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SOLVABILITY FOR SECOND-ORDER BOUNDARY VALUE PROBLEM WITH INTEGRAL BOUNDARY CONDITIONS ON AN UNBOUNDED DOMAIN AT RESONANCE

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ABSTRACT. This paper deals with the second-order differential equation

$$(p(t)x'(t))' + g(t)f(t, x(t), x'(t)) = 0, \text{ a.e. in } (0, \infty)$$

with the boundary conditions

$$x(0) = \int_0^\infty g(s)x(s)ds, \ \lim_{t\to\infty} p(t)x'(t) = 0,$$

where $g \in L^1[0,\infty)$ with $g(t) > 0$ on $[0,\infty)$ and $\int_0^\infty g(s)ds = 1$, f is a g-Carathéodory function. By applying the coincidence degree theory, the existence of at least one solution is obtained.

1. INTRODUCTION

In this paper, we study the second-order boundary value problem with integral boundary condition on a half line

(1.1) $$(p(t)x'(t))' + g(t)f(t, x(t), x'(t)) = 0, \text{a.e. in } (0, \infty),$$

(1.2) $$x(0) = \int_0^\infty g(s)x(s)ds, \ \lim_{t\to\infty} p(t)x'(t) = 0.$$

Throughout we assume

(A1) $p \in C[0,\infty) \cap C^1(0,\infty)$ and $p(t) > 0$ on $[0,\infty)$, $\frac{1}{p} \in L^1[0,\infty)$,

(A2) $g \in L^1[0,\infty)$ with $g(t) > 0$ on $[0,\infty)$ and $\int_0^\infty g(s)ds = 1$.

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Due to the condition $\int_0^\infty g(s)ds = 1$, the differential operator $-\frac{1}{g}\frac{d}{dt}(p\frac{d}{dt}\cdot)$ is not invertible. In the literature, BVPs of this type are referred to problems at resonance.

The motivation for the present work stems from both practical and theoretical aspects. In fact, second-order BVPs on infinite intervals arising from the study of radially symmetric solutions of nonlinear elliptic equation and models of gas pressure in a semi-infinite porous medium [1], have received much attention. For an extensive collection of results on BVPs on unbounded domains, we refer the readers to a monograph by Agarwal and O'Regan [1]. Other recent results and methods can be found in [4-13] and the reference therein. In [9], N. Kosmanov considered the second-order nonlinear differential equation at resonance

$$(p(t)u'(t))' = f(t, u(t), u'(t)), \quad \text{a.e. in } (0, \infty)$$

with two sets of boundary conditions:

$$u'(0) = 0, \ \sum_{i=1}^{n} \kappa_i u_i(T_i) = \lim_{t\to\infty} u(t)$$

and

$$u(0) = 0, \ \sum_{i=1}^{n} \kappa_i u_i(T_i) = \lim_{t\to\infty} u(t).$$

The author established existence theorems by the coincidence degree theorem of Mawhin.

Lian and Ge [10] studied the following second-order BVPs on a half-line

$$\begin{cases} x''(t) = f(t, x(t), x'(t)), \ 0 < t < \infty, \\ x(0) = x(\eta), \ \lim_{t\to\infty} x'(t) = 0 \end{cases}$$

and

$$\begin{cases} x''(t) = f(t, x(t), x'(t)) + e(t), \ 0 < t < \infty, \\ x(0) = x(\eta), \ \lim_{t\to\infty} x'(t) = 0, \end{cases}$$

By using Mawhin's continuation theorem, they obtained the existence results.

However, as we all know, for the resonance case, there has no work done for the boundary value problems with integral boundary conditions on a half-line, such as the BVP (1.1)-(1.2). The aim of this paper is to fill the gap in the relevant literatures. In addition, we define a g-Carathéodory function (Def. 2.1) in this paper, this is the first time that this definition has been introduced.

2. Related Lemma

For the convenience of readers, we present here some definitions and lemmas.

