J. Korean Soc. Math. Educ. Ser. B: Pure Appl. Math. ISSN 1226-0657
Volume 17, Number 1 (February 2010), Pages 93–106

TRAVELING WAVE GLOBAL PRICE DYNAMICS OF LOCAL MARKETS WITH LOGISTIC SUPPLIES

Yong-In Kim

Abstract. We employ the methods of Lattice Dynamical System to establish a global model extending the Walrasian evolutionary cobweb model in an independent single local market to the global market evolution over an infinite chain of many local markets with interaction of each other through a diffusion of prices between them.

For brevity of the model, we assume linear decreasing demands and logistic supplies with naive predictors, and investigate the traveling wave behaviors of global price dynamics and show that their dynamics are conjugate to those of Hénon maps and hence can exhibit complicated behaviors such as period-doubling bifurcations, chaos, and homoclic orbits etc.

1. Introduction

Over the last decade, a new class of infinite dimensional dynamical systems, so called Lattice Dynamical Systems(LDS) have been introduced and studied by many researchers (e.g., [1], [3], [7], [8]). These LDS's have been proved to be one of the most efficient tools to analyze space-time behaviors of the extended systems.

To begin with, we define the phase space (or state space) of the LDS. Suppose that at each site j of a d-dimensional lattice $\mathbf{Z}^d$, we have a finite dimensional local dynamical system which is defined by some map $f_j : M_j \longrightarrow M_j$, where M_j is a local phase space at the site j. For simplicity and applicability to our model, we will confine our attention to an infinite chain ($d = 1$) and the identical local map, i.e., $f_j = f, M_j = \mathbf{R}^1\, \forall j \in \mathbf{Z}$, where $\mathbf{R}^1$ is a 1-dimensional real Euclidean space with ordinary inner product $(\cdot,\cdot)$ and the norm $|\cdot| = \sqrt{(\cdot,\cdot)}$. Then we have an infinite dimensional dynamical system on a space

$$M = \prod_{j\in\mathbf{Z}} M_j = \{p = \{p_j\} | p_j \in \mathbf{R}, j \in \mathbf{Z}\} \tag{1.1}$$

Received by the editors December 11, 2009. Accepted January 12, 2010.
2000 *Mathematics Subject Classification.* 22E46, 53C35, 57S20.
Key words and phrases. Cobweb model, lattice dynamical system, Hénon map.

where M is obviously a linear space with respect to componentwise addition and scalar multiplication. A point (or, a state) $p = \{p_j\} \in M$ can be thought of as a bi-infinite sequence of real numbers. To make the linear space M to be a Hilbert space, we equip M with the inner product defined by

$$(1.2) \qquad \langle p, q \rangle_\rho = \sum_{j \in \mathbf{Z}} \frac{(p_j, q_j)}{\rho^{|j|}} \quad \forall p, q \in M,$$

where $\rho > 1$ is some fixed number depending on the particular problem. Then the norm $\|\cdot\|_\rho$ is induced by

$$\|\cdot\|_\rho = \sqrt{\langle \cdot, \cdot \rangle_\rho}$$

and now we can define the phase space of our LDS by

$$(1.3) \qquad B_\rho = \{p \in M | \, \|p\|_\rho < \infty\}.$$

Then it can be easily shown that B_ρ is a Hilbert space (e.g., [1]). Next, we define the evolution operator on B_ρ in the following.

Definition 1.1. Define the evolution operator $\Phi : B_\rho \to B_\rho$ by

$$(1.4) \qquad (\Phi p)_j = F(\{p_j\}^s), \forall j \in \mathbf{Z},$$

where $\{p_j\}^s = \{p_i | \, |i - j| \leq s, \, s \geq 1 \text{ integer}\}$ for each $j \in \mathbf{Z}$, i.e., $\{p_j\}^s$ is the set of values p_i at the site i which are within the distance of radius s from the site j, and $F : \mathbf{R}^{2s+1} \to \mathbf{R}$ is a differentiable map of class C^2 such that

$$(1.5) \qquad \left| \frac{\partial F}{\partial p_i} \right| \leq K, \quad \left| \frac{\partial^2 F}{\partial p_i \partial p_k} \right| \leq K,$$

for any collection $\{p_j\}^s$ and some constant $K > 0$.

Then it is easy to verify that under the condition (1.5), $\Phi(B_\rho) \subset B_\rho$ and Φ is Lipschitz continuous with the constant $L = C(2s+1)^{\frac{3}{2}} \rho^{\frac{s}{2}}$ (e.g., [1]).

Definition 1.2. Given a state $p(n) = \{p_j(n)\}_{j=-\infty}^{\infty} \in B_\rho$ at the moment n, we can obtain via (1.4) the next state $p(n+1)$, that is,

$$(1.6) \qquad \begin{aligned} p(n+1) &= \Phi(p(n)), \quad \text{or}, \\ p_j(n+1) &= (\Phi(p(n)))_j = F(\{p_j(n)\}^s). \end{aligned}$$

The dynamical system $(\Phi^n, B_\rho)_{n \in \mathbf{Z}^+}$ is called a *Lattice Dynamical System*(LDS).

