{\nabla\nabla}(t) + f(t, y(t)) = 0, \quad t \in [a, b)_{\mathbb{T}}, \\ y^{\nabla\nabla}(t) + g(t, x(t)) = 0, \quad t \in [a, b)_{\mathbb{T}}, \\ \alpha_1 x(\rho(a)) - \beta_1 x^{\nabla}(\rho(a)) = \gamma_1 x(b) + \delta_1 x^{\nabla}(b) = 0, \\ \alpha_2 y(\rho(a)) - \beta_2 y^{\nabla}(\rho(a)) = \gamma_2 y(b) + \delta_2 y^{\nabla}(b) = 0, \end{array}\right. \tag{4.1}$$

where $\alpha_i,\ \beta_i,\ \gamma_i,\ \delta_i \geq 0$ and $\rho_i = \alpha_i\gamma_i(b - \rho(a)) + \alpha_i\delta_i + \gamma_i\beta_i > 0 (i = 1, 2)$, $f(t, y)$ may be singular at $t = b$, $y = 0$, and $g(t, x)$ may be singular at $t = b$.

Consider the following Banach space $B = \mathbb{C}[\rho^2(a), b]_{\mathbb{T}}$, with the norm $\|u\| = \sup_{t\in[\rho^2(a),b]_{\mathbb{T}}} |u(t)|$.

Define the normal cone, $K \subset B$, as

$$K := \{u \in B |\ u(t) \geq 0,\ t \in [\rho^2(a), b]_{\mathbb{T}}\}.$$

Moreover, define the function $h_1 : [\rho^2(a), b]_{\mathbb{T}} \to \mathbb{R}^+ (\mathbb{R}^+ = [0, \infty))$ by

$$h_1(t) = \left\{\begin{array}{ll} t - \rho^2(a), & \text{if } t \in [\rho^2(a), \dfrac{\rho^2(a) + b}{2}]_{\mathbb{T}}, \\ b - t, & \text{if } t \in [\sigma(\dfrac{\rho^2(a) + b}{2}), b]_{\mathbb{T}}. \end{array}\right.$$

Finally, for $\theta > 0$, let

$$h_\theta = \theta \cdot h_1. \tag{4.2}$$

We make the following assumptions:

(H_1) $\alpha_i,\ \beta_i,\ \gamma_i,\ \delta_i \geq 0,\ \rho_i = \alpha_i\delta_i + \beta_i\gamma_i + \alpha_i\gamma_i(b - \rho(a)) > 0,\ a, b \in \mathbb{T}$ with b left dense;

(H_2) $f \in \mathbb{C}([\rho(a), b)_{\mathbb{T}} \times \mathbb{R}_0^+, \mathbb{R}_0^+)$ and $f(t, \cdot)$ is decreasing for every $t \in [\rho(a), b)_{\mathbb{T}}$, $g \in \mathbb{C}([\rho(a), b)_{\mathbb{T}} \times \mathbb{R}^+, \mathbb{R}_0^+)$ and $g(t, \cdot)$ is increasing for every $t \in [\rho(a), b)_{\mathbb{T}}$;

(H_3) $\lim_{y\to 0^+} f(t, y) = \infty,\ \lim_{y\to\infty} f(t, y) = 0,\ \lim_{x\to 0^+} g(t, x) = 0,\ \lim_{x\to\infty} g(t, x) = \infty$ uniformly on compact subsets of $[\rho(a), b)_{\mathbb{T}}$;

(H_4) $0 < \int_{\rho(a)}^{b} f(t, h_\theta(t))\nabla t < \infty$ for all $\theta > 0$ and $0 < \int_{\rho(a)}^{b} g(t, x)\nabla t < \infty$ for all $x \in \mathbb{R}^+$.

Lemma 4.1. *Suppose that* (H_1) *holds and* $e \in B$, *then linear boundary value problems*

$$(4.3)_i \qquad \begin{cases} u^{\nabla\nabla}(t) + e(t) = 0, \quad t \in (a, b]_{\mathbb{T}}, \\ \alpha_i u(\rho(a)) - \beta_i u^{\nabla}(\rho(a)) = 0, \ \gamma_i u(b) + \delta_i u^{\nabla}(b) = 0, \ i = 1, 2 \end{cases}$$

has a unique solution

$$(4.4) \qquad u_i(t) = \int_{\rho(a)}^{b} G_i(t, s)e(s)\nabla s,$$

where $G_i(t, s) : [\rho(a), b]_{\mathbb{T}} \times [a, b]_{\mathbb{T}} \to [0, \infty)$ *is defined by*

