J. Korean Soc. Math. Educ. Ser. B: Pure Appl. Math. ISSN 1226-0657
Volume 17, Number 1 (February 2010), Pages 1–27

ON THE SEMILOCAL CONVERGENCE OF A NEWTON-TYPE METHOD OF ORDER THREE

IOANNIS K. ARGYROS [a] AND SAÏD HILOUT [b]

ABSTRACT. Wu and Zhao [17] provided a semilocal convergence analysis for a Newton-type method on a Banach space setting following the ideas of Frontini and Sormani [9]-[11]. In this study first: we point out inconsistencies between the hypotheses of Theorem 1 and the two examples given in [17], and then, we provide the proof in affine invariant form for this result. Then, we also establish new convergence results with the following advantages over the ones in [17]: weaker hypotheses, and finer error estimates on the distances involved.

A numerical example is also provided to show that our results apply in cases other fail [17].

1. INTRODUCTION

In this study we are concerned with the problem of approximating a locally unique solution $x^\star$ of equation

$$F(x) = 0, \tag{1.1}$$

where F is a twice Fréchet–differentiable operator defined on a convex subset $\mathcal{D}$ of a Banach space $\mathcal{X}$ with values in a Banach space $\mathcal{Y}$.

A large number of problems in applied mathematics and also in engineering are solved by finding the solutions of certain equations. For example, dynamic systems are mathematically modeled by difference or differential equations, and their solutions usually represent the states of the systems. For the sake of simplicity, assume that a time–invariant system is driven by the equation $\dot{x} = Q(x)$, for some suitable operator Q, where x is the state. Then the equilibrium states are determined by solving equation (1.1). Similar equations are used in the case of discrete systems.

Received by the editors April 27, 2009. Accepted December 12, 2009.
2000 *Mathematics Subject Classification.* 65G99, 65J15, 47H17, 49M15.
Key words and phrases. Newton-type methods, Banach space, majorizing sequence, Fréchet-derivatives, semilocal convergence.

The unknowns of engineering equations can be functions (difference, differential, and integral equations), vectors (systems of linear or nonlinear algebraic equations), or real or complex numbers (single algebraic equations with single unknowns). Except in special cases, the most commonly used solution methods are iterative–when starting from one or several initial approximations a sequence is constructed that converges to a solution of the equation. Iteration methods are also applied for solving optimization problems. In such cases, the iteration sequences converge to an optimal solution of the problem at hand. Since all of these methods have the same recursive structure, they can be introduced and discussed in a general framework.

Frontini and Sormani [9]-[11] introduced a modified Newton method (MNM) with:

$$\int_{x_n}^{x} G'(t)\,dt \approx \frac{x-x_n}{2}\,(G'(x_n)-G'(x))$$

as follows:

$$x_{n+1} = x_n - 2\,(G'(x_n) + G'(x_n - G'(x_n)^{-1}\,G(x_n))^{-1}\,G(x_n),$$

where G is a real or complex function [9]-[11].

The cubical convergence of (MNM) was also established in [9]-[11] under various assumptions.

Recently, the cubical convergence of Newton–type method (NTM) related to (MNM), and given for initial iterate $x_0 \in \mathcal{D}$, by:

$$y_n = x_n - F'(x_n)^{-1}\,F(x_n) \quad (n \geq 0),$$

$$x_{n+1} = x_n - 2\,(F'(x_n) + F'(y_n))^{-1}\,F(x_n)$$

was established by Wu and Zhao in [17].

There are some inconsistencies between the main Theorem 1, and two numerical examples in [17].

Indeed, [17, Theorem 1] used the set of sufficient convergence conditions:

Conditions 1.1.

$$\| F'(x_0)^{-1} \| \leq \alpha,$$

$$\| F'(x_0)^{-1}\,F(x_0) \| \leq \eta,$$

$$\| F''(x) \| \leq M,$$

$$\| F''(x) - F''(y) \| \leq N\ \| x-y \|,$$

for all $x,\ y \in \mathcal{D}$,

$$M\left(1 + \frac{5\ N}{3\ M^2\ \alpha}\right) \leq K,$$

$$h = K\ \alpha\ \eta \leq \frac{1}{2},$$

and

$$\overline{U}(x_0, t^\star) = \{x \in \mathcal{X} : \parallel x - x_0 \parallel \leq t^\star\} \subseteq \mathcal{D},$$

where, $t^\star$ *is a uniquely determined point.*

If one now looks at Example 1 in [17] (similar observations for [17, Example 2]), where they consider $\mathcal{X} = \mathcal{Y}$, $\mathcal{D} = [1,3]$, $x_0 = 2$, and function F given by

$$F(x) = x^3 - 2\ x - 5.$$

Using the above conditions, we obtain $\alpha = .1$, $\eta = .1$, as they do but $M = 18$, $N = 6$, $K = 23.555$, and $h = .23555$, where as they get: $M = .18$, $N = .6$, $K = 1.85556$, and $h = .185556$, which were to be the correct values if the results were given in affine invariant form (instead of non–affine invariant form), $\alpha = 1$, and $F''(x)$, $F''(y)$ are replaced above by $F'(x_0)^{-1}\ F''(x)$, and $F'(x_0)^{-1}\ F''(y)$ $(x, y \in \mathcal{D})$.

That is why, we first show in Section 2, that the results in [17, Theorem 1] hold indeed in affine invariant form. The advantages of this approach have been explained in [3], [8]. Then, we show that under the same or weaker hypotheses, but the same condition on h, we can improve the estimates on the upper bounds of the distances $\parallel x_{n+1} - x_n \parallel$, $\parallel y_n - x_n \parallel$, $\parallel x_n - x^\star \parallel$, $\parallel y_n - x^\star \parallel$, $(n \geq 0)$.

Then, in Section 3, we provide sufficient convergence conditions which can be weaker that the ones given in Section 2.

Finally, numerical examples are provided to show that the results of Section 3 can apply, but the ones in Section 2 (or in [17]) do not.

Note that the set of Conditions 1.1 has been used by us in [3]-[7]. Other fast iterative methods have been studied in [1]-[4], [8]-[16].

2. Semilocal Convergence Analysis I for (NTM)

We show the main seimlocal convergence result for (NTM):

Theorem 2.1. *Let* $F : \mathcal{D} \subseteq \mathcal{X} \longrightarrow \mathcal{Y}$ *be a twice Fréchet–differentiable operator. Assume:*

there exist $x_0 \in \mathcal{D}$, *constants* $\eta \geq 0$, $M \geq 0$, $N \geq 0$, *and* $K > 0$, *such that for all* $x, y \in \mathcal{D}$:

$$F'(x_0)^{-1} \in \mathcal{L}(\mathcal{Y}, \mathcal{X}), \tag{2.1}$$

$$\| F'(x_0)^{-1} \; F(x_0) \| \leq \eta, \tag{2.2}$$

$$\| F'(x_0)^{-1} \; F''(x) \| \leq M, \tag{2.3}$$

$$\| F'(x_0)^{-1} \; (F''(x) - F''(y)) \| \leq N \; \| x - y \|, \tag{2.4}$$

$$M \left(1 + \frac{5\,N}{3\,M^2}\right) \leq K, \qquad M \neq 0 \tag{2.5}$$

$$h = K\,\eta \leq \frac{1}{2}, \tag{2.6}$$

and

$$\overline{U}(y_0, t^\star - \eta) \subseteq \mathcal{D}. \tag{2.7}$$

Then,

sequences $\{x_n\}$, $\{y_n\}$, $(n \geq 0)$ *generated by (NTM) are well defined, remain in* $\overline{U}(x_0, t^\star)$ *for all* $n \geq 0$, *and converge to a solution* $x^\star \in \overline{U}(x_0, t^\star)$ *of equation* $F(x) = 0$, *which is the unique solution of equation* $F(x) = 0$ *in* $U(x_0, t^{\star\star})$.