Formula (1.6) implies that given a state $p(n) \in B_\rho$, we can calculate its next state $p(n+1)$, so we can obtain the forward orbit of the evolution operator Φ, i.e.,

$$p(0), p(1) = \Phi(p(0)), p(2) = \Phi(p(1)) = \Phi^2(p(0)), \cdots.$$

Before ending this section, let us consider several kinds of basic motions (or solutions) in the LDS (1.6).

Definition 1.3. (i) A state (or solution) $p(n) = \{p_j(n)\}$ for the LDS (1.6) is *spatially homogeneous* if $p_j(n) = \psi(n) \, \forall j \in \mathbf{Z}$, i.e., a spatially homogeneous solution $\{\psi(n)\}$ does not depend on the space coordinates j and so has the same value at each site j.
(ii) A solution $p(n) = \{p_j(n)\}$ is *static* (or *stationary, steady state,standing wave*) if $p_j(n) = \phi_j \, \forall n \in \mathbf{Z}^+$, i.e., a static solution $\{\phi_j\}$ does not depend on time n, and is standing there along the space coordinates j at all times n.
(iii) A solution $p(n) = \{p_j(n)\}$ is a *traveling wave with wave velocity* m/l if $p_j(n) = \xi(lj + mn)$, where $l > 0, m \in \mathbf{Z}$ and $(l, m) = 1$(i.e., relatively prime). Here, the ratio m/l is called the *wave velocity* of the traveling wave.

For instance, suppose that the local system $f : M_j \to M_j$ has a fixed point p^*. Then the state $p = \{p_j\}, p_j = p^* \forall j \in \mathbf{Z}$ is a spatially homogeneous static solution, i.e., a fixed point of the evolution map Φ and also can be thought of as a traveling wave with arbitrary velocity.

2. The Cobweb Model

The Cobweb model for the local market dynamics has been well introduced and studied by many researchers (e.g., [4], [5], [6], [9]). The Cobweb model describes the dynamics of equilibrium prices in a single independent local market for a non-storable good, that takes one time period to produce, so that producers must form price expectations one period ahead using the past history of prices.

Let $p_n^e = H(\mathbf{P}_{n-1})$, where p_n^e is the expected price by the producers at time n and $\mathbf{P}_{n-1} = (p_{n-1}, p_{n-2}, \cdots, p_{n-L})$ is a vector of past prices of lag-length L and $H(\cdot) : \mathbf{R}^L \to \mathbf{R}$ is a real-valued function, so called a *predictor*. Let p_n be the actual market price at time n by the consumers, and let $D(p_n)$ be the consumer demand and $S(p_n^e)$ be the producer supply for the goods. The supply $S(p_n^e)$ is derived from producer's maximizing expected profit with a cost function $c(q)$, i.e.,

$$S(p_n^e) = arg \max_{q_n} \{p_n^e q_n - c(q_n)\}. \tag{2.1}$$

The demand function $D(\cdot)$ depends on the current market price p_n and is assumed to be strictly decreasing in the price p_n to ensure that its inverse D^{-1} is well-defined. The supply function $S(\cdot)$ depends on the expected price p_n^e and will be assumed to be quadratic in our paper. The intersection point p^* of the demand and supply curve such that $D(p^*) = S(p^*)$ is called the steady state equilibrium price.

If the beliefs of producers are homogeneous, i.e., all producers use the same predictor H, then the market equilibrium price dynamics in the cobweb model is described by

$$D(p_n) = S(H(\mathbf{P}_{n-1})), \quad \text{or,} \quad p_n = D^{-1}(S(H(\mathbf{P}_{n-1}))). \tag{2.2}$$

Thus, the actual equilibrium price dynamics in a local market depends on the demand D, the supply S, and the predictor H used by the producers.

Now, as our LDS model for the global market dynamics, we will take the following form:

$$\begin{aligned} p_j(n+1) &= (\Phi(p(n)))_j \\ &= (1-\alpha)p_j(n) + \alpha f(p_j(n)) + \varepsilon(p_{j-1}(n) - 2p_j(n) + p_{j+1}(n)), \end{aligned} \tag{2.3}$$

where a solution $p_j(n), j \in \mathbf{Z}, n \in \mathbf{Z}^+$ represents the price of a good at the site (or local market)j at the time n, and $f : \mathbf{R} \to \mathbf{R}$ is a Walrasian local market price dynamics at each site j, and $\alpha \in [0,1]$ is a parameter denoting the weighted average between $p_j(n)$ and $f(p_j(n))$, and the parameter ε is a diffusion coefficient measuring the intensity of interaction between the neighboring local markets. Thus, in this global market model, the price $p_j(n+1)$ at site j and at time $n+1$ is determined by several factors, i.e., the previous price $p_j(n)$, the local market dynamics f, the weight $\alpha \in [0,1]$ of the average between them, and the diffusion coefficient $\varepsilon > 0$. Notice that the parameter α plays a role of controlling each local market in such a way that if $\alpha = 1$ then $p_j(n+1)$ is determined completely by the local market dynamics f together with diffusion term and if $\alpha = 0$ then the local market dynamics is suppressed completely and $p_j(n+1)$ depends only on the present price and diffusion term.

