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RELATED FIXED POINT THEOREM ON TWO INTUITIONISTIC FUZZY METRIC SPACES

BHAVANA DESHPANDE [a] AND ROHIT PATHAK [b]

ABSTRACT. We prove a related fixed point theorem for two pairs of mappings on two intuitionistic fuzzy metric spaces. Our result is maiden in this line.

1. INTRODUCTION

Motivated by the potential applicability of fuzzy topology to quantum particle physics particularly in connection with both string and $e^{(\infty)}$ theory developed by El Naschie [7, 8], Park introduced and discussed in [24] a notion of intuitionistic fuzzy metric spaces which is based both on the idea of intuitionistic fuzzy set due to Atanassov [1] and the concept of fuzzy metric spaces given by George and Veeramani in [14]. Actually, Park's notion is useful in modelling some phenomena where it is necessary to study relationship between probability function. It has a direct physics motivation in the context of the two slit experiment as foundation of $E-$infinity of high energy physics, recently studied by El Naschie in [9, 10].

Alaca et al. [2] using the idea of intuitionistic fuzzy sets, they defined the notion of intuitionistic fuzzy metric space as Park [24] with the help of continuous t-norms and continuous t-conorms as a generalization of fuzzy metric space due to Kramosil and Michalek [19]. Further, they introduced the notion of Cauchy sequences in intuitionistic fuzzy metric spaces and proved the well known fixed point theorems of Banach [3] and Edelstein [6] extended to intuitionistic fuzzy metric spaces with the help of Grabiec [15].

Gregory et al. [16], Saadati and Park [25] studied the concept of intuitionistic fuzzy metric spaces and its applications. Many authors proved fixed point theorems in intuitionistic fuzzy metric spaces including Sharma and Deshpande [27].

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Fisher [11, 12], Nung [23], Cho et al. [4], Fisher and Murthy [13] proved related fixed point theorems on two or three metric spaces. Sharma et al. [28] proved related fixed point theorem on two fuzzy metric spaces.

However, so far the related fixed point theorems on intuitionistic fuzzy metric spaces have not been proved. Our work is maiden in this line.

In this paper, we prove a related fixed point theorem for two pairs of mappings on two intuitionistic fuzzy metric spaces. We intuitionistically fuzzify the results of Sharma et al. [28] and many others.

2. PRELIMINARIES

Definition 1 ([26]). A binary operation $*: [0,1] \times [0,1] \to [0,1]$ is a continuous t−norm if $*$ is satisfying the following conditions:

(i) $*$ is commutative and associative,
(ii) $*$ is continuous,
(iii) $a * 1 = a$ for all $a \in [0,1]$,
(iv) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0,1]$.

Definition 2 ([26]). A binary operation $\diamond: [0,1] \times [0,1] \to [0,1]$ is a continuous t−conorm if $\diamond$ satisfying the following conditions:

(i) $\diamond$ is commutative and associative,
(ii) $\diamond$ is continuous,
(iii) $a \diamond 0 = a$ for all $a \in [0,1]$,
(iv) $a \diamond b \leq c \diamond d$ whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0,1]$.

Remark 1. The concepts of triangular norms (t−norms) and triangular conorms (t−conorms) are known as axiomatic skeletons that we use for characterizing fuzzy intersections and unions respectively. These concepts were originally introduced by Menger [21] in his study of statistical metric spaces. Several examples for these concepts were proposed by many authors ([5], [17], [18], [29]).