Definition 2.1. $f : [0,\infty) \times \mathbb{R}^2 \to \mathbb{R}$ is called a g-Carathéodory function if and only if

(B1) for each $(u,v) \in \mathbb{R}^2$, the mapping $t \mapsto f(t,u,v)$ is Lebesgue measurable on $[0,\infty)$,

(B2) for a.e. $t \in [0,\infty)$, the mapping $(u,v) \mapsto f(t,u,v)$ is continuous on $\mathbb{R}^2$,

(B3) for each $l > 0$ and $g \in L^1[0,\infty)$, there exists a function $\phi_l : [0,\infty) \to [0,\infty)$ satisfying $\int_0^\infty g(s)\phi_l(s)ds < \infty$ such that

$$\max\{|u|,|v|\} \le l \ \text{ implies } \ |f(t,u,v)| \le \phi_l(t) \ \text{ for a.e. } t \in [0,\infty).$$

Theorem 2.1 ([2]). *Let X and Y be two Banach spaces with norms $||\cdot||_X$ and $||\cdot||_Y$, respectively, and $\Omega \subset X$ an open and bounded set. Suppose $L : X \cap \mathrm{dom}L \to Y$ is a Fredholm operator of index zero and $N : \overline{\Omega} \to Y$ is L-compact. In addition, if*

(C1) *$Lx \ne \lambda Nx$ for $\lambda \in (0,1)$, $x \in (\mathrm{dom}L \setminus \ker L) \cap \partial\Omega$;*

(C2) *$Nx \notin \mathrm{Im}L$ for $x \in \ker L \cap \partial\Omega$;*

(C3) *$\deg\{JQN|_{\overline{\Omega}\cap\ker L}, \Omega \cap \ker L, 0\} \ne 0$, where $Q : Y \to Y$ is a projection such that $\mathrm{Im}L = \ker Q$ and $J : \mathrm{Im}Q \to \ker L$ is an isomorphism.*

Then the abstract equation $Lx = Nx$ has at least one solution in $\overline{\Omega}$.

Let $AC[0,\infty)$ denotes the space of absolutely continuous function on the interval $[0,\infty)$. In this paper, we work in the Banach spaces

$$X = \Big\{x \in C[0,\infty) : x, px' \in AC[0,\infty), \lim_{t\to\infty} x(t) \text{ and } \lim_{t\to\infty} x'(t) \text{ exist}, (px')' \in L^1[0,\infty)\Big\},$$

$$Y = \Big\{y : [0,\infty) \to \mathbb{R} : \int_0^\infty g(t)|y(t)|dt < \infty\Big\}$$

with the norms $||x||_X = \max\{||x||_\infty, ||x'||_\infty\}$, where $||x||_\infty = \sup_{t\in[0,\infty)} |x(t)|$ and $||y||_Y = \int_0^\infty g(t)|y(t)|dt$. Let $L : \mathrm{dom}L \to Y$ with

$$\mathrm{dom}L = \Big\{x \in X : gx \in L^1[0,\infty),\ x(0) = \int_0^\infty x(s)g(s)ds,\ \lim_{t\to\infty} p(t)x'(t) = 0\Big\}$$

be defined by $Lx(t) = -\frac{1}{g(t)}(p(t)x'(t))'$.

Lemma 2.1. *$L : \mathrm{dom}L \subset X \to Y$ is a Fredholm operator of index zero.*

Proof. It is easy to see that $\ker L = \{x \in \mathrm{dom}L : x(t) \equiv c \text{ on } [0,\infty)\}$.
Consider the following equation for $x \in \mathrm{dom}L$,

$$-\frac{1}{g(t)}(p(t)x'(t))' = y(t). \tag{2.1}$$

Then $y \in Y$. From (2.1) and (1.2), one gets

$$x(t) = \int_0^t \frac{1}{p(s)} \int_s^\infty g(\tau)y(\tau)d\tau ds + x(0). \tag{2.2}$$

Since $\int_0^\infty g(s)ds = 1$, it follows from (1.2) that

$$\int_0^\infty g(t) \int_0^t \frac{1}{p(s)} \int_s^\infty g(\tau)y(\tau)d\tau ds dt = 0. \tag{2.3}$$

Thus,

$$\mathrm{Im}L \subset \left\{ y \in Y : \int_0^\infty g(t) \int_0^t \frac{1}{p(s)} \int_s^\infty g(\tau)y(\tau)d\tau ds dt = 0 \right\}.$$

Conversely, if (2.3) holds, we take candidate of $x \in \mathrm{dom}L$ as given by (2.2), then $(p(t)x'(t))' + g(t)y(t) = 0$ for $t \in (0,\infty)$ and (1.2) is satisfied. In fact, we have

$$\mathrm{Im}L = \left\{ y \in Y : \int_0^\infty g(t) \int_0^t \frac{1}{p(s)} \int_s^\infty g(\tau)y(\tau)d\tau ds dt = 0 \right\}. \tag{2.4}$$