Remark 2.1. For a solution $p_j(n)$ of our model (2.3) to have a meaning in economic sense, we impose a boundary condition at infinity that $p_j(n)$ must be bounded, i.e., $|p_j(n)| \le C \ \forall j \in \mathbf{Z}, n \in \mathbf{Z}^+$ for some $C > 0$. Also, we require that a solution $p_j(n)$ must have nonnegative value for all $j \in \mathbf{Z}, n \in \mathbf{Z}^+$. If a solution of (2.3) does not satisfy these conditions, then it would not be an admissible solution for our model.

Remark 2.2. Besides the solutions given in the Definition 1.3, there can also be many other solutions, e.g., spatially and/or temporally periodic solutions, spatially and/or temporally chaotic bounded solutions, and so on. In this paper, we restrict our attention only to those periodic solutions or bounded chaotic solutions which are the basic solutions mentioned in the Definition 1.3, e.g., spatially periodic static solutions, temporally periodic spatially homogeneous solutions, spatially and temporally periodic traveling wave solutions, etc.

3. Traveling Wave Global Price Dynamics

We assume that the predictor H is naive, the demand D is linear decreasing, and the supply S is logistic, that is, they are given by

$$\begin{aligned} p_n^e &= H(\mathbf{P}_{n-1}) = p_{n-1}, \quad D(p_n) = 1 - p_n, \\ S(p_n^e) &= S(p_{n-1}) = 4p_{n-1}(1 - p_{n-1}). \end{aligned} \tag{3.1}$$

respectively. Note that the price p_n in (3.1) is a scaled price such that $0 \le p_n \le 1$, and the quadratic supply function $S(x) = 4x(1-x)$ is the so called *logistic map*, which is known to exhibit chaotic dynamics on the whole interval $[0,1]$ (e.g., [11]). This kind of non-monotonic supply curve can be justified in an actual market, e.g., by an *income effect* in an agricultural market (e.g., [15], pp 339). This income effect, of course, may be applied to our fish market as well. In other words, as prices of fish are getting higher, the income of fishermen is getting higher, and so the production of fish might be getting less due to their taking more leisure time.

Now, with these choices of H, D, and S, the local market equilibrium price dynamics, $D(p_n) = S(p_n^e)$, is given by

$$\begin{aligned} 1 - p_n &= 4p_{n-1}(1 - p_{n-1}), \quad \text{or,} \\ p_n &= 1 - 4p_{n-1}(1 - p_{n-1}). \end{aligned} \tag{3.2}$$

Hence, our local market dynamics f for the global market model (2.3) is given by

$$f(x) = 1 - 4x(1-x) = (1-2x)^2, \tag{3.3}$$

where x may be assumed to be restricted to the interval $0 \le x \le 1$, since for $x \notin [0,1]$, the dynamics of f is very simple, i.e., $f^n(x) \to +\infty$ as $n \to +\infty$, and so only for $x \in [0,1]$, $f^n(x) \in [0,1]$ for all $n \in \mathbf{Z}^+$ and exhibits interesting chaotic dynamics.

The map f has two fixed points, one at $p^* = \frac{1}{4}$ where $f'(p^*) = -2 < -1$, and the other at $q^* = 1$ where $f'(q^*) = 4 > 1$, and so both fixed points are repellors. Note that at $p^* = \frac{1}{4}$, $D(\frac{1}{4}) = S(\frac{1}{4}) = \frac{3}{4}$, and prices near p^* diverges in an oscillatory way from p^*, while at $q^* = 1$, $D(1) = S(1) = 0$, and prices near and less than q^* decreases in a monotone way from q^* and so prices fluctuates between these two repellors p^* and q^* in a chaotic way.

Now, let us consider the traveling wave solutions with wave velocity 1 of the global market dynamics (2.3), where the local market dynamics f is given by (3.3), i.e.,

$$(3.4)\quad p_j(n+1) = (1-\alpha)p_j(n) + \alpha\{1-2p_j(n)\}^2 + \varepsilon\{p_{j-1}(n) - 2p_j(n) + p_{j+1}(n)\}.$$

Hereafter, we will slightly loosen our restriction $0 \le p_j(n) \le 1$ so that $p_j(n) \ge 0$ because interesting dynamics can occur for $p_j(n) > 1$ in the global market dynamics.

To obtain the traveling wave solutions, we set $p_j(n) = \xi(j+n)$ in (3.4), then we have

$$(3.5)\qquad \begin{aligned}\xi(j+n+1) = (1-\alpha)\xi(j+n) + \alpha\{1-2\xi(j+n)\}^2 \\ + \varepsilon\{\xi(j+n-1) - 2\xi(j+n) + \xi(j+n+1)\}.\end{aligned}$$

Letting $k = j+n$, (3.5) becomes

$$(3.6)\quad \xi(k+1) = (1-\alpha)\xi(k) + \alpha\{1-2\xi(k)\}^2 + \varepsilon\{\xi(k-1) - 2\xi(k) + \xi(k+1)\}.$$