Moreover the following estimates hold for all $n \geq 0$:

$$\| x_n - x^\star \| \leq t^\star - t_n = \frac{(1 - \theta^2)\,\eta}{1 - \theta^{3^n}}\,\theta^{3^n - 1}, \qquad \text{for } h < \frac{1}{2},$$

and

$$\| x_n - x^\star \| \leq t^\star - t_n \leq \frac{2\,\eta}{3^n} \qquad \text{for } h = \frac{1}{2},$$

where,

$$t_0 = 0, \quad s_n = t_n - f'(t_n)^{-1}\,f(t_n),$$

$$t_{n+1} = t_n - 2\,(f'(t_n) + f'(s_n))^{-1}\,f(t_n),$$

$$t^\star = \frac{1 - \sqrt{1 - 2\,h}}{h}\,\eta, \qquad t^{\star\star} = \frac{1 + \sqrt{1 - 2\,h}}{h}\,\eta, \quad h \neq 0,$$

and

$$\theta = \frac{t^\star}{t^{\star\star}}.$$

Proof. We shall show using induction on k:

$$
(2.8)\qquad
\begin{aligned}
& x_k \in \overline{U}(x_0, t_k),\\
& \| F'(x_k)^{-1}\ F'(x_0) \| \leq -f'(t_k)^{-1},\\
& \| y_k - x_k \| \leq s_k - t_k,\\
& \| 2\ (F'(x_k) + F'(y_k))^{-1}\ F'(x_0) \| \leq -2\ (f'(t_k) + f'(s_k))^{-1},\\
& \| x_{k+1} - y_k \| \leq t_{k+1} - s_k,\\
\text{and}\quad & \| x_{k+1} - x_k \| \leq t_{k+1} - t_k.
\end{aligned}
$$

Estimates (2.8) hold for $k = 0$ by the initial conditions. Assume (2.8) hold for all $m \leq k$.

Then, we have by the induction hypotheses:

$$
\| x_{k+1} - x_0 \| \leq \| x_{k+1} - x_k \| + \| x_k - x_0 \| \leq t_{k+1} - t_k + t_k = t_{k+1},
$$

and

$$
(2.9)\qquad
\begin{aligned}
\| F'(x_0)^{-1}\ (F'(x_{k+1}) - F'(x_0)) \| &\leq M\ \| x_{k+1} - x_0 \|\\
&\leq M\ t_{k+1}\\
&< K\ t^\star = K\ \eta\ \frac{1 - \sqrt{1 - 2\ h}}{h} \leq 1.
\end{aligned}
$$

In view of (2.9), and the Banach lemma on invertible operators [7], [15],

$$
F'(x_{k+1})^{-1},
$$

and

$$
(2.10)\qquad
\begin{aligned}
\| F'(x_{k+1})^{-1}\ F'(x_0) \| &\leq \frac{1}{1 - \| F'(x_0)^{-1}\ (F'(x_{k+1}) - F'(x_0)) \|}\\
&\leq \frac{1}{1 - M\ \| x_{k+1} - x_0 \|}\\
&\leq \frac{1}{1 - K\ \| x_{k+1} - x_0 \|} \leq \frac{1}{1 - K\ t_{k+1}} = -f'(t_{k+1})^{-1}.
\end{aligned}
$$

Using (2.4), and the induction hypotheses, we obtain:

$$\begin{aligned}(2.11)\quad & \| F'(x_0)^{-1} \left(\int_0^1 F''(x_k + t\,(y_k - x_k))\,(1-t)\,dt \right. \\ & \left. - \frac{1}{2} \int_0^1 F''(y_k + t\,(y_k - x_k))\,dt \right) \| \\ & \leq \| F'(x_0)^{-1} \left(\int_0^1 F''(x_k + t\,(y_k - x_k)) - F''(x_k) \right) (1-t)\,dt \| \\ & + \| \frac{1}{2} F'(x_0)^{-1} \left(\int_0^1 F''(y_k + t\,(y_k - x_k)) - F''(x_k) \right) dt \| \\ & \leq \frac{N}{6} \| y_k - x_k \| + \frac{N}{4} \| y_k - x_k \| = \frac{5\,N}{12} \| y_k - x_k \| .\end{aligned}$$

Using the Ostrowski–type approximation [1]–[8], [15], [17]:

$$(2.12)\quad \begin{aligned} F(x_{k+1}) &= \int_0^1 F''(y_k + t\,(x_{k+1} - y_k))\,(1-t)\,dt\,(x_{k+1} - y_k)^2 \\ &+ \frac{1}{2} \int_0^1 F''\,(x_k + t(y_k - x_k))\,dt\,(y_k - x_k)\,(x_{k+1} - y_k) \\ &+ \int_0^1 F''\,(x_k + t(y_k - x_k))\,(1-t)\,dt\,(y_k - x_k)^2 \\ &- \frac{1}{2} \int_0^1 F''\,(x_k + t(y_k - x_k))\,dt\,(y_k - x_k)^2, \end{aligned}$$

the estimate [17]:

$$(2.13)\quad \begin{aligned} \frac{(s_k - t_k)^2}{t_{k+1} - s_k} &= \frac{(a_k\,b_k)^2}{(a_k + b_k)^2}\,\frac{(a_k^2 + b_k^2 + a_k\,b_k)\,(a_k + b_k)}{a_k^2\,b_k^2} \\ &\leq a_k + b_k \leq t^\star + t^{\star\star} = \frac{2}{K}, \end{aligned}$$

and (2.11), where

$$(2.14)\qquad a_k = t^\star - t_k, \qquad b_k = t^{\star\star} - t_k,$$

we obtain in turn:

(2.15)

$$\| F'(x_0)^{-1}\,F(x_{k+1}) \| \leq \frac{M}{2} \| x_{k+1} - y_k \|^2 + \frac{M}{2} \| y_k - x_k \| \, \| x_{k+1} - y_k \|$$

$$
\begin{aligned}
& + \frac{5\,N}{12} \; \| \, y_k - x_k \, \|^3 \\
\leq \; & \frac{M}{2} \, (t_{k+1} - s_k)^2 + \frac{M}{2} \, (s_k - t_k) \, (t_{k+1} - s_k) + \frac{5\,N}{12} \, (s_k - t_k)^3 \\
\leq \; & \frac{M}{2} \, (t_{k+1} - s_k)^2 + \frac{1}{2} \left(M + \frac{5\,N}{12} \, \frac{2\,(s_k - t_k)^2}{t_{k+1} - s_k} \right) (s_k - t_k) \, (t_{k+1} - s_k) \\
\leq \; & \frac{M}{2} \, (t_{k+1} - s_k)^2 + \frac{1}{2} \left(M + \frac{5\,N}{12} \right) (s_k - t_k) \, (t_{k+1} - s_k) \\
\leq \; & \frac{K}{2} \, (t_{k+1} - s_k)^2 + \frac{K}{2} \, (s_k - t_k) \, (t_{k+1} - s_k) = f(t_{k+1}).
\end{aligned}
$$

In view of (NTM), (2.10), and (2.15), we get:

$$
\begin{aligned}
\| \, y_{k+1} - x_{k+1} \, \| & = \| -F'(x_{k+1})^{-1} \, F(x_{k+1}) \, \| \\
& \leq \| \, F'(x_{k+1})^{-1} \, F'(x_0) \, \| \; \| \, F'(x_0)^{-1} \, F(x_{k+1}) \, \| \\
& \leq -f'(t_{k+1})^{-1} \, f(t_{k+1}) = s_{k+1} - t_{k+1}.
\end{aligned}
\tag{2.16}
$$

As in (2.10), we need similar estimate for $\| \, 2 \, (F'(x_{k+1}) + F'(y_{k+1}))^{-1} \, F'(x_0) \, \|$. We have using (2.3), and the induction hypotheses:

(2.17)

$$
\begin{aligned}
& \| \, F'(x_0)^{-1} \left(\frac{F'(x_{k+1}) + F'(y_{k+1})}{2} - F'(x_0) \right) \| \\
& \leq \frac{1}{2} \left(\| \, F'(x_0)^{-1} \, (F'(x_{k+1}) - F'(x_0)) \, \| + \| \, F'(x_0)^{-1} \, (F'(y_{k+1}) - F'(x_0)) \, \| \right) \\
& \leq \frac{1}{2} \, M \left(\| \, x_{k+1} - x_0 \, \| + \| \, y_{k+1} - x_0 \, \| \right) \\
& \leq \frac{1}{2} \, M \, (t_{k+1} + s_{k+1}) < K \, t^\star \leq 1.
\end{aligned}
$$

In view of (2.17), and the lemma on invertible operators $(F'(x_{k+1}) + F'(y_{k+1}))^{-1}$ exists, and

$$
\| \, 2 \, (F'(x_{k+1}) + F'(y_{k+1}))^{-1} \, F'(x_0) \, \| \; \leq \; \frac{1}{1 - \dfrac{K}{2} \, (t_{k+1} + s_{k+1})}
\tag{2.18}
$$

$$\le \frac{2}{1 - K\ t_{k+1} + 1 - K\ s_{k+1}}$$

$$= -2\ (f'(t_{k+1}) + f'(s_{k+1}))^{-1}.$$

Using (NTM), (2.16), and (2.18), we obtain in turn:

(2.19)

$$\| x_{k+2} - y_{k+1} \|$$

$$= \| F'(x_{k+1})^{-1}\ F(x_{k+1}) - \left(\frac{F'(x_{k+1}) + F'(y_{k+1})}{2}\right)^{-1}\ F(x_{k+1}) \|$$

$$\le \| \left(\frac{F'(x_{k+1}) + F'(y_{k+1})}{2}\right)^{-1}\ F'(x_0) \|$$

$$\times \| F'(x_0)^{-1} \left(\frac{F'(x_{k+1}) + F'(y_{k+1})}{2} - F'(x_{k+1})\right) \|\ \| F'(x_{k+1})^{-1}\ F(x_{k+1}) \|$$

$$\le -2\ (f'(t_{k+1}) + f'(s_{k+1}))^{-1}\ \frac{M}{2}\ (s_{k+1} - t_{k+1})^2$$

$$\le -K\ (f'(t_{k+1}) + f'(s_{k+1}))^{-1}\ (s_{k+1} - t_{k+1})^2$$

$$= 2\ (f'(t_{k+1}) + f'(s_{k+1}))^{-1} \left(\frac{f'(t_{k+1}) + f'(s_{k+1})}{2} - f'(t_{k+1})\right) f'(t_{k+1})^{-1}\ f(t_{k+1})$$

$$= f'(t_{k+1})^{-1}\ f'(t_{k+1}) - 2\ (f'(t_{k+1}) + f'(s_{k+1}))^{-1}\ f(t_{k+1})$$

$$= t_{k+2} - s_{k+1}.$$

Moreover, using (2.16), and (2.19), we get

$$\begin{aligned} \| x_{k+2} - x_{k+1} \| &\le \| x_{k+2} - y_{k+1} \| + \| y_{k+1} - x_{k+1} \| \\ &\le t_{k+2} - t_{k+1}. \end{aligned} \tag{2.20}$$

The induction for estimates (2.8) is now complete.

It follows that sequence $\{x_n\}$ is Cauchy in a Banach space $\mathcal{X}$, and as such it converges to some $x^\star \in \overline{U}(x_0, t^\star)$ (since $\overline{U}(x_0, t^\star)$ is a closed set).

By letting $k \longrightarrow \infty$ in (2.15), we get $F(x^\star) = 0$.

To show uniqueness, let $y^\star \in \overline{U}(x_0, t^\star)$, with $F(y^\star) = 0$.

Using (2.3), we get in turn

$$
\begin{aligned}
& \| F'(x_0)^{-1} \left(\int_0^1 F'(x^\star + t\ (y^\star - x^\star)) - F'(x_0) \right) dt \| \\
& \leq M \int_0^1 \| x^\star + t\ (y^\star - x^\star) - x_0 \|\ dt \\
& \leq M \int_0^1 \Big((1-t)\ \| x^\star - x_0 \| + t\ \| y^\star - x_0 \| \Big)\ dt \\
& < \frac{1}{2}\ (t^\star + t^{\star\star}) \leq 1.
\end{aligned} \tag{2.21}
$$

It view of (2.21), and the Banach lemma on invertible operators that:

$$
\mathcal{M}^{-1} = \left(\int_0^1 F'(x^\star + t\ (y^\star - x^\star))\ dt \right)^{-1}
$$

exists.

In view of the identity:

$$
F(x^\star) - F(y^\star) = \mathcal{M}\ (y^\star - x^\star), \tag{2.22}
$$

we deduce $x^\star = y^\star$.

The rest of the proof as identical to the one in [17] is omitted.

That completes the proof of Theorem 2.1. □

Remark 2.2. In view of (2.3), there exists M_0, such that

$$
\| F'(x_0)^{-1}\ (F'(x) - F'(x_0)) \| \leq M_0\ \| x - x_0 \|, \quad \text{for all } x \in \mathcal{D}. \tag{2.23}
$$

Note that

$$
M_0 \leq M \leq K \tag{2.24}
$$

holds in general, and $\frac{K}{M_0}$ can be arbitrarily large [2], [3], [7].

Let us define function g by:

$$
g(t) = \frac{1}{2}\ M_0\ t^2 - t + \eta, \tag{2.25}
$$

and sequences $\{\bar{s}_n\}$, $\{\bar{t}_n\}$ by:

$$
\bar{t}_0 = 0, \quad \bar{s}_n = \bar{t}_n - g'(\bar{t}_n)^{-1}\ f(\bar{t}_n), \tag{2.26}
$$

$$
\bar{t}_{n+1} = \bar{t}_n - 2\ (g'(\bar{t}_n) + g'(\bar{s}_n))^{-1}\ f(\bar{t}_n). \tag{2.27}
$$

Note that in estimates (2.9), (2.10), (2.17), (2.18), and (2.21), M_0 can replace M, and K.

Then, by comparing sequences $\{s_n\}$, $\{t_n\}$ to $\{\overline{s}_n\}$, $\{\overline{t}_n\}$, using (2.24), and induction, we get:

A simple induction argument, shows that if $M_0 < K$, then,

$$\overline{s}_n \leq s_n, \tag{2.28}$$

$$\overline{t}_n \leq t_n, \tag{2.29}$$

$$\overline{s}_{n+1} - \overline{s}_n \leq s_{n+1} - s_n, \tag{2.30}$$

and

$$\overline{t}_{n+1} - \overline{t}_n \leq t_{n+1} - t_n. \tag{2.31}$$

Note also that if $M_0 < M$, then strict inequality holds in estimates (2.28)–(2.31) for $n > 1$.

Hence, under the same hypotheses, and computational cost as in Theorem 2.1, we showed more precise sequences $\{\overline{s}_n\}$, $\{\overline{t}_n\}$ can be used. Note also that the computation of M requires that of M_0.

If $t^{\star\star} < R \leq \dfrac{2}{M_0} - t^\star$, then the uniqueness ball is extended from $U(x_0, t^{\star\star})$ to $U(x_0, R)$, since M_0 can replace M in (2.21).

Note that estimates of the form (2.5) have been used on the cubically convergent midpoint method, and other Newton–type methods [1]-[7].

We provide three examples where $M_0 < M$, so the advantages can take place.