$$(4.5) \quad G_i(t, s) = \frac{1}{\rho_i} \begin{cases} (\beta_i + \alpha_i s - \alpha\rho(a))(\gamma_i b + \delta_i - \gamma_i t), \ \rho(a) \le s \le \sigma(s) \le t \le b, \\ (\beta_i + \alpha_i t - \alpha\rho(a))(\gamma_i b + \delta_i - \gamma_i s), \ \rho(a) \le t \le s \le \sigma(s) \le b. \end{cases}$$

Remark 4.1. If (H_1) holds, then $G_i(t, s) > 0$ for $(t, s) \in (\rho(a), b)_{\mathbb{T}} \times (a, b)_{\mathbb{T}}$.

Lemma 4.2. *Suppose that* (H_1) *holds and* $e \in B$, $e(t) > 0$ *on* $t \in [\rho(a), b)_{\mathbb{T}}$. *If* $u_i (i = 1, 2)$ *is a solution of* $BVP(4.3)_i$, *then*

(i) u_i *is concave;*

(ii) *there exists some* $\theta > 0$ *such that* $u_i(t) \ge h_\theta(t)$ *for all* $t \in [\rho(a), b]_{\mathbb{T}}$.

As we know, $(x, y) \in \mathbb{C}^2(\rho^2(a), b)_{\mathbb{T}} \times \mathbb{C}^2(\rho^2(a), b)_{\mathbb{T}}$ is a solution of system (4.1) if and only if $(x, y) \in \mathbb{C}[\rho^2(a), b]_{\mathbb{T}} \times \mathbb{C}[\rho^2(a), b]_{\mathbb{T}}$ is a solution of the following nonlinear integral equation systems

$$(4.6) \qquad \begin{cases} x(t) = \int_{\rho(a)}^{b} G_1(t, s)f(s, y(s))\nabla s, \\ y(t) = \int_{\rho(a)}^{b} G_2(t, s)g(s, x(s))\nabla s, \ t \in [\rho(a), b]_{\mathbb{T}}, \end{cases}$$

where $G_i(t, s) (i = 1, 2)$ are as defined in Lemma 4.1. Obviously, the above nonlinear system of integral equation can be regarded as the following nonlinear integral equation

$$(4.7) \qquad x(t) = \int_{\rho(a)}^{b} G_1(t, s)f\Big(s, \int_{\rho(a)}^{b} G_2(s, \tau)g(\tau, x(\tau))\nabla\tau\Big)\nabla s.$$

Define $D \subset K$ by

$$D := \{x \in K | \text{ there exists } \theta(x) > 0 \text{ such that } x(t) \ge h_\theta(t), \ t \in [\rho(a), b]_{\mathbb{T}}\},$$

and the operator $T : K \to K$ by

$$(Tx)(t) := \int_{\rho(a)}^{b} G_1(t,s) f\Big(s, \int_{\rho(a)}^{b} G_2(s,\tau) g(\tau, x(\tau)) \nabla\tau\Big) \nabla s. \tag{4.8}$$

Using arguments very similar to the previous section, we obtain an existence theorem.

Theorem 4.3. *Suppose that* (H_1)-(H_4) *hold, then system* (4.1) *has at least one positive solution.*

5. THE MIXED DELTA-NABLA PROBLEM

Finally, we extend these existence results to the following system of nonlinear singular boundary value problems on time scale

$$\begin{cases} x^{\Delta\nabla}(t) + f(t, y(t)) = 0, & t \in (a,b)_{\mathbb{T}}, \\ y^{\Delta\nabla}(t) + g(t, x(t)) = 0, & t \in (a,b)_{\mathbb{T}}, \\ \alpha_1 x(a) - \beta_1 x^{\Delta}(a) = \gamma_1 x(b) + \delta_1 x^{\Delta}(b) = 0, \\ \alpha_2 y(a) - \beta_2 y^{\Delta}(a) = \gamma_2 y(b) + \delta_2 y^{\Delta}(b) = 0, \end{cases} \tag{5.1}$$