Definition 3 ([2]). A 5−tuple $(X, M, N, *, \diamond)$ is said to be an *intuitionistic fuzzy metric space* if X is an arbitrary set, $*$ is a continuous t−norm, $\diamond$ is a continuous t−conorm and M, N are two fuzzy sets on $X^2 \times [0, \infty)$ satisfying the following conditions:

(i) $M(x, y, t) + N(x, y, t) \leq 1$ for all $x, y \in X$ and $t > 0$,
(ii) $M(x, y, 0) = 0$ for all $x, y \in X$,

(iii) $M(x, y, t) = 1$ for all $x, y \in X$ and $t > 0$ if and only if $x = y$,
(iv) $M(x, y, t) = M(y, x, t)$ for all $x, y \in X$ and $t > 0$,
(v) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$ for all $x, y, z \in X$ and $s, t > 0$,
(vi) for all $x, y \in X, M(x, y, .) : X^2 \times [0, \infty) \to [0, 1]$ is left continuous,
(vii) $\lim_{t \to \infty} M(x, y, t) = 1$ for all $x, y \in X$
(viii) $N(x, y, 0) = 1$ for all $x, y \in X$,
(ix) $N(x, y, t) = 0$ for all $x, y \in X$ and $t > 0$ if and only if $x = y$,
(x) $N(x, y, t) = N(y, x, t)$ for all $x, y \in X$ and $t > 0$,
(xi) $N(x, y, t) \diamond N(y, z, s) \geqslant N(x, z, t + s)$ for all $x, y, z \in X$ and $s, t > 0$,
(xii) for all $x, y \in X, N(x, y, .) : X^2 \times [0, \infty) \to [0, 1]$ is right continuous,
(xiii) $\lim_{t \to \infty} N(x, y, t) = 0$ for all $x, y \in X$.

Then (M, N) is called an *intuitionistic fuzzy metric* on X. The functions $M(x, y, t)$ and $N(x, y, t)$ denote the degree of nearness and the degree of nonnearness between x and y with respect to t, respectively.

Remark 2. Every fuzzy metric space $(X, M, *)$ is an intuitionistic fuzzy metric space of the form $(X, M, 1 - M, *, \diamond)$ such that t−norm $*$ and t- conorm $\diamond$ are associated [21] i.e.,

$$x \diamond y = 1 - ((1 - x) * (1 - y)) \quad \text{for all } x, y \in X.$$

Example 1. Let (X, d) be a metric space. Define t−norm by $a * b = \min\{a, b\}$, t-conorm by $a \diamond b = \max\{a, b\}$ and for all $x, y \in X$ and $t > 0$,

$$M_d(x, y, t) = \frac{t}{t + d(x, y)}, \; N_d(x, y, t) = \frac{d(x, y)}{t + d(x, y)}.$$

Then $(X, M, N, *, \diamond)$ is an intuitionistic fuzzy metric space. We call this intuitionistic fuzzy metric (M, N) induced by the metric d, the standard intuitionistic fuzzy metric.

Remark 3. In an intuitionistic fuzzy metric space $(X, M, N, *, \diamond)$, $M(x, y, .)$ is non-decreasing and $N(x, y, .)$ is non-increasing for all $x, y \in X$.

Definition 4 ([2]). Let $(X, M, N, *, \diamond)$ be an intuitionistic fuzzy metric space, then
(i) a sequence $\{x_n\}$ in X is said to be Cauchy sequence if for all $t > 0$ and $p > 0$,

$$\lim_{n \to \infty} M(x_{n+p}, x_n, t) = 1, \; \lim_{n \to \infty} N(x_{n+p}, x_n, t) = 0,$$

(ii) a sequence $\{x_n\}$ in X is said to be convergent to a point $x \in X$ if for all $t > 0$,

$$\lim_{n\to\infty} M(x_n, x, t) = 1, \ \lim_{n\to\infty} N(x_n, x, t) = 0.$$

Since $*$ and $\diamond$ are continuous, the limit is uniquely determined from (v) and (xi) respectively.

Definition 5 ([2]). An intuitionistic fuzzy metric space $(X, M, N, *, \diamond)$ is said to be complete if and only if every Cauchy sequence in X is convergent.

Lemma A ([4]). *Let $(X, M, N, *, \diamond)$ be an intuitionistic fuzzy metric space and $\{y_n\}$ be a sequence in X. If there exists a number $k \in (0,1)$ such that*

(I) $M(y_{n+2}, y_{n+1}, kt) \geqslant M(y_{n+1}, y_n, t)$,

(II) $N(y_{n+2}, y_{n+1}, kt) \leqslant N(y_{n+1}, y_n, t)$

for all $t > 0$ and $n = 1, 2, \ldots\ldots$, then $\{y_n\}$ is a Cauchy sequence in X.