Example 2.3. Returning back to numerical example of Section 1, and using (2.23), we get:

$$13 = M_0 < M = 16. \tag{2.32}$$

In view of (2.32), the improvements stated in Remark 2.2 follow.

Example 2.4. Let $\mathcal{X} = \mathcal{Y} = \mathcal{C}[0,1]$ be the space of real–valued continuous functions defined on the interval $[0,1]$ with norm

$$\| x \| = \max_{0 \leq s \leq 1} |x(s)|.$$

Let $\theta \in [0,1]$ be a given parameter. Consider the "Cubic" integral equation

$$u(s) = u^3(s) + \lambda\, u(s) \int_0^1 q(s,t)\, u(t)\, dt + y(s) - \theta. \tag{2.33}$$

Here the kernel $q(s,t)$ is a continuous function of two variables defined on $[0,1] \times [0,1]$; the parameter λ is a real number called the "albedo" for scattering; $y(s)$ is a given continuous function defined on $[0,1]$ and $x(s)$ is the unknown function sought in $C[0,1]$. Equations of the form (2.33) arise in the kinetic theory of gasses [3]. For simplicity, we choose $u_0(s) = y(s) = 1$, and $q(s,t) = \dfrac{s}{s+t}$, for all $s \in [0,1]$, and $t \in [0,1]$, with $s+t \neq 0$. If we let $\mathcal{D} = U(u_0, 1-\theta)$, and define the operator F on $\mathcal{D}$ by

$$F(x)(s) = x^3(s) - x(s) + \lambda\, x(s) \int_0^1 q(s,t)\, x(t)\, dt + y(s) - \theta, \tag{2.34}$$

for all $s \in [0,1]$, then every zero of F satisfies equation (2.33). We have the estimates

$$\max_{0 \leq s \leq 1} \left| \int \frac{s}{s+t}\, dt \right| = \ln 2.$$

Therefore, if we set $\xi = \| F'(u_0)^{-1} \|$, then it follows from hypotheses of Theorem 2.1 that:

$$\eta = \xi\, (|\lambda| \ln 2 + 1 - \theta),$$

$$M = 2\,\xi\, (|\lambda| \ln 2 + 3\,(2-\theta)) \quad \text{and} \quad M_0 = \xi\, (2\,|\lambda| \ln 2 + 3\,(3-\theta)).$$

Note also that $M_0 < M$ for all $\theta \in [0,1]$.

Example 2.5. Consider the following nonlinear boundary value problem [3]

$$\begin{cases} u'' = -u^3 - \gamma\, u^2 \\ u(0) = 0, \quad u(1) = 1. \end{cases}$$

It is well known that this problem can be formulated as the integral equation

$$u(s) = s + \int_0^1 Q(s,t)\ (u^3(t) + \gamma\, u^2(t))\ dt \tag{2.35}$$

where, Q is the Green function:

$$Q(s,t) = \begin{cases} t\ (1-s), & t \leq s \\ s\ (1-t), & s < t. \end{cases}$$

We observe that

$$\max_{0 \leq s \leq 1} \int_0^1 |Q(s,t)| = \frac{1}{8}.$$

Let $\mathcal{X} = \mathcal{Y} = C[0,1]$, with norm

$$\| x \| = \max_{0 \le s \le 1} |x(s)|.$$

Then problem (2.35) is in the form (1.1), where, $F : \mathcal{D} \longrightarrow \mathcal{Y}$ is defined as

$$[F(x)]\,(s) = x(s) - s - \int_0^1 Q(s,t)\ (x^3(t) + \gamma\ x^2(t))\ dt.$$

It is easy to verify that the Fréchet derivative of F is defined in the form

$$[F'(x)v]\,(s) = v(s) - \int_0^1 Q(s,t)\ (3\ x^2(t) + 2\ \gamma\ x(t))\ v(t)\ dt.$$

If we set $u_0(s) = s$, and $\mathcal{D} = U(u_0, R)$, then since $\| u_0 \| = 1$, it is easy to verify that $U(u_0, R) \subset U(0, R+1)$. It follows that $2\ \gamma < 5$, then

$$\begin{aligned}
\| I - F'(u_0) \| &\le \frac{3\ \| u_0 \|^2 + 2\ \gamma\ \| u_0 \|}{8} = \frac{3 + 2\ \gamma}{8}, \\
\| F'(u_0)^{-1} \| &\le \frac{1}{1 - \dfrac{3 + 2\ \gamma}{8}} = \frac{8}{5 - 2\ \gamma}, \\
\| F(u_0) \| &\le \frac{\| u_0 \|^3 + \gamma\ \| u_0 \|^2}{8} = \frac{1 + \gamma}{8}, \\
\| F(u_0)^{-1}\ F(u_0) \| &\le \frac{1 + \gamma}{5 - 2\ \gamma}.
\end{aligned}$$

On the other hand, for $x, y \in \mathcal{D}$, we have:

$$[(F'(x) - F'(y))v]\,(s) = - \int_0^1 Q(s,t)\ (3\ x^2(t) - 3\ y^2(t) + 2\ \gamma\ (x(t) - y(t)))\ v(t)\ dt.$$

Consequently,

$$\begin{aligned}
\| F'(x) - F'(y) \| &\le \frac{\| x - y \|\ \ (2\ \gamma + 3\ (\| x \| + \| y \|))}{8} \\
&\le \frac{\| x - y \|\ \ (2\ \gamma + 6\ R + 6\ \ \| u_0 \|)}{8} \\
&= \frac{\gamma + 6\ R + 3}{4}\ \ \| x - y \|, \\
\| F'(x) - F'(u_0) \| &\le \frac{\| x - u_0 \|\ \ (2\ \gamma + 3\ (\| x \| + \| u_0 \|))}{8} \\
&\le \frac{\| x - u_0 \|\ \ (2\ \gamma + 3\ R + 6\ \ \| u_0 \|)}{8} \\
&= \frac{2\ \gamma + 3\ R + 6}{8}\ \ \| x - u_0 \|.
\end{aligned}$$

Therefore, conditions of Theorem 2.1 hold with

$$\eta = \frac{1+\gamma}{5-2\,\gamma}, \quad M = \frac{\gamma+6\,R+3}{4}, \quad M_0 = \frac{2\,\gamma+3\,R+6}{8}.$$

Note also that $M_0 < M$.

In the next section, we provide sufficient convergence conditions, which can hold where as (2.6) does not. This way the applicability of (NTM) can be extended even further.

3. Semilocal Convergence Analysis II of (NTM)

We need to define some constants and sequences:

Let $M_0 \geq 0$, $M \geq 0$, $M_1 \geq 0$, $N_0 \geq 0$, $N \geq 0$, and $\eta \geq 0$ be given constants. Define constants δ_0, δ_1, δ_2, and w_∞ by:

$$\delta_0 = \frac{M_1\,\eta}{1-\dfrac{M_0}{2}\,\eta} < 2, \quad M_0\,\eta \neq 2, \tag{3.1}$$

$$\delta_1 = \frac{2\,M}{M_0+M+\sqrt{(M_0+M)^2+12\,M_0\,M}} < 1, \quad M_0 \neq 0 \text{ or } M \neq 0, \tag{3.2}$$

(3.3)

$$\delta_2 = \frac{2\,(6\,M+5\,N\,\eta)}{5\,N\,\eta+12\,M_0+\sqrt{(5\,N\,\eta+12\,M_0)^2+24\,(M+2\,M_0)\,(6\,M+5\,N\,\eta)}} < 1,$$

$$w_\infty = \frac{1-M_0\,\eta}{1+M_0\,\eta}, \tag{3.4}$$

and, scalar sequences $\{t_n\}$, $\{s_n\}$ $(n \geq 0)$ by:

$$\begin{aligned} &t_0 = 0, \quad s_0 = \eta, \\ &t_{n+1} = s_n + \frac{M_2\,(s_n-t_n)^2}{2\left(1-\dfrac{M_0}{2}\,(s_n+t_n)\right)}, \end{aligned} \tag{3.5}$$