Note that in (3.6), when $\varepsilon = 1$, $\xi(k+1)$ on both sides are canceled and $\xi(k)$ is not well defined in terms of $\xi(k-1)$ and so we will assume that $\varepsilon \ne 1$ hereafter. When $\varepsilon \ne 1$, solving (3.6) for $\xi(k+1)$, we have

$$(3.7)\qquad \xi(k+1) = \frac{1}{1-\varepsilon}\{\alpha + \varepsilon\xi(k-1) + (1-5\alpha-2\varepsilon)\xi(k) + 4\alpha\xi^2(k)\}.$$

Equation (3.7) clearly has two fixed points $\xi(k) = \frac{1}{4}$ and $\xi(k) = 1$ if $0 < \alpha \le 1$ and $\xi(k) = \xi(0)$ if $\alpha = 0$, for all $\varepsilon > 0$ as before. Letting again $x_k = \xi(k-1), y_k = \xi(k)$, (3.7) is reduced to a 2D discrete dynamical system:

$$(3.8)\qquad \begin{aligned}x_{k+1} &= y_k,\\ y_{k+1} &= \delta + \omega x_k + (1-5\delta-\omega)y_k + 4\delta y_k^2,\end{aligned}$$

where $\delta = \frac{\alpha}{1-\varepsilon}$ and $\omega = \frac{\varepsilon}{1-\varepsilon}$.

Now, the system (3.8) is generated by a 2D map $S_{\delta,\omega} : \mathbf{R}^2 \to \mathbf{R}^2$ given by

$$(3.9)\qquad S_{\delta,\omega}(x,y) = (y, \delta + \omega x + (1-5\delta-\omega)y + 4\delta y^2).$$

We first show that this map is dynamically the same as the Hénon map defined as follows:

Definition 3.1. The Hénon map $H_{a,b} : \mathbf{R}^2 \to \mathbf{R}^2$ is defined by

$$H_{a,b}(x,y) = (a + by - x^2, x), \tag{3.10}$$

where a, b are real parameters.

Remark 3.1. The Hénon map was originally defined by Hénon himself in the form

$$T_{a,b}(x,y) = (1 + y - ax^2, bx), \tag{3.11}$$

which represents one of the canonical forms for general quadratic maps with constant Jacobian determinant. But, this map $T_{a,b}$ can be transformed into the form $H_{a,b}$ given in (3.10) with the parameter values a and b unchanged, by the simple scaling $x \to x/a, y \to by/a$ for $a \neq 0$ and $b \neq 0$. Note that if $a = 0$, then $T_{a,b}$ becomes a linear map, while in the map $H_{a,b}$ given in (3.10), $a = 0$ has no special significance.

Now, we first show the dynamical equivalence between our map and the Hénon map.

Lemma 3.1. *Our map $S_{\delta,\omega}$ given in (3.9) is topologically conjugate to the Hénon map defined by (3.10) with the parameters $a = \frac{1}{4}(3\delta + \omega - 1)(3\delta - \omega + 1)$ and $b = \omega$, via the affine transformation given by $h(x,y) = (c_1 y + c_2, c_1 x + c_2)$ with $c_1 = -4\delta$ and $c_2 = -\frac{1}{2}(1 - 5\delta - \omega)$ where $\delta = \frac{\alpha}{1-\varepsilon}$ and $\omega = \frac{\varepsilon}{1-\varepsilon}$.*

Remark 3.2. In fact, as Devaney and Nitecki ([12]) have already mentioned, we don't have to examine the dynamical behavior of $H_{a,b}$ for the b values with $|b| > 1$, since the dynamics of $H_{a,b}$ for $|b| > 1$ does not exhibit any new behavior. This is because the inverse Hénon map

$$H_{a,b}^{-1}(x,y) = (y, \frac{1}{b}(x - a + y^2))$$

with given parameter values $a = A, b = B \neq 0$ is conjugate to the forward map $H_{a,b}$ with $a = A/B^2, b = 1/B$ by the linear change of variables $x \to -By, y \to -Bx$. Therefore, without loss of generality, we may restrict ourselves to parameters a, b satisfying $|b| \leq 1$. Thus, using Lemma 3.1, it is enough for us to consider only the case $|\omega| = |\frac{\varepsilon}{1-\varepsilon}| \leq 1$. This means that we need only consider the case $0 < \varepsilon \leq \frac{1}{2}$, i.e., $0 < \omega \leq 1$ and $\delta > 0$. Note that the case $\varepsilon = \frac{1}{2}$ or $\omega = 1$ is the orientation reversing and area-preserving case.

Now, let us first consider the local bifurcation problem about the map $S_{\delta,\omega}$. Our map $S_{\delta,\omega}$ clearly has two fixed points $Q_1 = (1/4, 1/4)$ and $Q_2 = (1,1)$ as before for all $0 < \omega \leq 1$ and $\delta > 0$. Contrary to the case in Section 7.2, $Q_2 = (1,1)$ is always unstable (saddle) for $0 < \omega \leq 1$ and $\delta > 0$, while $Q_1 = (1/4, 1/4)$ is stable (node)