Lemma B ([22]). *Let $(X, M, N, *, \diamond)$ be an intuitionistic fuzzy metric space and for all $x, y \in X,\ t > 0$ and if for a number $k \in (0,1)$,*

$$M(x, y, kt) \geqslant M(x, y, t)$$

and $N(x, y, kt) \leqslant N(x, y, t)$, then $x = y$.

3. MAIN RESULTS

Theorem 1. *Let $(X, M_1, N_1, *, \diamond)$ and $(Y, M_2, N_2, *, \diamond)$ be two complete intuitionistic fuzzy metric spaces. Let A, B be mappings from X into Y and let S, T be mappings from Y into X satisfying the inequalities:*

$$\begin{aligned} &M_1(SAx, TBx', kt) \\ &\quad \geqslant M_1(x, x', t) * M_1(x, SAx, t) * M_1(x', TBx', t) * M_1(SAx, TBx', t) \end{aligned} \tag{1.1}$$

and

$$\begin{aligned} &N_1(SAx, TBx', kt) \\ &\quad \leqslant N_1(x, x', t) \diamond N_1(x, SAx, t) \diamond N_1(x', TBx', t) \diamond N_1(SAx, TBx', t), \end{aligned}$$

$$\begin{aligned} &M_2(BSy, ATy', kt) \\ &\quad \geqslant M_2(y, y', t) * M_2(y, BSy, t) * M_2(y', ATy', t) * M_2(BSy, ATy', t) \end{aligned} \tag{1.2}$$

and

$$\begin{aligned} &N_2(BSy, ATy', kt) \\ &\quad \leqslant N_2(y, y', t) \diamond N_2(y, BSy, t) \diamond N_2(y', ATy', t) \diamond N_2(BSy, ATy', t) \end{aligned}$$

for all $x, x' \in X,\ y, y' \in Y, t > 0$ and $k \in (0,1)$. If one of the mappings A, B, S, T is continuous, then SA and TB have a unique common fixed point z in X and BS

and AT have a unique common fixed point w in Y. Further $Az = Bz = w$ *and* $Sw = Tw = z$.

Proof. Let $x = x_0$ be an arbitrary point in X and define sequences $\{x_n\}$ and $\{y_n\}$ in X and Y respectively as follows:

Choose a point $y_1 = Ax_0$, a point $x_1 = Sy_1$, a point $y_2 = Bx_1$ and a point $x_2 = Ty_2$. In general, having chosen x_{2n-2} in X, choose a point $y_{2n-1} = Ax_{2n-2}$, a point $x_{2n-1} = Sy_{2n-1}$, a point $y_{2n} = Bx_{2n-1}$, and a point $x_{2n} = Ty_{2n}$ for all $n = 1, 2, 3, 4......$Then applying (1.1), we get

$$\begin{aligned} M_1(x_{2n+1}, x_{2n}, kt) &= M_1(SAx_{2n}, TBx_{2n-1}, kt) \\ &\geqslant M_1(x_{2n}, x_{2n-1}, t) * M_1(x_{2n}, SAx_{2n}, t) * M_1(x_{2n-1}, TBx_{2n-1}, t) \\ &\quad * M_1(SAx_{2n}, TBx_{2n-1}, t) \\ &= M_1(x_{2n}, x_{2n-1}, t) * M_1(x_{2n}, x_{2n+1}, t) * M_1(x_{2n-1}, x_{2n}, t) \\ &\quad * M_1(x_{2n+1}, x_{2n}, t) \\ &\geqslant M_1(x_{2n}, x_{2n-1}, t) * M_1(x_{2n}, x_{2n+1}, t) \end{aligned} \tag{1.3}$$

