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CHARACTERIZATIONS OF AN INNER PRODUCT SPACE BY GRAPHS

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ABSTRACT. The graph of the parallelogram law is well known, which gives rise to the characterization of an inner product space among normed linear spaces [6]. In this paper we will sketch graphs of its deformations according to our previous paper [7, Theorem 3.1 and 3.2]; each one of which characterizes an inner product space among normed linear spaces. Consequently, the graphs of some classical characterizations of an inner product space follow easily.

1. INTRODUCTION

Throughout this paper $(X, \parallel \cdot \parallel)$ denotes a complex (or real) normed linear space. The most well-known characterization of a complex (or real) inner product space among complex (or real) normed linear spaces in terms of two vectors is *the parallelogram law* [2], which is *the classical Jordan-Neumann condition* [6], [2, p. 175]. In other words, X is an inner product space if and only if

$$\text{(A)} \quad \parallel x+y \parallel^2 + \parallel x-y \parallel^2 = 2 \parallel x \parallel^2 + 2 \parallel y \parallel^2$$

holds for every $x, y \in X$. As is well-known, the parallelogram law (A) can be easily given the geometrical interpretations in the unitary space from which its name is drawn. In fact, a vector x is represented by an arrow definning the direction, and its norm $\parallel x \parallel$ is the length of x. For vectors x and y, $x+y$ represents the resolvent of the arrows x and y, and (A) may be sketched as follows:

Conversely, from Fig. (A) we obtain the condition (A) by *the law of cosines* (cf. the next section). More precisely, the sum of the squares of the lengths of the diagonals in a parallelogram equals the sum of the squares of the lengths of the four sides.

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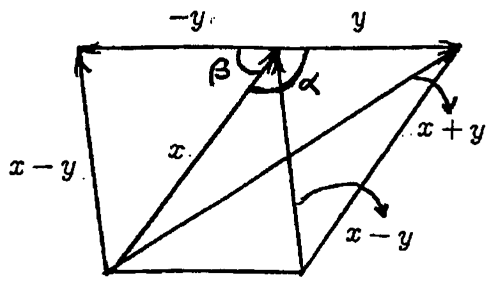


$\alpha + \beta = \pi$

Fig. (A)

For a generalization of the condition (A) we have

Lemma 1 ([7, Theorem 3.1 and 3.2]). *X is an inner product space if and only if*

$$\text{(B)} \quad r \parallel sx + ty \parallel^2 + s \parallel tx - ry \parallel^2 = (rs + t^2)(s \parallel x \parallel^2 + r \parallel y \parallel^2)$$

holds for every $x, y \in X$, *and for any nonzero real numbers* r, s, *and* t *such that* $rs + t^2 \neq 0$.

The proof of Lemma 1 was a direct approach. A direct computation for necessity, and for sufficiency we proved that all conditions for the inner product are satisfied. Remark that many classical necessary and sufficient conditions are special cases of (B) [7, Remarks]; such as conditions (1.1), (2.7), (4.24), (11.10), (11.12), (11.13), (4.23), and (11.11) in the book [1]. Nevertheless, the condition (B) may also be derived from other more general characterization with different assumptions, and this will be proved in the section three.

Naturally one might ask a question: For any two vectors in X, is the graph of a parallelogram in the unitary space the only characterization for an inner product space? We show that the answer is negative. The aim of this paper is in fact to sketch the graphs of (B) which contain exactly four deformations of a parallelogram, and each one characterizes an inner product space. Finally, it is shown consequently that the graphs of some classical conditions follow easily.

2. Graphs of Condition (B)

Theorem 1. *In the space* $(X, \parallel \cdot \parallel)$ *the graphs of the condition (B) contain exactly four deformations of a parallelogram in the unitary space, and each graph characterizes an inner product space* X.

Proof. We are going to sketch all possible graphs of the condition (B), and conversely from each graph we obtain an equation which is a special case of the condition (B).

Depending on the coefficients of the condition (B) we have to consider the following eight cases: (1) $r, s > 0$, and $t > 0$; (2) $r, s > 0$, and $t < 0$; (3) $r > 0$, $s < 0$, and $t > 0$; (4) $r > 0$, $s < 0$, and $t < 0$; (5) $r < 0$, $s > 0$, and $t > 0$; (6) $r < 0$, $s > 0$, and $t < 0$; (7) $r, s < 0$, and $t > 0$; and (8) $r, s < 0$, and $t < 0$. To this end, we let $r, s, t > 0$ and rewrite (B) into eight different equalities as follows.