and

$$\begin{cases} x^{\nabla\Delta}(t) + f(t, y(t)) = 0, & t \in (a,b)_{\mathbb{T}}, \\ y^{\nabla\Delta}(t) + g(t, x(t)) = 0, & t \in (a,b)_{\mathbb{T}}, \\ \alpha_1 x(a) - \beta_1 x^{\nabla}(a) = \gamma_1 x(b) + \delta_1 x^{\nabla}(b) = 0, \\ \alpha_2 y(a) - \beta_2 y^{\nabla}(a) = \gamma_2 y(b) + \delta_2 y^{\nabla}(b) = 0, \end{cases} \tag{5.2}$$

where α_i, β_i, γ_i, $\delta_i \geq 0$ and $\rho_i = \alpha_i\gamma_i + \alpha_i\delta_i + \gamma_i\beta_i > 0 (i = 1,2)$, $f(t,y)$ may be singular at $t = a$, b and $y = 0$, and $g(t,x)$ may be singular at $t = a$, b.

Consider the following Banach space $B = \mathbb{C}[a,b]_{\mathbb{T}}$, with the norm

$$\|u\| = \sup_{t\in[a,b]_{\mathbb{T}}} |u(t)|.$$

Define the normal cone, $K \subset B$, as

$$K := \{u \in B |\ u(t) \geq 0,\ t \in [a,b]_{\mathbb{T}}\}.$$

Moreover, define the function $h_1 : [a,b]_{\mathbb{T}} \to \mathbb{R}^+ (\mathbb{R}^+ = [0,\infty))$ by

$$h_1(t) = \begin{cases} t - a, & \text{if } t \in \left[a, \frac{a+b}{2}\right]_{\mathbb{T}}, \\ b - t, & \text{if } t \in \left[\sigma(\frac{a+b}{2}), b\right]_{\mathbb{T}}. \end{cases}$$

Finally, for $\theta > 0$, let

$$h_\theta = \theta \cdot h_1. \tag{5.3}$$

We make the following assumptions:

(H_1) $\alpha_i,\ \beta_i,\ \gamma_i,\ \delta_i \geq 0,\ \rho_i = \alpha_i\delta_i + \beta_i\gamma_i + \alpha_i\gamma_i(b-a) > 0,\ a, b \in \mathbb{T}$ with a right dense and b left dense;

(H_2) $f \in \mathbb{C}((a,b)_{\mathbb{T}} \times \mathbb{R}_0^+, \mathbb{R}_0^+)$ and $f(t,\cdot)$ is decreasing for every $t \in (a,b)_{\mathbb{T}}$, $g \in \mathbb{C}((a,b)_{\mathbb{T}} \times \mathbb{R}^+, \mathbb{R}_0^+)$ and $g(t,\cdot)$ is increasing for every $t \in (a,b)_{\mathbb{T}}$;

(H_3) $\lim\limits_{y\to 0^+} f(t,y) = \infty,\ \lim\limits_{y\to\infty} f(t,y) = 0,\ \lim\limits_{x\to 0^+} g(t,x) = 0,\ \lim\limits_{x\to\infty} g(t,x) = \infty$ uniformly on compact subsets of $(a,b)_{\mathbb{T}}$;

(H_{4a}) $0 < \int_a^b f(t, h_\theta(t))\nabla t < \infty$ for all $\theta > 0$ and $0 < \int_a^b g(t,x)\nabla t < \infty$ for all $x \in \mathbb{R}^+$.

(H_{4b}) $0 < \int_a^b f(t, h_\theta(t))\Delta t < \infty$ for all $\theta > 0$ and $0 < \int_a^b g(t,x)\Delta t < \infty$ for all $x \in \mathbb{R}^+$.

Lemma 5.1. *Suppose that* (H_1) *holds and* $e \in B$, *then linear boundary value problems*

$$(5.4)_i \qquad \begin{cases} u^{\Delta\nabla}(t) + e(t) = 0, & t \in (a,b)_{\mathbb{T}}, \\ \alpha_i u(a) - \beta_i u^{\Delta}(a) = 0, & \gamma_i u(b) + \delta_i u^{\Delta}(b) = 0,\ i = 1,2 \end{cases}$$

and

$$(5.5)_i \qquad \begin{cases} u^{\nabla\Delta}(t) + e(t) = 0, & t \in (a,b)_{\mathbb{T}}, \\ \alpha_i u(a) - \beta_i u^{\nabla}(a) = 0, & \gamma_i u(b) + \delta_i u^{\nabla}(b) = 0,\ i = 1,2 \end{cases}$$