Define the projection $Q : Y \to Y$ by

$$(Qy)(t) = \frac{1}{\omega} \int_0^\infty g(s) \int_0^s \frac{1}{p(\tau)} \int_\tau^\infty g(r)y(r)dr d\tau ds \tag{2.5}$$

where $\omega = \int_0^\infty g(s) \int_0^s \frac{1}{p(\tau)} \int_\tau^\infty g(r)dr d\tau ds$. (2.4) and (2.5) imply that $\mathrm{Im}L = \ker Q$. Then codim $\mathrm{Im}L = \dim \mathrm{Im}Q = 1 = \dim \ker L$. As a result, L is a Fredholm operator of index zero. □

Let $P : X \to X$ be a projection defined by $(Px)(t) = x(0)$ for $t \in [0,\infty)$. Set $L_p = L|_{\mathrm{dom}L \cap \ker P}$ and $K_p : \mathrm{Im}L \to \mathrm{dom}L \cap \ker P$ denotes the inverse of L_p. Define

$$K_p y(t) = \int_0^\infty k(t,s)g(s)y(s)ds$$

where

$$k(t,s) := \begin{cases} \int_0^s \frac{1}{p(\tau)} d\tau, & 0 \le s < t < \infty, \\ \int_0^t \frac{1}{p(\tau)} d\tau, & 0 \le t \le s < \infty. \end{cases}$$

Let the nonlinear operator $N : X \to Y$ be defined by

$$Nx(t) = f(t, x(t), x'(t)), \ t \in [0, \infty).$$

Then the BVP (1.1)-(1.2) can be written as $Lx = Nx$, which is equivalent to

$$\begin{aligned} &x = Px + K_p(I - Q)Nx, \\ &JQNx = 0, \end{aligned}$$

where $J : \mathrm{Im}Q \to \ker L$ is an isomorphism.

In order to apply Theorem 2.1, we have to prove that N is L-compact, that is, QN and $K_p(I - Q)N$ are compact on every bounded subset of X. Because the Arzelà-Ascoli theorem fails to apply to noncompact interval case, we will use the following criterion.

Theorem 2.2 ([1]). *Let X be the space of all bounded continuous vector-valued functions on $[0, \infty)$ and $S \subset X$. Then S is relatively compact if the following conditions hold:*

(D1) *S is bounded in X,*

(D2) *all functions from S are equicontinuous on any compact subinterval of $[0, \infty)$,*

(D3) *all functions from S are equiconvergent at infinity, that is, for any given $\varepsilon > 0$, there exists a $T = T(\varepsilon) > 0$ such that $||\phi(t) - \phi(\infty)||_{\mathbb{R}^n} < \varepsilon$ for all $t > T$ and $\phi \in S$.*

Lemma 2.2. *If f is a g-Carathéodory function, then $N : X \to Y$ is L-compact.*

Proof. Suppose that $\Omega \subset X$ is a bounded set. Then there exists $l > 0$ such that $||x||_X \le l$ for $x \in \overline{\Omega}$. Since f is an g-Carathéodory function, there exists a function ϕ_l satisfying $\phi_l(t) \ge 0$ on $[0, \infty)$ and $\int_0^\infty g(s)\phi_l(s)ds < \infty$ such that for a.e. $t \in [0, \infty)$, $|f(t, x(t))| \le \phi_l(t)$ for $x \in \overline{\Omega}$. Then for $x \in \overline{\Omega}$,

$$\begin{aligned} ||QNx||_Y &= \frac{1}{\omega}\int_0^\infty g(t)|\int_0^\infty g(s)\int_0^s \frac{1}{p(\tau)}\int_\tau^\infty g(r)f(r, x(r), x'(r))drd\tau ds|dt \\ &\le \frac{1}{\omega}\int_0^\infty \frac{1}{p(\tau)}d\tau \int_0^\infty g(s)|f(s, x(s), x'(s))|ds \\ &\le \frac{1}{\omega}\int_0^\infty \frac{1}{p(\tau)}d\tau \int_0^\infty g(s)\phi_l(s)ds \\ &= \frac{1}{\omega}\left\|\frac{1}{p}\right\|_{L^1} \cdot ||g\phi_l||_{L^1} =: M_1 < \infty, \end{aligned}$$