(1) $$r \| sx + ty \|^2 + s \| tx - ry \|^2 = (rs + t^2)(s \| x \|^2 + r \| y \|^2).$$
Equivalently,
$$\| \sqrt{r}sx + \sqrt{r}ty \|^2 + \| \sqrt{s}tx - \sqrt{s}ry \|^2$$
$$= (\| \sqrt{r}sx \|^2 + \| \sqrt{r}ty \|^2) + (\| \sqrt{s}tx \|^2 + \| \sqrt{s}ry \|^2).$$

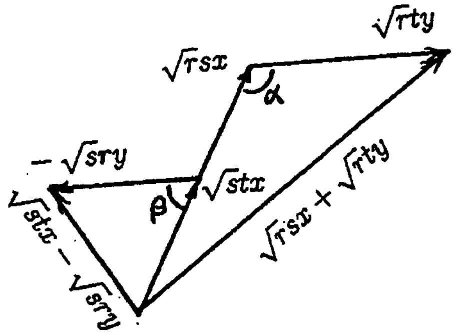
(2) $$r \| sx - ty \|^2 + s \| -tx - ry \|^2 = (rs + t^2)(s \| x \|^2 + r \| y \|^2).$$
Equivalently,
$$\| \sqrt{r}sx - \sqrt{r}ty \|^2 + \| \sqrt{s}tx + \sqrt{s}ry \|^2$$
$$= (\| \sqrt{r}sx \|^2 + \| \sqrt{r}ty \|^2) + (\| \sqrt{s}tx \|^2 + \| \sqrt{s}ry \|^2).$$

(3) $$r \| -sx + ty \|^2 - s \| tx - ry \|^2 = (-rs + t^2)(-s \| x \|^2 + r \| y \|^2).$$
Equivalently,
$$\| \sqrt{r}sx - \sqrt{r}ty \|^2 - \| \sqrt{s}tx - \sqrt{s}ry \|^2$$
$$= (\| \sqrt{r}sx \|^2 + \| \sqrt{r}ty \|^2) - (\| \sqrt{s}tx \|^2 + \| \sqrt{s}ry \|^2).$$

(4) $$r \| -sx - ty \|^2 - s \| -tx - ry \|^2 = (-rs + t^2)(-s \| x \|^2 + r \| y \|^2).$$
Equivalently,
$$\| \sqrt{r}sx + \sqrt{r}ty \|^2 - \| \sqrt{s}tx + \sqrt{s}ry \|^2$$
$$= (\| \sqrt{r}sx \|^2 + \| \sqrt{r}ty \|^2) - (\| \sqrt{s}tx \|^2 + \| \sqrt{s}ry \|^2).$$

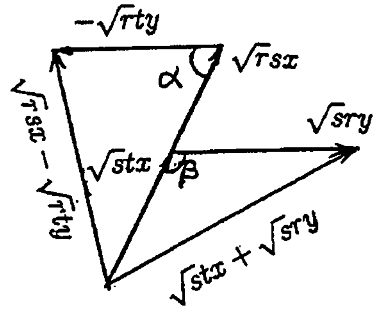
(5) $$-r \| sx + ty \|^2 + s \| tx + ry \|^2 = (-rs + t^2)(s \| x \|^2 - r \| y \|^2).$$
Equivalently,
$$\| \sqrt{r}sx + \sqrt{r}ty \|^2 - \| \sqrt{s}tx + \sqrt{s}ry \|^2$$
$$= (\| \sqrt{r}sx \|^2 + \| \sqrt{r}ty \|^2) - (\| \sqrt{s}tx \|^2 + \| \sqrt{s}ry \|^2).$$

(6) $$-r \| sx - ty \|^2 + s \| -tx + ry \|^2 = (-rs + t^2)(s \| x \|^2 - r \| y \|^2)$$
Equivalently,
$$\| \sqrt{r}sx - \sqrt{r}ty \|^2 - \| -\sqrt{s}tx + \sqrt{s}ry \|^2$$
$$= (\| \sqrt{r}sx \|^2 + \| \sqrt{r}ty \|^2) - (\| \sqrt{s}tx \|^2 + \| \sqrt{s}ry \|^2).$$



$\alpha + \beta = \pi$

Fig. (1)



$\alpha + \beta = \pi$

Fig. (2)