(3.6)

$$s_{n+1} = t_{n+1} + \frac{6\,M\,(t_{n+1}-s_n)^2+6\,M\,(s_n-t_n)\,(t_{n+1}-s_n)+5\,N_1\,(s_n-t_n)^3}{12\,(1-M_0\,t_{n+1})},$$

where

$$M_2 = \begin{cases} M_1 & if \quad n=0 \\ M & if \quad n>0 \end{cases}, \qquad N_1 = \begin{cases} N_0 & if \quad n=0 \\ N & if \quad n>0 \end{cases}.$$

Define constant η_0 by:

$$\eta_0 = \frac{1}{M_0} \min \left\{ \frac{1-\delta_2}{1+\delta_2}, \frac{1-\delta_1}{1+\delta_1}, \frac{2-\delta_0}{2+\delta_2}, \frac{2\ M_0}{M_0+M_1} \right\}.$$

Then, if $\eta \leq \eta_0$, we have $\delta_0 \in [0,2)$, and the set:

$$I = [\max\{2\ \delta_1,\ 2\ \delta_2,\ \delta_0\},\ 2\ w_\infty] \neq \emptyset.$$

Choose

$$\delta \in I. \tag{3.7}$$

In view of (3.6), for $n=0$, there exists $\eta_1 > 0$, such that:

$$M_0\ t_1 < 1$$

and

$$s_1 - t_1 \leq \frac{\delta}{2}\ (s_0 - t_0)$$

for all $\eta \in [0,\ \eta_1]$.

Define constant α by:

$$\alpha = \min\{\eta_0,\ \eta_1\}. \tag{3.8}$$

We can show the following result on majorizing sequences for (NTM):

Lemma 3.1. *Assume:*

$$h_A = \overline{K}\ \eta \leq \frac{1}{2} \tag{3.9}$$

where,

$$\overline{K} = \frac{1}{2\ \alpha}, \qquad \alpha \neq 0. \tag{3.10}$$

Then, sequences $\{t_n\}$, $\{s_n\}$ $(n \geq 0)$ *given by* (3.5) *and* (3.6), *are non–decreasing, bounded by:*

$$t^{\star\star} = \frac{2\ \eta}{2-\delta}, \tag{3.11}$$

and converge to their unique least upper bound $t^\star$ *satisfying:*

$$t^\star \in [0, t^{\star\star}]. \tag{3.12}$$

Moreover the following estimates hold for all $n \geq 0$*:*

$$t_n \leq s_n \leq t_{n+1} \leq s_{n+1}, \tag{3.13}$$

and

$$0 \leq s_{n+1} - t_{n+1} \leq \frac{\delta}{2}\,(s_n - t_n) \leq \left(\frac{\delta}{2}\right)^{n+1}\,\eta. \tag{3.14}$$

Proof. We shall show using induction on k:

$$0 \leq t_{k+1} - s_k \leq \frac{\delta}{2}\,(s_k - t_k), \tag{3.15}$$

$$M_0\,(t_k + s_k) < 2, \tag{3.16}$$

$$0 \leq s_{k+1} - t_{k+1} \leq \frac{\delta}{2}\,(s_k - t_k), \tag{3.17}$$

and

$$M_0\,t_{k+1} < 1. \tag{3.18}$$

Estimates (3.15)-(3.18) hold by the initial conditions, and the choice of η.

Assume (3.15)-(3.18) hold for all $m \leq k$.

By the induction hypotheses, we have:

$$\begin{aligned} s_m &\leq t_m + \frac{\delta}{2}\,(s_{m-1} - t_{m-1}) \\ &\leq s_{m-1} + \frac{\delta}{2}\,(s_{m-1} - t_{m-1}) + \frac{\delta}{2}\,(s_{m-1} - t_{m-1}) \\ &\leq \eta + 2\left(\frac{\delta}{2}\,\eta + \left(\frac{\delta}{2}\right)^2\eta + \cdots + \left(\frac{\delta}{2}\right)^m\eta\right) \\ &= \eta + \frac{1 - \left(\frac{\delta}{2}\right)^m}{1 - \frac{\delta}{2}}\,\delta\,\eta, \end{aligned} \tag{3.19}$$

and

$$\begin{aligned} t_{m+1} &\leq s_m + \frac{\delta}{2}\,(s_m - t_m) \\ &\leq \eta + \frac{1 - \left(\frac{\delta}{2}\right)^m}{1 - \frac{\delta}{2}}\,\delta\cdot\eta + \left(\frac{\delta}{2}\right)^{m+1}\eta. \end{aligned} \tag{3.20}$$

Estimates (3.15), and (3.16) will certainly hold if

$$M\ (s_{m+1}-t_{m+1}) \leq \delta\left(1-\frac{M_0}{2}\ (t_{m+1}+s_{m+1})\right)$$

or

(3.21)

$$M\left(\frac{\delta}{2}\right)^{m+1}\eta+\frac{\delta\ M_0}{2}\left\{2+\delta\ \frac{1-\left(\frac{\delta}{2}\right)^{m+1}}{1-\frac{\delta}{2}}+\delta\ \frac{1-\left(\frac{\delta}{2}\right)^{m}}{1-\frac{\delta}{2}}+\left(\frac{\delta}{2}\right)^{m+1}\right\}\eta-\delta\leq 0.$$

Estimate (3.21) motivates us to set $w \geq \frac{\delta}{2}$, define functions f_m:

(3.22)
$$\begin{aligned} f_m(w) &= M\ w^m\ \eta+M_0\ \Big(2+2\ w\ (1+w+\cdots+w^m) \\ &\quad +2\ w\ (1+w+\cdots+w^{m-1})+w^{m+1}\Big)\ \eta-2, \quad m\geq 1. \end{aligned}$$

We need a relationship between two consecutive f_m:

(3.23)
$$\begin{aligned} f_{m+1}(w) &= M\ w^{m+1}\ \eta+M_0\ \big(2+2\ w\ (1+w+\cdots+w^m)+ \\ &\quad +2\ w^{m+2}+2\ w\ (1+w+\cdots+w^{m-1})+2\ w^{m+1} \\ &\quad +w^{m+1}-w^{m+1}+w^{m+2}\big)\ \eta-2+M\ w^m\ \eta-M\ w^m\ \eta \\ &= f_m(w)+(3\ M_0\ w^2+(M_0+M)\ w-M)\ w^m\ \eta \\ &= f_m(w)+g_1(w)\ w^m\ \eta \end{aligned}$$

where,

(3.24)
$$g_1(w)=3\ M_0\ w^2+(M_0+M)\ w-M.$$

Note that δ_1 given by (3.2) is the unique positive zero of quadratic polynomial g_1.

We have:

(3.25)
$$f_m(0)=2\ (M_0\ \eta-1)<0$$

and

(3.26)
$$f_m(w)>0$$

for sufficiently large $w>0$.

It then follows from (3.25), and (3.26), and the intermediate value theorem that there exists a positive zero w_m for each function f_m $(m \geq 1)$. The zero's s_m are unique in $[0, \infty)$, since: $f'_m(w) > 0$ $(w > 0)$, $(m \geq 1)$.

Estimate (3.21) certainly holds if:

$$f_m(w) \leq 0 \quad \text{for all } w \in [0, w_m], \quad (m \geq 1). \tag{3.27}$$

If there exists $m \geq 0$, such that $w_{m+1} \geq \delta_1$, then using (3.23), we get:

$$f_{m+1}(w_{m+1}) \;=\; f_m(w_{m+1}) + g_1(w_{m+1})\; w_{m+1}^m\; \eta,$$

or

$$f_m(w_{m+1}) \leq 0, \tag{3.28}$$

since $f_{m+1}(w_{m+1}) = 0$, and $g_1(w_{m+1})\; w_{m+1}^m\; \eta \geq 0$, which imply:

$$w_{m+1} \leq w_m. \tag{3.29}$$

We can certainly choose the last of the w_m's denoted by w_∞ (obtained from (3.21) by letting $m \longrightarrow \infty$, and given by (3.4)), to be w_{m+1}.