which implies that QN is bounded on $\overline{\Omega}$. Noticing that dim Im$Q = 1$, it follows that QN is a compact operator. Next, we show that $K_p(I-Q)N$ is compact, i.e., $K_p(I-Q)N$ maps bounded sets into relatively compact ones. Furthermore, denote $K_{P,Q} = K_p(I-Q)$(see [9,10]). For $x \in \overline{\Omega}$, one gets

$$\begin{aligned}|(K_{P,Q}Nx)(t)| &\le \int_0^\infty |k(t,s)g(s)[f(s,x(s),x'(s)) - (QNx)(s)]|ds \\ &\le \int_0^\infty \frac{1}{p(\tau)}d\tau\left[\int_0^\infty g(s)|f(s,x(s),x'(s))|ds + \int_0^\infty g(s)|(QNx)(s)|ds\right] \\ &\le \left\|\frac{1}{p}\right\|_{L^1}(\|g\phi_l\|_{L^1} + \|QNx\|_Y) =: M_2 < \infty\end{aligned}$$

and

$$\begin{aligned}|(K_{P,Q}Nx)'(t)| &= \left|\frac{1}{p(t)}\int_t^\infty g(s)[f(s,x(s),x'(s)) - (QNx)(s)]ds\right| \\ &\le \frac{1}{p(t)}\left[\int_0^\infty g(s)|f(s,x(s),x'(s))|ds + \int_0^\infty g(s)|(QNx)(s)|ds\right] \\ &\le \left\|\frac{1}{p}\right\|_\infty(\|g\phi_l\|_{L^1} + \|QNx\|_Y) =: M_3 < \infty,\end{aligned}$$

that is, $K_{P,Q}N(\overline{\Omega})$ is uniformly bounded. Meanwhile, for any $t_1, t_2 \in [0,T]$ with T a positive constant and $t_1 < t_2$, we have

$$\begin{aligned}|(K_{P,Q}Nx)(t_2) - (K_{P,Q}Nx)(t_1)| &= \left|\int_{t_1}^{t_2}(K_{P,Q}Nx)'(s)ds\right| \\ &\le M_3|t_2 - t_1| \to 0, \text{ uniformly as } |t_2 - t_1| \to 0,\end{aligned}$$

$$\begin{aligned}&|(K_{P,Q}Nx)'(t_2) - (K_{P,Q}Nx)'(t_1)| \\ &= \left|\frac{1}{p(t_2)}\int_{t_2}^\infty g(s)[f(s,x(s),x'(s)) - (QNx)(s)]ds\right. \\ &\quad \left.-\frac{1}{p(t_1)}\int_{t_1}^\infty g(s)[f(s,x(s),x'(s)) - (QNx)(s)]ds\right| \\ &\le \frac{1}{p(t_2)}\left|\int_{t_2}^{t_1} g(s)[f(s,x(s),x'(s)) - (QNx)(s)]ds\right| \\ &\quad + \left|\frac{1}{p(t_2)} - \frac{1}{p(t_1)}\right|\int_{t_1}^\infty g(s)\left|f(s,x(s),x'(s)) - (QNx)(s)\right|ds \\ &\le \left\|\frac{1}{p}\right\|_\infty\left(\int_{t_1}^{t_2} g(s)\phi_l(s)ds + \int_{t_1}^{t_2} g(s)|(QNx)(s)|ds\right) \\ &\quad + \left|\frac{1}{p(t_2)} - \frac{1}{p(t_1)}\right|(\|g\phi_l\|_{L^1} + \|QNx\|_Y) \to 0, \text{ uniformly as } |t_2 - t_1| \to 0,\end{aligned}$$

which means that $K_{P,Q}N(\overline{\Omega})$ is equicontinuous. In addition, we claim that $K_{P,Q}N(\overline{\Omega})$ is equiconvergent at infinity. In fact,

$$\begin{aligned}
&|(K_{P,Q}Nx)(\infty) - (K_{P,Q}Nx)(t)| \\
= &\left| \int_t^{\infty} \int_t^{s} \frac{1}{p(\tau)} d\tau g(s)(f(s, x(s), x'(s)) - (QNx)(s)) ds \right| \\
\leq &\int_t^{\infty} \frac{1}{p(\tau)} d\tau \left(\int_t^{\infty} g(s)\phi_l(s) ds + \int_t^{\infty} g(s)|(QNx)(s)| ds \right) \to 0, \\
&\text{uniformly as } t \to \infty
\end{aligned}$$