(7) $$-r \parallel -sx + ty \parallel^2 - s \parallel tx + ry \parallel^2 = (rs + t^2)(-s \parallel x \parallel^2 - r \parallel y \parallel^2).$$

Equivalently,

$$\begin{aligned} &\parallel \sqrt{r}sx - \sqrt{r}ty \parallel^2 + \parallel \sqrt{s}tx + \sqrt{s}ry \parallel^2 \\ &= (\parallel \sqrt{r}sx \parallel^2 + \parallel \sqrt{r}ty \parallel^2) + (\parallel \sqrt{s}tx \parallel^2 + \parallel \sqrt{s}ry \parallel^2). \end{aligned}$$

(8) $$-r \parallel -sx - ty \parallel^2 - s \parallel -tx + ry \parallel^2 = (rs + t^2)(-s \parallel x \parallel^2 - r \parallel y \parallel^2).$$

Equivalently,

$$\begin{aligned} &\parallel \sqrt{r}sx + \sqrt{r}ty \parallel^2 + \parallel \sqrt{s}tx - \sqrt{s}ry \parallel^2 \\ &= (\parallel \sqrt{r}sx \parallel^2 + \parallel \sqrt{r}ty \parallel^2) + (\parallel \sqrt{s}tx \parallel^2 + \parallel \sqrt{s}ry \parallel^2). \end{aligned}$$

We see that the equalities (1), (2), (3) and (4) are exactly the same as the equalities (8), (7), (6) and (5), respectively; and so we need only to sketch graphs of (1), (2), (3) and (4). Note that in the graphs below we assume that $\sqrt{r}s > \sqrt{s}t$ and $\sqrt{r}t \neq \sqrt{s}r$ for $r, s, t > 0$.

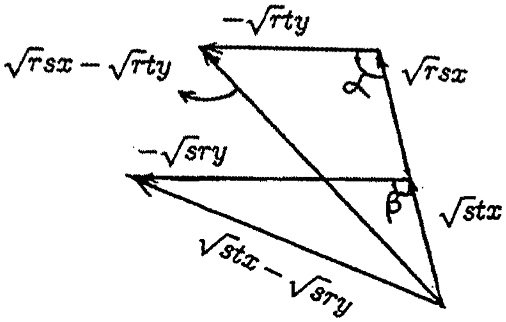


Fig. (3)

$\alpha = \beta$

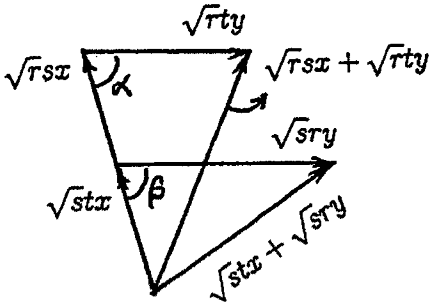


Fig. (4)

$\alpha = \beta$

Conversely, we next show that every equality (1) through (8) can be obtained by its graph and the law of cosines. Recall *the law of cosines*: In any triangle, with angles α, β, and γ and corresponding opposite sides a, b, and c, the relation $a^2 = b^2 + c^2 - 2bc \cos \alpha$ holds (the other two relations can be formed similarly). Let us consider Fig. (1) and (4) only.

By Fig. (1), $\alpha + \beta = \pi$ and $\cos \beta = -\cos \alpha$. Also,

$$\begin{aligned} \| \sqrt{r}sx + \sqrt{r}ty \|^2 &= \| \sqrt{r}sx \|^2 + \| \sqrt{r}ty \|^2 - 2 \| \sqrt{r}sx \| \| \sqrt{r}ty \| \cos\alpha \\ &= \| \sqrt{r}sx \|^2 + \| \sqrt{r}ty \|^2 - 2rst \| x \| \| y \| \cos\alpha; \end{aligned}$$

and

$$\| \sqrt{s}tx - \sqrt{s}ry \|^2 = \| \sqrt{s}tx \|^2 + \| \sqrt{s}ry \|^2 + 2rst \| x \| \| y \| \cos\alpha.$$

By Fig. (4), $\alpha = \beta$, Also,

$$\| \sqrt{r}sx + \sqrt{r}ty \|^2 = \| \sqrt{r}sx \|^2 + \| \sqrt{r}ty \|^2 - 2rst \| x \| \| y \| \cos\alpha;$$

and

$$\| \sqrt{s}tx + \sqrt{s}ry \|^2 = \| \sqrt{s}tx \|^2 + \| \sqrt{s}ry \|^2 - 2rst \| x \| \| y \| \cos\alpha.$$

Whence equalities (1) and (4) follow easily. All others can be similarly obtained, and the proof of the theorem is now completed.