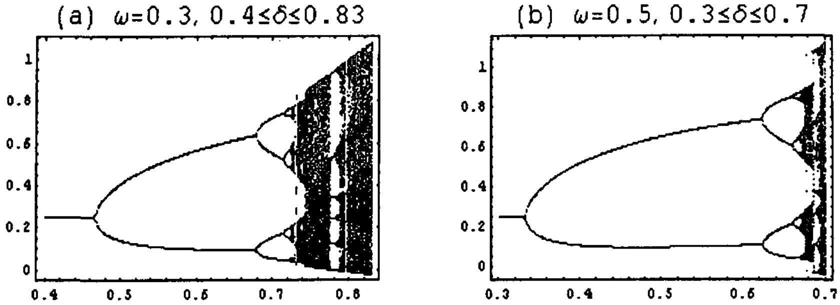

Figure 1. (a) Bifurcation diagram when $\omega = 0.3$; (b) Bifurcation diagram when $\omega = 0.5$. After this successive bifurcation ends, orbits escape to infinity. Note that chaotic motion is confined in a bounded region (black region).

for $0 < \omega < 1$ and $0 < \delta < \frac{2}{3}(1 - \omega)$, and unstable (saddle) for $0 < \omega \leq 1$ and $\delta > \frac{2}{3}(1 - w)$. Note that at $\omega = 1$, $Q_1 = (1/4, 1/4)$ is unstable for $\delta > 0$. In this case, near the fixed point $Q_1 = (1/4, 1/4)$, we have a period-doubling bifurcation as can be shown in the following lemma. See Figure 1.

Lemma 3.2. *For each fixed* $0 < \omega < 1$, $S_{\delta,\omega}$ *has an attracting fixed point* $Q_1 = (1/4, 1/4)$ *for* $0 < \delta < \frac{2}{3}(1 - \omega)$ *and this fixed point undergoes a period-doubling bifurcation as* δ *passes through the value* $\delta = 2(1 - \omega)/3$.

Now, let us consider the global bifurcation problem about the map $S_{\delta,\omega}$.

Hénon([14]) investigated the dynamics of the map $T_{a,b}$ given in Remark 3.1 by numerical experiments. Recall that $T_{a,b}$ is conjugate to the $H_{a,b}$ without changing the values of the parameters a and b. He considered four crucial a-values, say, A_0, A_1, A_2 and A_3. Here, $A_0 = -\frac{1}{4}(1 - b)^2$ and $A_1 = \frac{3}{4}(1 - b)^2$ are the values such that two fixed points exist for $a > A_0$, and one is always unstable and the other is unstable for $a > A_1$; while $A_2 = 1.06$ and $A_3 = 1.55$ are the numerical values obtained when $b = 0.3$. To concentrate on investigating the existence of a strange attractor, he fixed the contraction parameter b appropriately as $b = 0.3$ through many experiments and showed that:

(i) for $a < A_0$ or $a > A_3$, the points in the plane always escape to infinity (by using the Lemma 7.3.1, for our map $S_{\delta,\omega}$, $a < A_0$ implies $\delta^2 < 0$ and so has no meaning, and $b = 0.3, a > A_3$ corresponds to $\omega = 0.3, \delta > 0.862168$ or, in terms of the original parameters, $\varepsilon \approx 0.230769, \alpha > 0.663206$).

(ii) for $A_0 < a < A_3$, depending on the initial point (x_0, y_0), either the point escape to infinity or converges to an attractor (i.e., $\omega = 0.3, 0 < \delta < 0.862168$ or,