and

$$\begin{aligned} &N_1(x_{2n+1}, x_{2n}, kt) \\ &= N_1(SAx_{2n}, TBx_{2n-1}, kt) \\ &\leqslant N_1(x_{2n}, x_{2n-1}, t) \diamond N_1(x_{2n}, SAx_{2n}, t) \diamond N_1(x_{2n-1}, TBx_{2n-1}, t) \\ &\quad \diamond N_1(SAx_{2n}, TBx_{2n-1}, t) \\ &= N_1(x_{2n}, x_{2n-1}, t) \diamond N_1(x_{2n}, x_{2n+1}, t) \diamond N_1(x_{2n-1}, x_{2n}, t) \\ &\quad \diamond N_1(x_{2n+1}, x_{2n}, t) \\ &\leqslant N_1(x_{2n}, x_{2n-1}, t) \diamond N_1(x_{2n}, x_{2n+1}, t). \end{aligned} \tag{1.4}$$

Similarly, we have

$$M_1(x_{2n+2}, x_{2n+1}, kt) \geqslant M_1(x_{2n+1}, x_{2n}, t) * M_1(x_{2n+1}, x_{2n+2}, t) \tag{1.5}$$

and

$$N_1(x_{2n+2}, x_{2n+1}, kt) \leqslant N_1(x_{2n+1}, x_{2n}, t) \diamond N_1(x_{2n+1}, x_{2n+2}, t). \tag{1.6}$$

Thus, from (1.3) – (1.6), it follows that

$$M_1(x_{n+1}, x_{n+2}, kt) \geqslant M_1(x_n, x_{n+1}, t) * M_1(x_{n+1}, x_{n+2}, t)$$

and

$$N_1(x_{n+1}, x_{n+2}, kt) \leqslant N_1(x_n, x_{n+1}, t) \diamond N_1(x_{n+1}, x_{n+2}, t),$$

for $n = 1, 2, 3......$ Using the above two inequalities, we obtain the following with the help of simple induction

$$M_1(x_{n+1}, x_{n+2}, kt) \geqslant M_1(x_n, x_{n+1}, t) * M_1\left(x_{n+1}, x_{n+2}, \frac{t}{k^p}\right)$$

and

$$N_1(x_{n+1}, x_{n+2}, kt) \leqslant N_1(x_n, x_{n+1}, t) \diamond N_1\left(x_{n+1}, x_{n+2}, \frac{t}{k^p}\right),$$

for positive integers n and p.

Thus since $M_1\left(x_{n+1}, x_{n+2}, \frac{t}{k^p}\right) \to 1$ and $N_1\left(x_{n+1}, x_{n+2}, \frac{t}{k^p}\right) \to 0$ as $p \to \infty$, we have

$$M_1(x_{n+1}, x_{n+2}, kt) \geqslant M_1(x_n, x_{n+1}, t)$$

and

$$N_1(x_{n+1}, x_{n+2}, kt) \leq N_1(x_n, x_{n+1}, t).$$

Similarly applying inequalities (1.2), we get

$$\begin{aligned} M_2(y_{2n}, y_{2n+1}, kt) &= M_2(BSy_{2n-1}, ATy_{2n}, kt) \\ &\geqslant M_2(y_{2n-1}, y_{2n}, t) * M_2(y_{2n-1}, y_{2n}, t) * M_2(y_{2n}, y_{2n+1}, t) \\ &\quad * M_2(y_{2n}, y_{2n+1}, t) \\ &\geqslant M_2(y_{2n-1}, y_{2n}, t) * M_2(y_{2n}, y_{2n+1}, t) \end{aligned} \tag{1.7}$$

and

$$\begin{aligned} N_2(y_{2n}, y_{2n+1}, kt) &= N_2(BSy_{2n-1}, ATy_{2n}, kt) \\ &\leqslant N_2(y_{2n-1}, y_{2n}, t) \\ &\quad \diamond N_2(y_{2n-1}, y_{2n}, t) \diamond N_2(y_{2n}, y_{2n+1}, t) \diamond N_2(y_{2n}, y_{2n+1}, t) \\ &\leqslant N_2(y_{2n-1}, y_{2n}, t) \diamond N_2(y_{2n}, y_{2n+1}, t) \end{aligned} \tag{1.8}$$