has a unique solution

$$u_i(t) = \int_a^b G_i(t,s)e(s)\nabla s,$$

and

$$u_i(t) = \int_a^b G_i(t,s)e(s)\Delta s,$$

respectively, where $G_i(t,s) : [a,b]_{\mathbb{T}} \times [a,b]_{\mathbb{T}} \to [0,\infty)$ *is defined by*

$$(5.6) \qquad G_i(t,s) = \frac{1}{\rho_i}\begin{cases} (\beta_i + \alpha_i s - \alpha_i a)(\gamma_i b + \delta_i - \gamma_i t), & a \leq s \leq \sigma(s) \leq t \leq b, \\ (\beta_i + \alpha_i t - \alpha_i a)(\gamma_i b + \delta_i - \gamma_i s), & a \leq t \leq s \leq \sigma(s) \leq b. \end{cases}$$

Remark 5.1. If (H_1) holds, then $G_i(t,s) > 0$ for $(t,s) \in (a,b)_{\mathbb{T}} \times (a,b)_{\mathbb{T}}$.

Lemma 5.2. *Suppose that* (H_1) *holds and* $e \in B$, $e(t) > 0$ *on* $t \in (a,b)_{\mathbb{T}}$. *If* $u_i (i = 1,2)$ *is a solution of* $BVP(5.4)_i$ (*or* $(5.5)_i$), *then*

(i) u_i *is concave;*

(ii) *there exists some* $\theta > 0$ *such that* $u_i(t) \geq h_\theta(t)$ *for all* $t \in [a,b]_{\mathbb{T}}$.

As we know, $(x,y) \in \mathbb{C}^2(a,b)_{\mathbb{T}} \times \mathbb{C}^2(a,b)_{\mathbb{T}}$ is a solutions of system (5.1), system (5.2) if and only if $(x,y) \in \mathbb{C}[a,b]_{\mathbb{T}} \times \mathbb{C}[a,b]_{\mathbb{T}}$ is a solution of the following nonlinear integral equation systems

$$(5.7) \qquad \begin{cases} x(t) = \int_a^b G_1(t,s) f(s, y(s)) \nabla s, \\ y(t) = \int_a^b G_2(t,s) g(s, x(s)) \nabla s, \ t \in [a,b]_{\mathbb{T}}, \end{cases}$$

and

$$(5.8) \qquad \begin{cases} x(t) = \int_a^b G_1(t,s) f(s, y(s)) \Delta s, \\ y(t) = \int_a^b G_2(t,s) g(s, x(s)) \Delta s, \ t \in [a,b]_{\mathbb{T}}, \end{cases}$$

respectively, where $G_i(t,s) (i = 1,2)$ are as defined in Lemma 4.1. Obviously, the above nonlinear system of integral equation can be regarded as the following non-linear integral equation

$$(5.9) \qquad x(t) = \int_a^b G_1(t,s) f(s, \int_a^b G_2(s,\tau) g(\tau, x(\tau)) \nabla \tau) \nabla s$$

and

$$(5.10) \qquad x(t) = \int_a^b G_1(t,s) f(s, \int_a^b G_2(s,\tau) g(\tau, x(\tau)) \Delta \tau) \Delta s,$$

respectively.

Define $D \subset K$ by

$$D := \{x \in K | \text{ there exists } \theta(x) > 0 \text{ such that } x(t) \geq h_\theta(t), \ t \in [a,b]_{\mathbb{T}}\},$$

and the operator $T : K \to K$ by

$$(5.11) \qquad (Tx)(t) := \int_a^b G_1(t,s) f(s, \int_a^b G_2(s,\tau) g(\tau, x(\tau)) \nabla \tau) \nabla s, \quad \text{for (5.1)},$$

and

$$(5.12) \qquad (Tx)(t) := \int_a^b G_1(t,s) f(s, \int_a^b G_2(s,\tau) g(\tau, x(\tau)) \Delta \tau) \Delta s, \quad \text{for (5.2)}.$$

Using arguments very similar to section 3, we obtain an existence theorem.

Theorem 5.3. *Suppose that* (H_1)-(H_{4a}) *hold, then system* (5.1) *has at least one positive solution.*

Theorem 5.4. *Suppose that* (H_1)-(H_{4b}) *hold, then system* (5.2) *has at least one positive solution.*

6. Appendix

Let $\mathbb{T}$ be a time scale, which is a closed subset of $\mathbb{R}$, the set of real numbers, with the subspace topology inherited from the Euclidean topology on $\mathbb{R}$. An alternative terminology for time scale is measure chain.