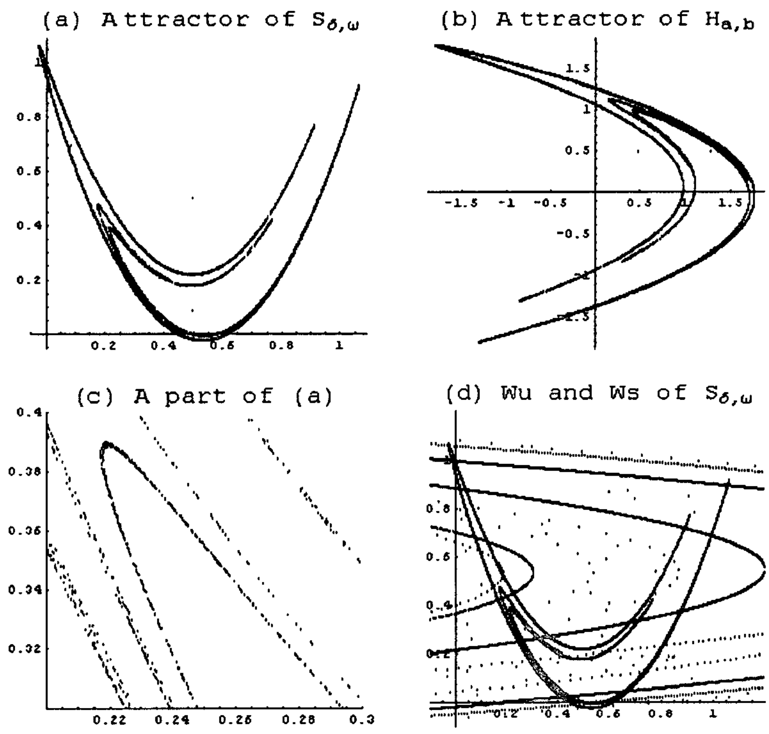

Figure 2. (a) Strange attractor of $S_{\delta,\omega}$ for $\omega = 0.3, \delta \approx 0.822598$; (b) Strange attractor of $H_{a,b}$ for $b = 0.3, a = 1.4$; (c) Fractal structure of the strange attractor of $S_{\delta,\omega}$. It is an example of 2D Cantor set, i.e., the Cartesian product of 1D Cantor set and 1D manifold (curve); (d) The stable (horizontal one) and the unstable manifold (vertical one) of $S_{\delta,\omega}$ at the fixed point $(\frac{1}{4}, \frac{1}{4})$ for $\omega = 0.3, \delta \approx 0.822598$. It appears that the strange attractor is a part of the unstable manifold. Note that the unstable manifold is quadratically tangent to the stable manifold at the bottom part.

$\varepsilon \approx 0.230769, 0 < \alpha < 0.663206$ for our map $S_{\delta,\omega}$).
(iii) for $A_0 < a < A_1$, the attractor is simply the stable fixed point. But, as a is increased over A_1, the attractor consists of a periodic set of p points and the value of p increases through successive bifurcations as a increases and p seems to go to infinity as a approaches to $A_2 = 1.06$ (i.e., $A_0 < a < A_1$ corresponds to $\omega = 0.3, 0 < \delta < \frac{2}{3}(1-\omega) \approx 0.466667$ or, $0 < \alpha < 0.358974$ for our map and $A_2 = 1.06$ corresponds to $\delta \approx 0.724952$ or,$\alpha \approx 0.557655$). See Figure 1(a)(b).
(iv) for $A_2 < a < A_3$, the attractor is no more simple, and the behavior of the points become erratic. He finally chose the parameter value $a = 1.4, b = 0.3$ (i.e., $\omega = 0.3, \delta \approx 0.822598$ or, $\varepsilon \approx 0.230769, \alpha \approx 0.632767$) and demonstrated the celebrated

Hénon attractor. See Figure 2. In this figure, we can confirm again the conjugacy between $S_{\delta,\omega}$ and $H_{a,b}$.

Shortly later, Curry ([10]) has shown that for Hénon's values of the parameters ($a = 1.4, b = 0.3$), one of the fixed points has a transversal homoclinic orbit and hence there is a horseshoe embedded in the dynamics of the map $H_{a,b}$. Feit ([13]) has shown that for $a > 0, 0 < b < 1$ (i.e., $0 < \omega < 1, \delta > \frac{1}{3}(1-\omega)$ for our case), the non-wandering set $\Omega(H_{a,b})$ is contained in a compact set, and all points outside this set escape to infinity.

Devaney and Nitecki's results apparently cover the above results and so we again convert their results to the case of our map $S_{\delta,\omega}$ in the following lemma. Note that their results also include the case $b = 1$, i.e., the case $\omega = 1$ for $S_{\delta,\omega}$.

Lemma 3.3. *For any fixed $0 < \omega \le 1$, the map $S_{\delta,\omega}$, depending on the values of $\delta > 0$, shows the following global dynamics:*

(i) *For $\delta > 0$, the nonwandering set $\Omega(S_{\delta,\omega})$ is contained in a square*

$$V = \left\{(x,y) \middle| \frac{5}{8} - r_1 \le x \le \frac{5}{8} + r_2, \frac{5}{8} - r_1 \le y \le \frac{5}{8} + r_2\right\},$$

where $r_1 = \frac{2+\sqrt{9\delta^2+4\omega}}{8\delta}$ and $r_2 = \frac{2\omega+\sqrt{9\delta^2+4\omega}}{8\delta}$.

(ii) *For $\delta \ge \delta_1 = \frac{1}{3}\sqrt{9(1+\omega)^2 - 4\omega}$, $\Lambda = \cap_{n\in\mathbf{Z}} S^n_{\delta,\omega}(V)$ is a topological horseshoe, i.e., there exists a continuous semi-conjugacy of $\Omega(S_{\delta,\omega}) \subset \Lambda$ onto the 2-shift.*

(iii) *For $\delta > \delta_2 = \frac{1}{3}\sqrt{(1+\sqrt{5})^2(1+\omega)^2 - 4\omega}$, $\Lambda = \Omega(S_{\delta,\omega})$ has a hyperbolic structure and is topologically conjugate to the 2-shift.*

Note that the square V in Lemma 3.3 contains the fixed points $(\frac{1}{4}, \frac{1}{4})$ and $(1,1)$ and as $\delta \to +\infty$, the r_1 and r_2 approaches to $\frac{3}{8}$, and so V shrinks to the square $\{(x,y) | \frac{1}{4} \le x \le 1, \frac{1}{4} \le y \le 1\}$. Hence, all the bifurcating periodic orbits and the compact invariant set Λ are contained in V.

Kirchgraber and Stoffer([16]) also studied the 2-parameter family Hénon map $H_{a,b}$ and have proved that:
for $|b| \in (0,1]$ and $a \ge 20 + \frac{91}{20}b - \frac{19}{400}b^2$, the Hénon map $H_{a,b}$ admits a transversal homoclinic point.

We restate this result in terms of our map $S_{\delta,\omega}$ in the following.