and

$$\begin{aligned}
&|(K_{P,Q}Nx)'(\infty) - (K_{P,Q}Nx)'(t)| \\
= &\left| \frac{1}{p(t)} \int_t^{\infty} g(s)[f(s, x(s), x'(s)) - (QNx)(s)] ds \right| \\
\leq &\frac{1}{p(t)} \left[\int_t^{\infty} g(\tau)\phi_l(\tau) d\tau + \int_t^{\infty} g(\tau)|(QNx)(\tau)| d\tau \right] \to 0, \text{ uniformly as } t \to \infty.
\end{aligned}$$

Hence, Theorem 2.2 implies that $K_p(I-Q)N(\overline{\Omega})$ is relatively compact. Furthermore, since f is a g-Carathéodory function, the continuity of QN and $K_p(I-Q)N$ on $\overline{\Omega}$ follows from the Lebesgue dominated convergence theorem. So we can complete the proof. □

3. Main Results

In this section, we establish two existence results for the BVP (1.1)-(1.2) by applying Mawhin's continuation theorem.

Theorem 3.1. *If f is a g–Carathéodory function, suppose*
(H1) *there exists a constant $A > 0$ such that for $x \in \text{dom}L \setminus \ker L$, if $|x(t)| > A$ for all $t \in [0, \infty)$, then*

$$\int_0^{\infty} g(s) \int_0^{s} \frac{1}{p(\tau)} \int_{\tau}^{\infty} g(r) f(r, x(r), x'(r)) dr d\tau ds \neq 0; \tag{3.1}$$

(H2) *there exist nonnegative functions $\alpha, \beta, \gamma, \rho \in Y$ and a constant $\sigma \in [0, 1)$ such that for $(u, v) \in \mathbb{R}^2$ and a.e. $t \in [0, \infty)$, one has*

$$|f(t, u, v)| \leq \alpha(t)|u| + \beta(t)|v| + \gamma(t)|u|^{\sigma} + \rho(t), \tag{3.2}$$

we denote $\alpha_1 = ||\alpha||_Y$, $\beta_1 = ||\beta||_Y$, $\gamma_1 = ||\gamma||_Y$, $\rho_1 = ||\rho||_Y$;
(H3) *there exists a constant $B > 0$ such that for $b \in \mathbb{R}$ with $|b| > B$, we have either*

$$b\cdot\int_0^\infty g(s)\int_0^s\frac{1}{p(\tau)}\int_\tau^\infty g(r)f(r,b,0)drd\tau ds<0 \tag{3.3}$$

or

$$b\cdot\int_0^\infty g(s)\int_0^s\frac{1}{p(\tau)}\int_\tau^\infty g(r)f(r,b,0)drd\tau ds>0. \tag{3.4}$$

Then BVP(1.1) *has at least one solution provided*

$$\max\left\{\beta_1\|\frac{1}{p}\|_\infty,\ \frac{\alpha_1\|\frac{1}{p}\|_{L^1}}{1-\beta_1\|\frac{1}{p}\|_\infty}\right\}<1. \tag{3.5}$$

Proof. Let $U_1=\{x\in \mathrm{dom}L\setminus\ker L: Lx=\lambda Nx \text{ for some } \lambda\in(0,1)\}$. For $x\in U_1$, $\lambda Nx=Lx\in \mathrm{Im}L=\ker Q$, so $QNx=0$, then

$$\int_0^\infty g(s)\int_0^s\frac{1}{p(\tau)}\int_\tau^\infty g(r)f(r,x(r),x'(r))drd\tau ds=0.$$

It follows from (H1) that there exists $t_0\in[0,\infty)$ such that $|x(t_0)|\le A$. Then,

$$|x(t)|=\left|x(t_0)+\int_{t_0}^t x'(s)ds\right|\le A+\|x'\|_{L^1},$$

that is,

$$\|x\|_\infty\le A+\|x'\|_{L^1}. \tag{3.6}$$

Since

$$x'(t)=\frac{1}{p(t)}\int_t^\infty \lambda g(s)f(s,x(s),x'(s))ds,$$

$$\begin{aligned}\|x'\|_\infty &= \sup_{t\in[0,\infty)}\left|\frac{1}{p(t)}\int_t^\infty \lambda g(s)f(s,x(s),x'(s))ds\right|\\ &\le \left\|\frac{1}{p}\right\|_\infty\int_0^\infty g(s)[\alpha(s)|x(s)|+\beta(s)|x'(s)|+\gamma(s)|x(s)|^\sigma+\rho(s)]ds\\ &\le \left\|\frac{1}{p}\right\|_\infty(\alpha_1\|x\|_\infty+\beta_1\|x'\|_\infty+\gamma_1\|x\|_\infty^\sigma+\rho_1).\end{aligned}$$