Similarly, we also have

$$M_2(y_{2n+1}, y_{2n+1}, kt) \geqslant M_2(y_{2n}, y_{2n+1}, t) * M_2(y_{2n+1}, y_{2n+2}, t) \tag{1.9}$$

and

$$N_2(y_{2n+1}, y_{2n+1}, kt) \leqslant N_2(y_{2n}, y_{2n+1}, t) \diamond N_2(y_{2n+1}, y_{2n+2}, t). \tag{1.10}$$

Thus from $(1.7) - (1.10)$, it follows that

$$M_2(y_{n+1}, y_{n+2}, kt) \geqslant M_2(y_n, y_{n+1}, t) * M_2(y_{n+1}, y_{n+2}, t)$$

and

$$N_2(y_{n+1}, y_{n+2}, kt) \leqslant N_2(y_n, y_{n+1}, t) \diamond N_2(y_{n+1}, y_{n+2}, t).$$

for $n = 1, 2, 3.........$ Using the above two inequalities, we obtain the following with the help of simple induction

$$M_2(y_{n+1}, y_{n+2}, kt) \geqslant M_2(y_n, y_{n+1}, t) * M_2\Big(y_{n+1}, y_{n+2}, \frac{t}{k^p}\Big)$$

and

$$N_2(y_{n+1}, y_{n+2}, kt) \leqslant N_2(y_n, y_{n+1}, t) \diamond N_2\Big(y_{n+1}, y_{n+2}, \frac{t}{k^p}\Big),$$

for positive integers n and p.

Thus since

$$M_2\Big(y_{n+1}, y_{n+2}, \frac{t}{k^p}\Big) \longrightarrow 1$$

and

$$N_2\Big(y_{n+1}, y_{n+2}, \frac{t}{k^p}\Big) \longrightarrow 0$$

as $p \longrightarrow \infty$, we have

$$M_2(y_{n+1}, y_{n+2}, kt) \geqslant M_2\Big(y_n, y_{n+1}, t\Big)$$

and

$$N_2(y_{n+1}, y_{n+2}, kt) \leqslant N_2\Big(y_n, y_{n+1}, t\Big).$$

By Lemma A, the sequence $\{x_n\}$ is therefore a Cauchy sequence in complete intuitionistic fuzzy metric space X and so has a limit z in X. It follows similarly that the sequence $\{y_n\}$ is a Cauchy sequence in complete intuitionistic fuzzy metric space $Y$and so has a limit w in Y. Using (1.1), we have

$$\begin{aligned}
M_1(SAx_{2n}, z, kt) &\geqslant M_1\Big(SAx_{2n}, x_{2n}, \frac{kt}{2}\Big) * M_1\Big(x_{2n}, z, \frac{kt}{2}\Big)\\
&= M_1\Big(SAx_{2n}, TBx_{2n-1}, \frac{kt}{2}\Big) * M_1\Big(x_{2n}, z, \frac{kt}{2}\Big)\\
&\geqslant M_1\Big(x_{2n}, x_{2n-1}, \frac{t}{2}\Big) * M_1\Big(x_{2n}, SAx_{2n}, \frac{t}{2}\Big)\\
&\quad * M_1\Big(x_{2n-1}, TBx_{2n-1}, \frac{t}{2}\Big) * M_1\Big(SAx_{2n}, TBx_{2n-1}, \frac{t}{2}\Big)\\
&\quad * M_1\Big(x_{2n}, z, \frac{kt}{2}\Big)\\
&\geqslant M_1\Big(x_{2n}, x_{2n-1}, \frac{t}{2}\Big) * M_1\Big(x_{2n}, x_{2n+1}, \frac{t}{2}\Big)\\
&\quad * M_1\Big(x_{2n-1}, x_{2n}, \frac{t}{2}\Big) * M_1\Big(x_{2n+1}, x_{2n}, \frac{t}{2}\Big) * M_1\Big(x_{2n}, z, \frac{kt}{2}\Big)
\end{aligned}$$