Lemma 3.4. *For $0 < \omega \le 1$ and $\delta \ge 3 + 0.3\omega$, our map $S_{\delta,\omega}$ admits a transversal homoclinic point to the hyperbolic fixed point $Q_1 = (\frac{1}{4}, \frac{1}{4})$. Consequently, near the* *hyperbolic fixed point* $Q_1 = (\frac{1}{4}, \frac{1}{4})$, *there exists an invariant Cantor set on which the dynamics of* $S_{\delta,\omega}$ *is topologically conjugate to a 2-shift and so is chaotic.*

Moreover, Arai and Mischaikow ([2]) have shown by numerically aided proof that for parameter values close to $a = 1.4, b = 0.3$ ($\delta \approx 0.822598, \omega = 0.3$ for $S_{\delta,\omega}$), there exist homoclinic tangencies, which imply the existence of the abundant strange attractors ([17]) and the occurrence of the Newhouse phenomena (i.e., the successive bifurcation to infinitely many attracting periodic orbits) ([18]). See Figure 2(d).

Now, again combining the results of the above four Lemmas and the result of Arai and Mischaikow, and converting the parameters δ, ω to the original parameters α, ε, we can state the following Theorem for $0 < \varepsilon \le \frac{1}{2}$ (For $\varepsilon > \frac{1}{2}$, see Remark 3.1).

Theorem 3.5. *The traveling waves* $p_j(n) = \xi(j+n) = \xi(k)$ *with wave velocity 1, satisfy the 2nd order nonlinear difference equation given by (3.9):*

$$\xi(k+1) = \frac{1}{1-\varepsilon}\{\alpha + \varepsilon\xi(k-1) + (1 - 5\alpha - 2\varepsilon)\xi(k) + 4\alpha\xi^2(k)\},$$

which has two spatially homogeneous static traveling wave $\xi(k) = \frac{1}{4}$ *and* $\xi(k) = 1$ *for* $0 < \alpha \le 1$ *and* $\xi(k) = \xi_0$ *for* $\alpha = 0$. *The traveling wave solutions show the following dynamics:*

(i) *For each fixed* $0 < \varepsilon < \frac{1}{2}$, *the spatially homogeneous static traveling wave solution* $\xi(k) = \frac{1}{4}$ *is attracting for* $0 < \alpha < \frac{2}{3}(1 - 2\varepsilon)$ *and undergoes a period-doubling bifurcation as* α *passes through the value* $\alpha = \frac{2}{3}(1 - 2\varepsilon)$. *At* $\varepsilon = 1/2$, *the fixed point* $\xi(k) = \frac{1}{4}$ *becomes hyperbolic and so there is no local bifurcation.*

(ii) *For each fixed* $0 < \varepsilon \le \frac{1}{2}$, *if* $\alpha > 0$, *then all the bifurcating spatially-periodic traveling waves and the spatially-chaotic traveling waves are confined to the infinite strip*

$$I = \left\{\{\xi(k)\} \mid \frac{5}{8} - s_1 \le \xi(k) \le \frac{5}{8} + s_2, k \in \mathbf{Z}\right\},$$

where

$$s_1 = \frac{1}{8\alpha}\{\varepsilon(1-\varepsilon) + \sqrt{9\alpha^2 + 4\varepsilon(1-\varepsilon)}\}$$

and

$$s_2 = \frac{1}{8\alpha}\{2\varepsilon + \sqrt{9\alpha^2 + 4\varepsilon(1-\varepsilon)}\}.$$

(iii) *For values close to* $\varepsilon \approx 0.230769, \alpha \approx 0.632767$, *there exist abundant strange attractors and successive bifurcation to infinitely many attracting periodic orbits.*

(iv) *If* $\alpha \geq \alpha_1$, *the motion of the spatially-chaotic traveling waves is semi-conjugate to the shift on two symbols, where* $\alpha_1 = \frac{1}{3}\sqrt{9 - 4\varepsilon(1-\varepsilon)}$.

(v) *If* $\alpha \geq \alpha_2$, *the motion of the spatially-chaotic traveling waves is conjugate to the* 2*-shift, where* $\alpha_2 = \frac{1}{3}\sqrt{(1+\sqrt{5})^2 - 4\varepsilon(1-\varepsilon)}$.

(vi) *If* $\alpha \geq 3 - 2.7\varepsilon$, *then there exist traveling wave solutions which converge to the spatially homogeneous static solution* $\xi(k) = \frac{1}{4}$ *as* $k \to \pm\infty$ *and compact invariant Cantor sets of real numbers on which the motions of traveling wave solutions* $\xi(k)$ *exhibit spatial chaos along the coordinates* k *and are topologically conjugate to the* 2*-shift.*

Therefore, according to Theorem 3.5, the bounded traveling wave solutions are as follows:
(i) the static spatially homogeneous solutions, i.e., $\{\xi(k)\} = \{\frac{1}{4}\}$ and $\{\xi(k)\} = \{1\}$ for $0 < \alpha \leq 1$, and $\{\xi(k)\} = \{\xi(0)\}$ for $\alpha = 0$.
(ii) the infinitely many spatially (and temporally)-periodic traveling wave solutions created through the period-doubling bifurcation of the fixed point $\xi(k) = \frac{1}{4}$ or the successive bifurcation of periodic sinks due to the existence of homoclinic tangencies.
(iii) the homoclinic traveling wave solutions which converge to the spatially homogeneous solution $\xi(k) = \frac{1}{4}$ as $k \to \pm\infty$.
(iv) the bounded spatially-chaotic traveling wave solutions created right after the period-doubling bifurcation of the fixed point $\xi(k) = \frac{1}{4}$ or the successive bifurcation of the periodic sinks, or created by the points in the strange attractors.
(v) the bounded spatially-chaotic traveling wave solutions created due to the transversal homoclinic orbits to the fixed point $\xi(k) = \frac{1}{4}$.