It then follows, sequence $\{w_m\}$ is non–increasing, bounded below by zero, and as such it converges to its unique maximum lowest bound $w^\star$ satisfying $w^\star \geq w_\infty$.

Then, estimate (3.27) certainly holds, if

$$\delta_1 \leq w_\infty, \tag{3.30}$$

which holds by the choice of δ.

That completes the induction for (3.15), and (3.16).

Instead of showing (3.17), and (3.18) for $m \geq 1$, it suffices (by (3.15), and (3.16)):

$$\begin{aligned} 0 &\leq s_{m+1} - t_{m+1} \\ &\leq \frac{6\; M \left(\dfrac{\delta}{2}\right)^2 (s_m - t_m)^2 + 6\; M \left(\dfrac{\delta}{2}\right) (s_m - t_m)^2 + 5\; N\; (s_m - t_m)^3}{12\; (1 - M_0\; t_{m+1})} \\ &\leq \frac{(3\; M\; \delta^2 + 6\; M\; \delta + 10\; N\; (s_m - t_m))\; (s_m - t_m)^2}{24\; (1 - M_0\; t_{m+1})} \\ &\leq \frac{\delta}{2}\; (s_m - t_m), \end{aligned}$$

or

$$\left(3\ M\ \delta^2 + 6\ M\ \delta + 10\ N\left(\frac{\delta}{2}\right)^m \eta\right)\left(\frac{\delta}{2}\right)^m \eta$$

$$+12\ M_0\ \delta\left\{1 + \frac{1 - \left(\frac{\delta}{2}\right)^m}{1 - \frac{\delta}{2}}\ \delta + \left(\frac{\delta}{2}\right)^{m+1}\right\}\eta - 12\ \delta \leq 0. \tag{3.31}$$

As in (3.22), we define recurrent functions p_m on $[0, +\infty)$ $(m \geq 1)$ by:

$$\begin{aligned} p_m(w) &= (6\ M\ w^{m+1} + 6\ M\ w^m + 5\ N\ w^m\ \eta)\ \eta \\ &\quad +12\ M_0\ (1 + 2\ w\ (1 + w + \cdots + w^{m-1}) + w^{m+1})\ \eta - 12, \end{aligned} \tag{3.32}$$

from which, we get as in (3.23):

$$p_{m+1}(w) \;=\; p_m(w) + g_2(w)\ w^m\ \eta, \tag{3.33}$$

where,

$$g_2(w) = 6\ (M + 2\ M_0)\ w^2 + (12\ M_0 + 5\ N\ \eta)\ w - (5\ N\ \eta + 6\ M). \tag{3.34}$$

Note that δ_2 given by (3.3) is the unique positive zero of function g_2. We also have from (3.31) that:

$$w_\infty^1 = w_\infty. \tag{3.35}$$

Note that:

$$p_m(0) = 12\ (M_0\ \eta - 1) < 0 \qquad (m \geq 1),$$

and

$$p_m(w) > 0 \qquad (w > 0), \quad (m \geq 1)$$

for sufficiently large $w > 0$.

Hence, each p_m $(m \geq 1)$ has a unique zero w_m^1 in $[0, \infty)$, since:

$$p'_m(w) > 0 \qquad (w > 0), \quad (m \geq 1). \tag{3.36}$$

Estimate (3.31) certainly hold if:

$$p_m(w) \leq 0 \qquad \text{for all } w \in [0, w_m^1], \quad (m \geq 1). \tag{3.37}$$

If there exists $m \geq 0$, such that $w_{m+1}^1 \geq \delta_2$, then using (3.33), we obtain:

$$p_{m+1}(w_{m+1}^1) \;=\; p_m(w_{m+1}^1) + g_2(w_{m+1}^1)\ {w_{m+1}^1}^m\ \eta,$$

or

$$p_m(w^1_{m+1}) \le 0, \tag{3.38}$$

since $p_{m+1}(w^1_{m+1}) = 0$, and $g_2(w^1_{m+1})\ {w^1_{m+1}}^m\ \eta \ge 0$, which imply:

$$w^1_{m+1} \le w^1_m \qquad (m \ge 0). \tag{3.39}$$

We can certainly choose the last of the w^1_m's denoted by w^1_∞ (obtained by letting $k \longrightarrow \infty$), to be s^1_{m+1}.

It then follows, sequence $\{w^1_m\}$ is non–increasing, bounded below by zero, and as such it converges to its unique maximum lowest bound $w^{\star 1}$ satisfying $w^{\star 1} \ge w^1_\infty = w_\infty$.

Then, estimate (3.37) holds, if (3.30) does. The induction is now completed.

Finally, sequences $\{t_n\}$, $\{s_n\}$ are non–decresing, bounded above by $t^{\star\star}$, and as such they converge to their unique, common least upper bound $t^\star$, satisfying (3.12).

That completes the proof of Lemma 3.1. □

Note that according to Theorem 2.1, $\{t_n\}$, $\{s_n\}$ given by (3.5) and (3.6) are majorizing sequences for $\{x_n\}$, $\{y_n\}$. Therefore, in view of Lemma 3.1, we arrived at the analog of Theorem 2.1:

Theorem 3.2. *Let $F : \mathcal{D} \subseteq \mathcal{X} \longrightarrow \mathcal{Y}$ be a twice Fréchet–differentiable operator.*

Assume:

there exist $x_0 \in \mathcal{D}$, nonnegative constants $M_0 \ge 0$, $M \ge 0$, $M_1 \ge 0$, $N_0 \ge 0$, $N \ge 0$, and $\eta \ge 0$, such that for all $x, y \in \mathcal{D}$:

$$F'(x_0)^{-1} \in \mathcal{L}(\mathcal{Y}, \mathcal{X}),$$

$$\| F'(x_0)^{-1}\ F(x_0) \| \le \eta,$$

$$\| F'(x_0)^{-1}\ F''(x) \| \le M, \quad \| F'(x_0)^{-1}\ F''(x_0) \| \le M_1,$$

$$\| F'(x_0)^{-1}\ (F'(x) - F'(x_0)) \| \le M_0\ \ \| x - x_0 \|,$$

$$\| F'(x_0)^{-1}\ (F''(x) - F''(y)) \| \le N\ \ \| x - y \|,$$

$$\| F'(x_0)^{-1}\ (F''(x) - F''(x_0)) \| \le N_0\ \ \| x - x_0 \|,$$

$$h_A = \overline{K}\ \eta\ \le \frac{1}{2},$$

where $\overline{K}$ is given in (3.10),

and

$$\overline{U}(y_0, t^\star - \eta) \subseteq \mathcal{D},$$

Then sequences $\{x_n\}$, $\{y_n\}$ *generated by (NTM) are well defined, remain in* $\overline{U}(x, t^\star)$ *for all* $n \geq 0$, *and converge to a solution* $x^\star$ *in* $\overline{U}(x, t^\star)$ *of equation* $F(x) = 0$, *which is unique in* $U(x_0, R)$, *provided*

$$R \in \left(t^{\star\star}, \frac{2}{M_0} - t^\star\right] \neq \emptyset.$$

Moreover, the following estimates hold:

$$\| x_{n+1} - y_n \| \leq t_{n+1} - s_n,$$

$$\| x_n - y_n \| \leq s_n - t_n,$$

$$\| x_n - x^\star \| \leq t^\star - t_n,$$

$$\| y_n - x^\star \| \leq t^\star - s_n.$$

Remark 3.3. Note that majorizing sequences $\{t_n\}$, $\{s_n\}$ given by (3.5) and (3.6) simply replace the corresponding ones in the proof of Theorem 2.1.