and

$$\begin{aligned}
N_1(SAx_{2n}, z, kt) &\leqslant N_1\Big(SAx_{2n}, x_{2n}, \frac{kt}{2}\Big) \diamond N_1\Big(x_{2n}, z, \frac{kt}{2}\Big) \\
&= N_1\Big(SAx_{2n}, TBx_{2n-1}, \frac{kt}{2}\Big) \diamond N_1\Big(x_{2n}, z, \frac{kt}{2}\Big) \\
&\leqslant N_1\Big(x_{2n}, x_{2n-1}, \frac{t}{2}\Big) \diamond N_1\Big(x_{2n}, SAx_{2n}, \frac{t}{2}\Big) \\
&\quad \diamond N_1\Big(x_{2n-1}, TBx_{2n-1}, \frac{t}{2}\Big) \diamond N_1\Big(SAx_{2n}, TBx_{2n-1}, \frac{t}{2}\Big) \\
&\quad \diamond N_1\Big(x_{2n}, z, \frac{kt}{2}\Big) \\
&\leqslant N_1\Big(x_{2n}, x_{2n-1}, \frac{t}{2}\Big) \diamond N_1\Big(x_{2n}, x_{2n+1}, \frac{t}{2}\Big) \diamond N_1\Big(x_{2n-1}, x_{2n}, \frac{t}{2}\Big) \\
&\quad \diamond N_1\Big(x_{2n+1}, x_{2n}, \frac{t}{2}\Big) \diamond N_1\Big(x_{2n}, z, \frac{kt}{2}\Big).
\end{aligned}$$

Taking limit $n \to \infty$, we have $M_1(SAx_{2n}, z, kt) \to 1$ and $N_1(SAx_{2n}, z, kt) \to 0$. Thus we have

$$\lim_{n\to\infty} SAx_{2n} = z = \lim_{n\to\infty} Sy_{2n+1}. \tag{1.11}$$

Similarly we can prove that

$$\lim_{n\to\infty} TBx_{2n-1} = z = \lim_{n\to\infty} Ty_{2n}. \tag{1.12}$$

$$\lim_{n\to\infty} BSy_{2n-1} = w = \lim_{n\to\infty} Bx_{2n-1}. \tag{1.13}$$

$$\lim_{n\to\infty} ATy_{2n} = w = \lim_{n\to\infty} Ax_{2n}. \tag{1.14}$$

Now suppose A is continuous. Thus

$$\lim_{n\to\infty} Ax_{2n} = Az = w. \tag{1.15}$$

Using (1.1), we have

$$\begin{aligned}
M_1(SAz, TBx_{2n-1}, kt) &\geqslant M_1(z, x_{2n-1}, t) * M_1(z, SAz, t) \\
&\quad * M_1(x_{2n-1}, TBx_{2n-1}, t) * M_1(SAz, TBx_{2n-1}, t)
\end{aligned}$$

and

$$\begin{aligned}
N_1(SAz, TBx_{2n-1}, kt) &\leqslant N_1(z, x_{2n-1}, t) \diamond N_1(z, SAz, t) \\
&\quad \diamond N_1(x_{2n-1}, TBx_{2n-1}, t) \diamond N_1(SAz, TBx_{2n-1}, t).
\end{aligned}$$

Letting $n \to \infty$ and using (1.12), we get

$$M_1(SAz, z, kt) \geqslant M_1(z, SAz, t)$$

and

$$N_1(SAz, z, kt) \leqslant N_1(z, SAz, t).$$

Therefore, by Lemma B, we get

$$SAz = z = Sz \tag{1.16}$$

Applying (1.2), we get

$$M_2(BSw, ATy_{2n}, kt) \geqslant M_2(w, y_{2n}, t) * M_2(w, BSw, t) * M_2(y_{2n}, ATy_{2n}, t) * M_2(BSw, ATy_{2n}, t)$$

and

$$N_2(BSw, ATy_{2n}, kt) \leqslant N_2(w, y_{2n}, t) \diamond N_2(w, BSw, t) \diamond N_2(y_{2n}, ATy_{2n}, t) \diamond N_2(BSw, ATy_{2n}, t).$$