The theory of time scale was introduced and developed by Aulbach and Hilger [2] to unify continuous and discrete analysis. Now, there have been many publications (see [7, 8, 14, 18]) relating difference equations with differential equations. A result for a dynamic equation contains simultaneously a corresponding result for a differential equation, one for a difference equation, as well as results for other dynamic equations in arbitrary time scales. Time scales theory present us with the tools necessary to understand and explain the mathematical structure underpinning the theories of discrete and continuous dynamical systems and allows us to connect them. That is certainly the goal with this work. The following definitions on time scales can be found in papers [7, 8].

Definition 6.1. Define the interval in $\mathbb{T}$

$$[a, b]_{\mathbb{T}} := \{t \in \mathbb{T} \text{ such that } a \leq t \leq b\}.$$

Open intervals and half-open intervals etc. are defined accordingly.

Definition 6.2. A time scale may or may not be connected, so we define the forward jump operator and backward jump operator σ, ρ by

$$\sigma(t) := \inf\{\tau > t;\ \tau \in \mathbb{T}\} \in \mathbb{T}, \quad \rho(t) := \sup\{\tau < t;\ \tau \in \mathbb{T}\} \in \mathbb{T},$$

for all $t \in \mathbb{T}$ with $t < \sup \mathbb{T}$.

An element $t \in \mathbb{T}$ is left-dense, right-dense, left-scattered, right-scattered if $\rho(t) = t$, $\sigma(t) = t$, $\rho(t) < t$, $\sigma(t) > t$, respectively. Also, $\inf \emptyset =:= \sup \mathbb{T}$ and $\sup \emptyset := \inf \mathbb{T}$. If $\mathbb{T}$ has a right scattered minimum m, then $\mathbb{T}_k = \mathbb{T} - \{m\}$, otherwise $\mathbb{T}_k = \mathbb{T}$. If $\mathbb{T}$ has a left-scattered maximum M, then $\mathbb{T}^k = \mathbb{T} - \{M\}$, otherwise $\mathbb{T}^k = \mathbb{T}$.

Definition 6.3. Assume $x : \mathbb{T} \to \mathbb{R}$ and fix $t \in \mathbb{T}^k$. Then we define $x^{\Delta}(t)$ to be the number (provided it exists) with the property that given any $\varepsilon > 0$, there is a neighborhood U of t such that

$$|[x(\sigma(t)) - x(s)] - x^{\Delta}(t)[\sigma(t) - s]| \leq \varepsilon|\sigma(t) - s|,$$

for all $s \in U$. We call $x^{\Delta}(t)$ the delta derivative of $x(t)$ at $t \in \mathbb{T}^k$. The second derivative of $x(t)$ is defined by $x^{\Delta\Delta}(t) = (x^{\Delta})^{\Delta}(t)$.

Definition 6.4. Assume $x : \mathbb{T} \to \mathbb{R}$ and fix $t \in \mathbb{T}_k$. Then we define $x^{\nabla}(t)$ to be the number (provided it exists) with the property that given any $\varepsilon > 0$, there is a neighborhood U of t such that

$$|[x(\rho(t)) - x(s)] - x^{\nabla}(t)[\rho(t) - s]| \leq \varepsilon|\rho(t) - s|,$$

for all $s \in U$. We call $x^{\nabla}(t)$ the nabla derivative of $x(t)$ at $t \in \mathbb{T}_k$. The second derivative of $x(t)$ is defined by $x^{\nabla\nabla}(t) = (x^{\nabla})^{\nabla}(t)$.

ACKNOWLEDGEMENT

The authors thank the referee for his/her careful reading of the manuscript and valuable suggestions.