Remark that a graph of deformation of a parallelogram appeared in [7, p. 131], which is merely a special case of Fig. (2).

3. A DIFFERENT PROOF OF CONDITION (B)

In this section we prove that the condition (B) in Lemma 1 may be derived from *the carlsson's condition* [3] in Lemma 2 below for $m = 3$, and conversely.

Lemma 2 ([3], or (1.16) in [1, p. 13]). *X is an inner product space if and only if for $m \geq 1$, $k = 0, 1, ..., m$, there are $a_k \neq 0$ and pair-wise linearly independent set $\{(b_k, c_k)\}_{k=0}^{m}$ such that, for all $x, y \in X$, the equality $\sum_{k=0}^{m} a_k \parallel b_k x + c_k y \parallel^2 = 0$ holds.*

For our purposes we shall consider in particular only four terms in Lemma 2 for Theorem 2 below, and prove that it is equivalent to the condition (B). Thus, the condition (B) characterizes an inner product space as expected.

Theorem 2. *Given $x, y \in X$ and consider the condition*

$$\text{(C)} \quad a_0 \parallel b_0 x + c_0 y \parallel^2 + a_1 \parallel b_1 x + c_1 y \parallel^2 = -a_2 \parallel b_2 x + c_2 y \parallel^2 - a_3 \parallel b_3 x + c_3 y \parallel^2,$$

with $a_k \neq 0$, $k = 0, 1, 2, 3$, and pair-wise linearly independent set $\{(b_k, c_k)\}_{k=0}^{3}$.

Then the condition (C) is equivalent to the condition (B).

Proof. Due to both conditions (C) and (B) we may write correspondingly $a_0 = r$, $a_1 = s$, $a_2 = -(rs + t^2)s$, and $a_3 = -(rs + t^2)r$. Also, $(b_0, c_0) = (s, t)$, $(b_1, c_1) = (t, -r)$, $(b_2, c_2) = (1, 0)$, and $(b_3, c_3) = (0, 1)$. Now,

(C)$\Rightarrow$(B). It is clear that r, s and $rs+t^2$ are nonzero. If $t = 0$, then $(b_0, c_0) = (s, 0)$ and $(b_1, c_1) = (0, -r)$. It follows that (b_0, c_0) and (b_2, c_2), (b_1, c_1) and (b_3, c_3) as well, are linearly dependent. This contradicts the condition (C), and so $t \neq 0$.

(B)$\Rightarrow$(C). By (B) a_0, a_1, a_2 and a_3 are clearly nonzero. Next, suppose that (b_0, c_0) and (b_1, c_1) were linearly dependent, then $s = pt$ and $t = -pr$ for some constant p, and so $rs + t^2 = 0$ a cotradiction to a condition in (B). All other are obviously pair-wise linearly independent. Therefore, $\{(b_k, c_k)\}_{k=0}^{3}$ is a pair-wise linearly independent set, and the proof is completed. □

Notice that the advantage of the condition (B) over (C) is clear. Since the combined 12 coeficients of x and y in the condition (C) (i.e., a_i, b_i and c_i for $i = 0, 1, 2, 3$) has been significantly reduced to only 3 in (B) (i.e., r, s and t); and the pair-wise linearly independent set is not assumed in (B). Therefore, it is relatively easy to sketch graphs of (B).

4. Graphs of Some Classical Conditions

In this final section we will consider graphs of some classical characterizations of an inner product space in terms of two vectors. The following list of eight conditions with condition number is from the book [1]. Each one is a special case of the condition (B) as is described in the bracket $\{\cdots\}$ and hence we may omit sketching.

(1.1) *The Jordan-Neumann condition* [6]: For any $x, y \in X$,

$$\| x+y \|^2 + \| x-y \|^2 = 2 \| x \|^2 + 2 \| y \|^2 .$$

$\{$Let $r = s = t = 1 > 0$ in (B); a special case of (1)$\}$.

Its graph is precisely Fig. (A).