Letting $n \to \infty$ and using (1.14), we get

$$M_2(BSw, w, kt) \geqslant M_2(BSw, w, t)$$

and

$$N_2(BSw, w, kt) \leqslant N_2(BSw, w, t).$$

Therefore, by Lemma B, we get

$$BSw = w = Bz \tag{1.17}$$

Using inequality (1.1), we get

$$M_1(SAx_{2n}, TBz, kt) \geqslant M_1(x_{2n}, z, t) * M_1(x_{2n}, SAx_{2n}, t) * M_1(z, TBz, t) * M_1(SAx_{2n}, TBz, t)$$

and

$$N_1(SAx_{2n}, TBz, kt) \leqslant N_1(x_{2n}, z, t) \diamond N_1(x_{2n}, SAx_{2n}, t) \diamond N_1(z, TBz, t) \diamond N_1(SAx_{2n}, TBz, t).$$

Letting $n \to \infty$ and using (1.11), we get

$$M_1(z, TBz, kt) \geqslant M_1(z, TBz, t)$$

and

$$N_1(z, TBz, kt) \leqslant N_1(z, TBz, t).$$

Therefore, by Lemma B and using (1.15) – (1.17) we get $TBz = z = Tw$. Thus

$$ATw = Az = w \tag{1.18}$$

Therefore, we have

$$(1.19)\qquad \left.\begin{array}{r} SAz = TBz = Sw = Tw = z, \\ BSw = ATw = Az = Bz = w. \end{array}\right\}$$

By symmetry, (1.19) holds if one of the mappings B, S, T is continuous instead of A.

To prove uniqueness, suppose that SA and TB have a common fixed point z' also. Using (1.1), we get

$$M_1(SAz, TBz', kt) \geqslant M_1(z, z', t) * M_1(z, SAz, t) * M_1(z', TBz', t) * M_1(SAz, TBz', t)$$

and

$$N_1(SAz, TBz', kt) \leqslant N_1(z, z', t) \diamond N_1(z, SAz, t) \diamond N_1(z', TBz', t) \diamond N_1(SAz, TBz', t).$$

Therefore, we have

$$M_1(z, z', kt) \geqslant M_1(z, z', t)$$

and

$$N_1(z, z', kt) \leqslant N_1(z, z', t).$$

By Lemma B, we have $z = z'$. Similarly we can prove that w is unique common fixed point of BS and AT. This completes the proof. □

If we put $M_1 = M_2 = M$ and $N_1 = N_2 = N$ in Theorem 1, we get the following-

Corollary 2. *Let $(X, M, N, *, \diamond)$ and $(Y, M, N, *, \diamond)$ be two complete intuitionistic fuzzy metric spaces. Let A, B be mappings from X into Y and let S, T be mappings from Y into X satisfying the inequalities:*

$$(2.1)\qquad M(SAx, TBx', kt) \geqslant M(x, x', t) * M(x, SAx, t) * M(x', TBx', t) * M(SAx, TBx', t)$$

and

$$N(SAx, TBx', kt) \leqslant N(x, x', t) \diamond N(x, SAx, t) \diamond N(x', TBx', t) \diamond N(SAx, TBx', t)$$

$$(2.2)\qquad M(BSy, ATy', kt) \geqslant M(y, y', t) * M(y, BSy, t) * M(y', ATy', t) * M(BSy, ATy', t)$$

and

$$N(BSy, ATy', kt) \leqslant N(y, y', t) \diamond N(y, BSy, t) \diamond N(y', ATy', t) \diamond N(BSy, ATy', t)$$

for all $x, x' \in X$, $y, y' \in Y, t > 0$ and $k \in (0,1)$. If one of the mappings A, B, S, T is continuous, then SA and TB have a unique common fixed point z in X and BS and AThave a unique common fixed point w in Y. Further $Az = Bz = w$ and $Sw = Tw = z$.