Moreover, in view of the estimates $M_0 \leq M$, $M_2 \leq M$, and $N_1 \leq N_0$, majorizing sequences given by (3.5) and (3.6) are finer than the ones used in Section 2 (see also estimates (2.28)–(2.31) with $\{\overline{t}_n\}$, $\{\overline{s}_n\}$ replaced by the corresponding ones given by (3.5) and (3.6), respectively).

In practice, if hypotheses of Theorems 2.1 and 3.2 hold, we will use the majorizing sequences (3.5), (3.6) to compute estimates for $\| x_{n+1} - x_n \|$, $\| x_n - x^\star \|$, $(n \geq 0)$.

A direct comparison sufficient convergence conditions (2.6), and (3.9) is not possible, since they use different information. So it is possible for (3.9) to but not (2.6) or vice versa.

We provide a numerical example to show that condition (2.6) does not hold, but (3.9) does.

Example 3.4. Let $\mathcal{X} = \mathcal{Y} = \mathbb{R}$, $x_0 = 1$, $\mathcal{D} = \{x : \| x - x_0 \| \leq 1 - \beta\}$, $\beta \in [0, \frac{1}{2})$, and define function F on $\mathcal{D}$ by:

$$F(x) = x^3 - \beta. \tag{3.40}$$

Using (2.1)–(2.6), we get:

$$\eta = \frac{1}{3}\ (1 - \beta), \quad M_0 = 3 - \beta, \quad M = 2\ (2 - \beta), \quad N = 2,$$

and

$$h > \frac{1}{2} \quad \text{for all } \beta \in [0, \frac{1}{2}). \tag{3.41}$$

Therefore, there is no guarantee under our Theorem 2.1 or Theorem 1 in [17], that sequences $\{x_n\}$, $\{y_n\}$ converge to $x^\star = \sqrt[3]{\beta}$.

However, using (3.1)–(3.6), we have for $\beta = .49$:

$$M_1 = 2, \qquad \eta = .17, \quad M_0 = 2.51, \qquad M = 3.02,$$

$$\delta_0 = .542426746, \qquad \delta_2 = .391048897, \quad \delta_1 = .364851779,$$

$$w_\infty = .401836406, \qquad I = [.782097794,\ .803672812],$$

$$\frac{2}{M_0 + M} = .361663653,$$

$$\frac{1}{M_0}\left(\frac{1-\delta_1}{1+\delta_1}\right) = .185402623, \quad \frac{1}{M_0}\left(\frac{2-\delta_0}{2+\delta_0}\right) = .228406375,$$

and

$$\frac{1}{M_0}\left(\frac{1-\delta_2}{1+\delta_2}\right) = .174407961.$$

Set $\delta = 2\ \delta_2$, and for $\eta_1 = \eta_0 = .18$, we have:

$$t_1 = .206738066, \qquad t_1 - s_1 = .036738066,$$

$$M_0\ t_1 = .518912546 < 1,$$

and

$$s_1 - t_1 = .032349323 < .066478312 = \frac{\delta}{2}\ (s_0 - t_0),$$

respectively.

Hence, Hypotheses of Thoerem 3.2 are satisfied for all $\eta \in [0, \eta_1]$, and in particular for $\eta = .17 \in [0, \eta_0]$, we can set:

$$\alpha = \eta_0 = .18.$$

We also have by (3.10), and (3.9):

$$\overline{K} = 2.777,$$

and

$$h_A = .472222 < .5$$

respectively.

Hence, the conclusions of Theorem 3.2 hold, for equation (3.40).

Finally, note that the hypotheses in [9], [10], [14], given by

$$h_{EH} = M\ \eta \leq .300637,$$
$$h_{EH} = M\ \eta \leq .292246,$$
$$h_G = M\ \eta \leq .300637,$$

respectively, are viloated, since

$$M\ \eta = .5134.$$

Hence, the convergence of (NTM) cannot be guarantee under their conditions.

We can finally show the cubical convergence of sequences $\{s_n\}$, $\{t_n\}$.

Proposition 3.5. *Under the hypotheses of Lemma* 3.1, *for* $q > 0$, *define parameters* a, b, c, d *by:*

$$a = \frac{M^2\ \delta}{8} + \frac{M^2}{4} + \frac{5\ N\ \left(1 - \frac{M_0}{2}\ \eta\right)}{12},$$
$$b = \frac{\sqrt{a}}{q},$$
$$c = \frac{1-b}{M_0}, \qquad M_0 \neq 0,$$
$$d = \frac{M}{2\ b}, \qquad b \neq 0,$$

and function f *on* $\left[0, \frac{1}{q}\right)$ *by:*

$$f(s) = s + \frac{1}{q} + \frac{d}{q^2}\left(\frac{(q\ s)^2}{1-(q\ s)^2} + s^2\right).$$

Assume:

$$\sqrt{a}\ \eta < 1,$$

and fix

$$q \in (\sqrt{a}, \frac{1}{\eta}) \qquad \eta \neq 0;$$

(3.42) $$\min\{t_1, f(\eta)\} \leq c.$$

Then, the following estimates hold for all $k \geq 0$*:*

$$t_{k+1} - s_k \leq \frac{d}{q^2} \sqrt[3]{\left((q\ \eta)^2\right)^{3^{k+1}}},$$

and

$$s_k - t_k \leq \frac{1}{q} (q\ \eta)^{3^k}.$$

Proof. Under the hypotheses of Lemma 3.1, we have in turn by (3.5), and (3.6):

$$\frac{M}{2} (t_{m+1} - t_m)^2 + \frac{M}{2} (s_m - t_m) (t_{m+1} - s_m) + \frac{5\ N}{12} (s_m - t_m)^3$$

$$\leq \frac{M}{2} \left(\frac{M\ (s_m - t_m)}{2 - M_0\ (s_m + t_m)}\right)^2 + \frac{M}{2} \left(\frac{M\ (s_m - t_m)^2}{2 - M_0\ (s_m + t_m)}\right) (s_m - t_m)$$

$$+ \frac{5\ N}{12} (s_m - t_m)^3$$

$$\leq \frac{M^3}{8} \frac{(s_m - t_m)^4}{\left(1 - \frac{M_0}{2} (s_m + t_m)\right)^2} + \frac{M^2}{4} \frac{(s_m - t_m)^3}{1 - \frac{M_0}{2} (s_m + t_m)}$$

$$+ \frac{5\ N}{12} (s_m - t_m)^3$$

$$\leq \frac{M^2}{8} \frac{(s_m - t_m)\ (s_m - t_m)^2}{1 - \frac{M_0}{2} (s_m + t_m)} + \frac{M^2}{4} \frac{(s_m - t_m)^3}{1 - \frac{M_0}{2} (s_m + t_m)}$$

$$+ \frac{5\ N}{12} \frac{(s_m - t_m)^3 \left(1 - \frac{M_0}{2} (s_m + t_m)\right)}{1 - \frac{M_0}{2} (s_m + t_m)}$$

$$\leq \frac{a\ (s_m - t_m)^3}{1 - \frac{M_0}{2} (s_m + t_m)}$$

$$\leq \frac{a\ (s_m - t_m)^3}{1 - M_0\ t_{m+1}}.$$

We shall show:

$$s_{m+1} - t_{m+1} \leq \frac{a\ (s_m - t_m)^3}{(1 - M_0\ t_{m+1})^2} \leq q^2\ (s_m - t_m)^3$$

or

$$\frac{a}{q^2} \leq (1 - M_0\ t_{m+1})^2$$

or

$$t_{m+1} \leq c.$$

By hypothesis (3.42), we have: $t_1 \leq c$. Then $t_{m+1} \leq c$ holds for $m = 0$.