(2.7) *The Ficken's condition* [5]: For any unit vectors $x, y \in X$ and any real number $\lambda \neq 0$,

$$\| x + \lambda y \| = \| y + \lambda x \| .$$

$\{$Let $r = 1 > 0$, $s = -1 < 0$, and $t = \frac{-1}{\lambda}$ in (B); a special case of (4) if $\lambda > 0$, and of (3) if $\lambda < 0\}$.

Its graph is similar in shape to Fig. (4) if $\lambda > 0$, and to Fig. (3) if $\lambda < 0$.

(4.23) *The Oman's condition* [9]: For any $x, y \in X$ there exists a real number $\lambda \in (0,1)$ such that

$$\| \lambda x + (1-\lambda) y \|^2 + \lambda(1-\lambda) \| x - y \|^2 = \lambda \| x \|^2 + (1-\lambda) \| y \|^2 .$$

$\{$Let $r = t = 1 > 0$, and $s = \frac{\lambda}{1-\lambda} > 0$ in (B); a special case of (1)$\}$.

Its graph is similar to Fig. (1).

(11.10) For any unit vectors $x, y \in X$,

$$\| 3x - y \|^2 = 4 + 3 \| x - y \|^2 .$$

$\{$Let $r = t = 1 > 0$, and $s = -3 < 0$ in (B); a special case of (3)$\}$.

Its graph is similar to Fig. (3).

(11.12) For any unit vectors $x, y \in X$ and any real number $\lambda \neq 0$ such that $\|x-y\| = 1$,

$$\| y - \lambda x \|^2 = 1 - \lambda + \lambda^2.$$

$\{$Let $r = 1 > 0$, $s = -\lambda$, and $t = 1$ in (B); a special case of (3) if $\lambda > 0$, and of (1) if $\lambda < 0\}$.

Its graph is similar to Fig. (3) if $\lambda > 0$, and to Fig. (1) if $\lambda < 0$.

(11.13) For any unit vectors $x, y \in X$,

$$\| x + 2y \|^2 = 2 \| x + y \|^2 + 1.$$

{Let $r = -2 < 0$, and $s = t = 1 > 0$ in (B); a special case of (5)}.

Its graph is similar to Fig. (4).

(4.24) *The Day's and Oman's condition* [4,9]: For any $x, y \in X$ there exist real numbers $\lambda, \mu \neq 1$ and $\lambda, \mu \neq 0$ such that

$$\mu(1-\mu) \| \lambda x + (1-\lambda) y \|^2 + \lambda(1-\lambda) \| \mu x - (1-\mu) y \|^2$$
$$= (\lambda + \mu - 2\lambda\mu)[\lambda\mu \| x \|^2 + (1-\lambda)(1-\mu) \| y \|^2].$$

{Let $r = \frac{1-\mu}{\mu}$, $s = \frac{\lambda}{1-\lambda}$, and $t = 1 > 0$ in (B)}.

Now, we have to consider the following cases.

(a) Both $r = \frac{1-\mu}{\mu}$ and $s = \frac{\lambda}{1-\lambda} > 0$, a special case of (1), i.e., $0 < \mu, \lambda < 1$; then its graph is similar to Fig. (1).

(b) Both $r = \frac{1-\mu}{\mu}$ and $s = \frac{\lambda}{1-\lambda} < 0$, a special case of (7), i.e., $\mu, \lambda < 0$, or $\mu, \lambda > 1$; then its graph is similar to Fig. (2).

(c) $r = \frac{1-\mu}{\mu} > 0$ and $s = \frac{\lambda}{1-\lambda} < 0$, a special case of (3), i.e., $0 < \mu < 1$, $\lambda < 0$ or $\lambda > 1$; then its graph is similar to Fig. (3).

(d) $r = \frac{1-\mu}{\mu} < 0$ and $s = \frac{\lambda}{1-\lambda} > 0$, a special case of (5), i.e., $\mu < 0$ or $\mu > 1$, and $0 < \lambda < 1$; then its graph is similar to Fig. (4).

(11.11) For any unit vectors $x, y \in X$ such that $x \neq y$,

$$\left\| x + \frac{x-y}{\| x - y \|} \right\|^2 = 2 + \| x - y \| .$$

{Let $r = t = 1 > 0$, and $s = -(\| x - y \| + 1) < 0$ in (B); a special case of (3)}.

Its graph is similar to Fig. (3).

In conclusion, we remark that conversely, as was mentioned in the section two, each condition above can be obtained by its graph and the law of cosines. Also notice that the (a,b,c,d)-orthogonality characterization in a normed linear space could be found in our article [8].

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