If we put $A = B$ and $S = T$ in Theorem 1, we get the following-

Corollary 3. *Let $(X, M_1, N_1, *, \diamond)$ and $(Y, M_2, N_2, *, \diamond)$ be two complete intuitionistic fuzzy metric spaces. Let A be a mapping from X into Y and let S be a mapping from Y into X satisfying the inequalities:*

$$(3.1) \qquad \begin{aligned} M_1(SAx, SAx', kt) \geqslant & M_1(x, x', t) * M_1(x, SAx, t) \\ & * M_1(x', SAx', t) * M_1(SAx, SAx', t) \end{aligned}$$

and

$$\begin{aligned} N_1(SAx, SAx', kt) \leqslant & N_1(x, x', t) \diamond N_1(x, SAx, t) \\ & \diamond N_1(x', SAx', t) \diamond N_1(SAx, SAx', t) \end{aligned}$$

$$(3.2) \qquad \begin{aligned} M_2(ASy, ASy', kt) \geqslant & M_2(y, y', t) * M_2(y, ASy, t) \\ & * M_2(y', ASy', t) * M_2(ASy, AAy', t) \end{aligned}$$

and

$$\begin{aligned} N_2(ASy, ASy', kt) \leqslant & N_2(y, y', t) \diamond N_2(y, ASy, t) \\ & \diamond N_2(y', ASy', t) \diamond N_2(ASy, ASy', t) \end{aligned}$$

for all $x, x' \in X$, $y, y' \in Y, t > 0$ and $k \in (0,1)$. If one of the mappings A or S is continuous, then SA has a unique common fixed point z in X and AS has a unique common fixed point w in Y. Further $Az = w$ and $Sw = z$.

Example 2. Let $X = [0,2]$, $Y = [0,4]$. Define for all $t > 0$, $x, x' \in X$ and $y, y' \in Y$,

$$M_1(x, x', t) = \frac{t}{t + |x - x'|}, \quad N_1(x, x', t) = \frac{|x - x'|}{t + |x - x'|}$$

$$M_2(y, y', t) = e^{-\frac{|y - y'|}{t}}, \quad N_2(y, y', t) = \frac{e^{\frac{|y-y'|}{t}} - 1}{e^{\frac{|y-y'|}{t}}}$$

Let $a * b = ab$ and $a \diamond b = \min\{1, a + b\}$. Then $(X, M_1, N_1, *, \diamond)$ and $(Y, M_2, N_2, *, \diamond)$ are complete intuitionistic fuzzy metric spaces.

Define $A, B : X \to Y$ and $S, T : Y \to X$ by

$$Ax = 0.5, \quad Bx = \begin{cases} 0.5, & 0 \leqslant x \leqslant 1 \\ \frac{x}{4}, & 1 < x \leqslant 2 \end{cases},$$

$$Sy = \begin{cases} 0.5, & 0 \leqslant y \leqslant 1 \\ \frac{y}{5}, & 1 < y \leqslant 4 \end{cases}, \quad Ty = \begin{cases} 0.5, & 0 \leqslant y \leqslant 1 \\ \frac{y}{6}, & 1 < y \leqslant 4. \end{cases}$$

Then for $k = \frac{1}{4}$, all the conditions of Theorem 1 are satisfied. The common fixed point for SA and TB is $z = 0.5$ and that for BS and AT is $w = 0.5$. Also $Az = Bz = w$ and $Sw = Tw = z$.

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[a]DEPARTMENT OF MATHEMATICS, GOVT. ARTS AND SCIENCE P.G. COLLEGE, RATLAM (M.P.), INDIA
Email address: bhavnadeshpande@yahoo.com

[b]DEPARTMENT OF APPLIED MATHEMATICS, INSTITUTE OF ENGINEERING AND TECHNOLOGY, DAVV INDORE (M.P.), INDIA
Email address: rohitpathak8@yahoo.com