Assume:

$$t_m \leq c. \tag{3.43}$$

We have:

$$q\ (s_{m+1} - t_{m+1}) \leq (q\ (s_m - t_m))^3 \leq (q\ \eta)^{3^{m+1}},$$

or

$$s_{m+1} - t_{m+1} \leq \frac{1}{q}\ (q\ \eta)^{3^{m+1}}.$$

We also have:

$$\begin{aligned} t_{m+1} - s_m &\leq \frac{M\ (s_m - t_m)^2}{2\ (1 - M\ t_{m+1})^2} \\ &\leq d\ (s_m - t_m)^2 \leq \frac{d}{q^2}\ \sqrt[3]{\left((q\ \eta)^2\right)^{3^{m+1}}}. \end{aligned}$$

Moreover, we get in turn:

$$\begin{aligned} t_{m+1} &\leq (s_m - t_m) + (t_m - s_{m-1}) + \cdots + (t_1 - s_0) + s_0 + d\ (s_m - t_m)^2 \\ &\leq \eta + \frac{1}{q}\ (q\ \eta)^{3^m} + \frac{d}{q^2} \left\{ \left((q\ \eta)^{3^m}\right)^{2\frac{3}{3}} + \left((q\ \eta)^{3^{m-1}}\right)^{2\frac{3}{3}} + \cdots + \eta^2 \right\} \\ &= \eta + \frac{1}{q}\ (q\ \eta)^{3^m} + \frac{d}{q^2} \left(\left(((q\ \eta)^2)^{\frac{1}{3}}\right)^{3^{m+1}} + \left(((q\ \eta)^2)^{\frac{1}{3}}\right)^{3^m} \right. \\ &\quad \left. + \cdots + \left(((q\ \eta)^2)^{\frac{1}{3}}\right)^{3^1} + \eta^2 \right) \\ &\leq \eta + \frac{1}{q}\ (q\ \eta)^{3^m} + \frac{d}{q^2} \left((q\ \eta)^{2\,(m+1)} + (q\ \eta)^{2\,m} + \cdots + (q\ \eta)^2 + \eta^2 \right) \\ &\leq f(\eta) \leq c. \end{aligned}$$

That completes the induction for (3.43), and the proof of Proposition 3.5. □

Remark 3.6. (a) *Condition* (3.42) *can be replaced by a stronger, but easier to check:*

$$\frac{2\ \eta}{2 - \delta} \leq c, \tag{3.44}$$

for $\delta \in I$ *(see,* (3.7), *and* (3.11)).

Set

$$\delta_3 = \min\{2\ \delta_1,\ 2\ \delta_2,\ \delta_0\}.$$

The most appropriate choice for δ *seems to be* $\delta = \delta_3$.

Condition (3.44) can then be re-written as:

$$\eta \leq \frac{(2-\delta_3)\ c}{2}.$$

(b) *The ratio of convergence "$q\ \eta$" given in Proposition* 3.5 *can be smaller than* θ *given in Theorem* 2.1 *for* q *sufficiently close to* $\sqrt{a}$. *Assume* M, N *are not both zero, and* $\eta > 0$.

Set

$$h_0 = \frac{1}{2\ K}\left(1 - \left[\left(\frac{4\ K^2}{a}\right)^{\frac{1}{4}} - 1\right]^{\frac{1}{2}}\right).$$

By comparing $\alpha = \sqrt{a}\ \eta$ *to* θ, *we get:*

Case 1. *If*

$$4\ a < K^2$$

or

$$\frac{a}{4} \leq K^2 \leq 4\ a$$

and

$$\eta > h_0,$$

then, we have:

$$\alpha < \theta.$$

Case 2. *If*

$$K^2 < 4\ a$$

and

$$\eta < h_0,$$

then, we have:

$$\theta < \alpha.$$

Case 3. *If*

$$\eta = h_0,$$

then, we have:

$$\alpha = \theta.$$

CONCLUSION

We provided a semilocal convergence analysis for Newton–type method in order to approximate a locally unique solution of a nonlinear equation in a Banach space, involving a twice differentiable operator. We also established that the order of convergence is three.

Using our new idea of recurrent functions, we provided a semilocal convergence analysis with the following advantages over the work in [17]: larger convergence domain, and weaker sufficient convergence conditions. Note that these advantages are obtained under the same computational cost as in [17].

A numerical examples, and some favorable comparisons with the previous works are also provided.

REFERENCES

1. Argyros, I.K.: An improved error analysis for Newton–like methods under generalized conditions. *J. Comput. Appl. Math.* **157** (2003), 169-185.
2. ______: On the comparison of a weak variant of the Newton–Kantorovich and Miranda theorems. *J. Comput. Appl. Math.* **166** (2004), 585-589.
3. ______: *Computational theory of iterative methods, Series: Studies in Computational Mathematics*, 15, Editors: C.K. Chui and L. Wuytack, Elsevier Publ. Co., New York, USA, 2007.
4. Argyros, I.K. & Chen, D.: The midpoint method for solving nonlinear operator equations in Banach space. *Appl. Math. Lett.* **5** (1992), 7-9.
5. ______: Results on the Chebyshev method in Banach spaces. *Proyecciones* **12** (1993), 119-128.
6. ______: The midpoint method in Banach spaces and the Pták error estimates. *Appl. Math. Comput.* **62** (1994), 1-15.
7. ______: On the midpoint iterative method for solving nonlinear operator equations in Banach space and its applications in integral equations. *Rev. Anal. Numér. Théor. Approx.* **23** (1994), 139-152.
8. Deuflahard, P. & Heindl, G.: Affine invariant convergence theorems for Newton's method and extensions to related methods. *SIAM J. Numer. Anal.* **16** (1979), 1-10.
9. Frontini, M. & Sormani, E.: Some variant of Newton's method with third–order convergence. *Appl. Math. Comput.* **140** (2003), 419-426.
10. ______: Modified Newton's method with third–order convergence and multiple roots. *J. Comput. Appl. Math.* **156** (2003), 345-354.

11. ______: Third–order methods from quadrature formulae for solving systems of nonlinear equations, *Appl. Math. Comput.* **149** (2004), 771-782.
12. Gutiérrez, J.M. & Hernández, M.A.: An acceleration of Newton's method: super–Halley method. *Appl. Math. Comput.* **117** (2001), 223-239.
13. Hernández, M.A.: A modification of the classical Kantorovich conditions for Newton's method. *J. Comput. Appl. Math.* **137** (2001), 201-205.
14. Hernández, M.A. & Salanova, M.A.: Modification of the Kantorovich assumptions for semilocal convergence of the Chebyshev method. *J. Comput. Appl. Math.* 126 (2000), 131-143.
15. Kantorovich, L.V. & Akilov, G.P.: *Functional analysis in normed spaces.* Pergamon Press, Oxford, 1982.
16. Weerakoon, S. & Fernando, T.G.I.: A variant of Newton's method with accelerated third–order convergence, *Appl. Math. Lett.* **13** (2000), 87-93.
17. Wu, Q. & Zhao, Y.: Third–order convergence theorem by using majorizing function for a modified Newton method in Banach space. *Appl. Math. Comput.* **175** (2006), 1515-1524.

[a]CAMERON UNIVERSITY, DEPARTMENT OF MATHEMATICS SCIENCES, LAWTON, OK 73505, USA
Email address: `iargyros@cameron.edu`

[b]POITIERS UNIVERSITY, LABORATOIRE DE MATHÉMATIQUES ET APPLICATIONS, BD. PIERRE ET MARIE CURIE, TÉLÉPORT 2, B.P. 30179, 86962 FUTUROSCOPE CHASSENEUIL CEDEX, FRANCE
Email address: `said.hilout@math.univ-poitiers.fr`