REFERENCES

1. H. Amann: Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces. *SIAM Rev.* **18** (1976), 620-709.
2. B. Aulbach & S. Hilger: Linear dynamic processes with inhomogeneous time scale, in: Nonlinear dynamics and quantum dynamical systems. *Mathematical Research. Akademie Verlag, Berlin* **59** (1990), 9-20.
3. R.P. Agarwal & D. O'Regan: Nonlinear superlinear singular and nonsingular second order boundary value problems. *J. Differential Equations* **143** (1998), 60-95.
4. ______: Existence theory for single and multiple solutions to singular positone boundary value problems. *J. Differential Equations* **175** (2001), 393-414.
5. ______: Twin solutions to singular Dirichlet problems. *J. Math. Anal. Appl.* **240** (1999), 433-445.
6. R.P. Agarwal, V.O. Espinar, K. Perera & D.R. Vivero: Multiple positive solutions of singular Dirichlet problems on time scales via variational methods. *Nonlinear Anal.* **67** (2007), 368-381.
7. M. Bohner & A. Peterson (Eds.): *Dynamic equations on time scales: An introduction with applications.* Birkhäuser, Boston, 2001.
8. ______: *Advances in dynamic equations on time scales.* Birkhäuser, Boston, 2003.
9. J. Chu, P.J. Torres & M. Zhang: Periodic solutions of second order non-autonomous singular dynamical systems. *J. Differential Equations* **239** (2007) 196-212.
10. A. Cabada: Extremal solutions and Green's functions of higher order periodic boundary value problems in time scales. *J. Math. Anal. Appl* **290** (2004), 35-54.
11. J.J. DaCunha, J.M. Davis & P.K. Sing: Existence results for singular three point boundary value problems on time scales. *J. Math. Anal. Appl.* **295** (2004), 378-391.
12. J.A. Gatica, V. Oliker & P. Waltman: Singular nonlinear boundary value problems for

second-order ordinary differential equations. *J. Differential Equations* **79** (1989), 62-78.
13. ______: Iterative procedures for nonlinear second order boundary value problem. *Ann. Mat. Pura Appl.* **157** (1990), 1-25.
14. D. Jiang, J. Chu & M. Zhang: Multiplicity of positive periodic solutions to superlinear repulsive singular equations. *J. Differential Equations* **211** (2005) 282-302.
15. S. Hilger: Analysis on measure chains-a unified approach to continuous and discrete calculus. *Result. Math.* **18** (1990), 18-56.
16. K. Ha & Y. Lee: Existence of multiple positive solutions of singular boundary value problems. *Nonlinear Anal.* **28** (1997), 1429-1438.
17. Z. Hao, J. Liang & T. Xiao: Existence results for time scale boundary value problem. *J. Comput. Appl. Math.* **197** (2006), 156-168.
18. Z. Hao, T. Xiao & J. Liang: Existence of positive solutions for singular boundary value problem on time scales. *J. Math. Anal. Appl.* **325** (2007), 517-528.
19. B. Kaymakcalan, V. Lakshmikantham & S. Sivasundaram: *Dynamical systems on measure chains.* Kluwer Academic, Boston, 1996.
20. M.A. Krasnosel'skii: *Positive solutions of operator equations.* Noordhoff, Groningen, 1964.
21. K. Lan & J.L. Webb: Positive solutions of semilinear differential equations with singularities. *J. Differential Equations* **148** (1998) 407-421.
22. Y. Liu & A. Qi: Positive solutions of nonlinear singular boundary value problem in abstract space. *Comput. Appl. Math.* **47** (2004), 683-688.
23. S. Liang & J. Zhang: The existence of countably many positive solutions for nonlinear singular m-point boundary value problems on time scales. *J. Comput. Appl. Math.* **223** (2009), no. 1, 291-303.
24. D. Ma, J. Han & X. Chen: Positive solution of three-point boundary value problem for the one-dimensional p-Laplacian with singularities. *J. Math. Anal. Appl.* **324** (2006), 118-133.
25. D. Ma & W. Ge: Positive solution of multi-point boundary value problem for the one-dimensional p-Laplacian with singularities. *Rocky Mountain J. Math.* **137** (2007), no. 4, 1229-1250.
26. Y. Su, W. Li & H. Sun: Positive solutions of singular p-Laplacian dynamic equations with sign changing nonlinearity. *Appl. Math. Comput.* **200** (2008), no. 1, 352-368.
27. Y. Tian & W. Ge: Existence and uniqueness results for nonlinear first-order three-point boundary value problems on time scales. *Nonlinear Anal.* **69** (2008), 2833-2842.

[a]College of Science, Changchun University, Changchun 130022, P. R. China
Email address: `miaochunmei@yahoo.com.cn`

[b]College of Science, Tianjin University of Technology, Tianjin 300191, P. R. China
Email address: `jdh200298@163.com`

[c]DEPARTMENT OF MATHEMATICS, BEIJING INSTITUTE OF TECHNOLOGY, BEIJING 100081, P. R. CHINA
Email address: miaochunmei@yahoo.com.cn, zhao_junfang@163.com, gew@bit.edu.cn