APPENDIX

Proof of Lemma 3.1. By using the parameter relationships, we can immediately check that the commutativity relation $h \circ S_{\delta,\omega}(x,y) = H_{a,b} \circ h(x,y)$ holds. □

Proof of Lemma 3.2. For fixed $0 < \omega < 1$, if $\delta = 2(1-\omega)/3$, $Q_1 = (1/4, 1/4)$ has eigenvalues -1 and ω. So, by using the standard procedure of applying the center manifold reduction and normal form theory ([22]), we can easily show that $Q_1 = (1/4, 1/4)$ undergoes a period-doubling bifurcation as δ passes through the value $\delta = 2(1-\omega)/3$. □

Proof of Lemma 3.3. By using the parameter relationships $a = \frac{1}{4}\{9\delta^2 - (\omega-1)^2\}$ and

$b = \omega$ and the change of coordinates as in Lemma 3.3, we can get the results. □

Proof of Lemma 3.4. The parameter relationships $a = \frac{1}{4}\{9\delta^2 - (\omega - 1)^2\}$ and $b = \omega$ give the results. □

REFERENCES

1. Afraimovich, V. S. & Bunimovich, L. A.: Simplest structures in coupled map lattices and their stability. preprint in GIT (1992).
2. Arai, Z. & Mischaikow, K.: Rigorous Computations of Homoclinic Tangencies. Preprint. *Kyoto University* (2006).
3. Aranson, I.S., Afraimovich, V.S. & Rabinovich, M.I.: Stability of spatially homogeneous chaotic regimes in unidirectional chains. *Nonlinearity* **3** (1990), 639-651.
4. Brock, W. A. & Hommes, C. H.: A Rational Route to Randomness. *Econometrica* **65** (1997), no. 5, 1059-1095.
5. ______ : Heterogeneous Beliefs and Routes to Chaos in a Simple Asset Pricing Model. *Journal of Economic Dynamics and Control* **22** (1998), 1235-1274.
6. Brock, W. A., Hommes, C. H. & Wagner, F.O.O.: Evolutionary dynamics in markets with many trader types. *Journal of Mathematical Economics* **41** (2005), 7-42.
7. Bunimovich, L. A. et al: Trivial Maps. *Chaos* **2** (1992).
8. Bunimovich, L. A. & Sinai, Y. G.: 1988. Spacetime Chaos in Coupled Map Lattices. *Nonlinearity* **1** (1998), 491-516.
9. Choudhary, M. A. & Orszag, J. M.: A cobweb model with local externalities. *Journal of Economic Dynamics & Control* (2007).
10. Curry, J.: On the structure of the Hénon attracter. Preprint. *National center for atmospheric research* (1997).
11. Devaney, R. L.: *An introduction to Chaotic Dynamical Systems.* Addison-Wesley, 1989.
12. Devaney, R. L. & Nitecki, Z.: Shift Automorphisms in the Hénon Mapping. *Communications in Mathematical Physics* **67**, 137-146.
13. Feit, S.: Characteristic exponents and strange attracters. *Communications in Mathematical Physics* **61** (1978), 249-260.
14. Hénon, M.: A Two-dimensional Mapping with a Strange Attracter. *Communications in Mathematical Physics* **50** (1976), 69-77.
15. Hommes, C. H.: On the consistency of backward-looking expectations: The case of the cobweb. *Journal of Economic Behavior & Organization* **33** (1998), 333-362.
16. Kirchgraber, U. & Stoffer, D.: Transversal homoclinic points of the Hénon map. *Annali di Mathematica* **185** (2006), 187-204.
17. Mora, L. & Viana, M.: Abundance of Strange Attractors. *Acta Mathematica* **171** (1993), 1-71.

18. Robinson, C.: Bifurcation to Infinitely many sinks. *Communications in Mathematical Physics* **90** (1983), 433-459.
19. ______: *Dynamical Systems: Stability, Symbolic Dynamics, and Chaos.* CRC Press
20. Sterling, D., Dullin, H.R. & Meiss, J.D.: Homoclinic bifurcations for the Hénon map. *Physica D* **134** (1999), 153-184.
21. Wiggins S.: *Global Bifurcations and Chaos.* Springer-Verlag, New York, 1988.
22. ______: *Introduction to Applied Nonlinear Dynamical Systems and Chaos.* Springer-Verlag, New York, 1990

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ULSAN, ULSAN 689-749, KOREA
Email address: yikim@mail.ulsan.ac.kr