Noticing (3.5), one gets

$$\|x'\|_\infty\le\frac{(\alpha_1\|x\|_\infty+\gamma_1\|x\|_\infty^\sigma+\rho_1)\|\frac{1}{p}\|_\infty}{1-\beta_1\|\frac{1}{p}\|_\infty}. \tag{3.7}$$

$$
\begin{aligned}
\|x'\|_{L^1} &= \int_0^\infty \left| \frac{1}{p(t)} \int_t^\infty \lambda g(s) f(s, x(s), x'(s)) ds \right| dt \\
&\le \left\| \frac{1}{p} \right\|_{L^1} \int_0^\infty g(s)[\alpha(s)|x(s)| + \beta(s)|x'(s)| + \gamma(s)|x(s)|^\sigma + \rho(s)] ds \qquad (3.8)\\
&\le \left\| \frac{1}{p} \right\|_{L^1} (\alpha_1 \|x\|_\infty + \beta_1 \|x'\|_\infty + \gamma_1 \|x\|_\infty^\sigma + \rho_1).
\end{aligned}
$$

In view of (3.6), (3.7) and (3.8), we have

$$
\begin{aligned}
\|x\|_\infty &\le A + \left\| \frac{1}{p} \right\|_{L^1} [\alpha_1 \|x\|_\infty + \frac{\beta_1 \|\frac{1}{p}\|_\infty}{1 - \beta_1 \|\frac{1}{p}\|_\infty} (\alpha_1 \|x\|_\infty + \gamma_1 \|x\|_\infty^\sigma + \rho_1) \\
&\quad + \gamma_1 \|x\|_\infty^\sigma + \rho_1] \\
&= \frac{\alpha_1 \|\frac{1}{p}\|_{L^1}}{1 - \beta_1 \|\frac{1}{p}\|_\infty} \|x\|_\infty + \frac{\gamma_1 \|\frac{1}{p}\|_{L^1}}{1 - \beta_1 \|\frac{1}{p}\|_\infty} \|x\|_\infty^\sigma + \left(\frac{\rho_1 \|\frac{1}{p}\|_{L^1}}{1 - \beta_1 \|\frac{1}{p}\|_\infty} + A \right).
\end{aligned}
$$

Due to $0 \le \sigma < 1$ and (3.5), there exists a constant $E_1 > 0$ such that $\|x\|_\infty \le E_1$. Then

$$
\|x'\|_\infty \le \frac{(\alpha_1 E_1 + \gamma_1 E_1^\sigma + \rho_1) \|\frac{1}{p}\|_\infty}{1 - \beta_1 \|\frac{1}{p}\|_\infty} =: E_2.
$$

Hence, $\|x\|_X \le \max\{E_1, \ E_2\}$, that is, U_1 is bounded.

Define $U_2 = \{x \in \ker L : Nx \in \mathrm{Im} L\}$. For $x \in U_2$, then $x(t) \equiv c$ on $[0, \infty)$ and $Nx \in \mathrm{Im} L = \ker Q$. Thus

$$
\int_0^\infty g(s) \int_0^s \frac{1}{p(\tau)} \int_\tau^\infty g(r) f(r, c, 0) dr d\tau ds = 0.
$$

From (H3) we get that $\|x\|_X = |c| \le B$. So U_2 is bounded.

Let $U_3 = \{x \in \ker L : -\mu x + (1 - \mu) JQNx = 0, \ \mu \in [0, 1]\}$, where $J : \ker L \to \mathrm{Im} Q$ is an isomorphism given by $J(c) = c$ for $c \in \mathbb{R}$. For $c_0 \in U_3$, we obtain

$$
\mu c_0 = (1 - \mu) \frac{1}{\omega} \int_0^\infty g(s) \int_0^s \frac{1}{p(\tau)} \int_\tau^\infty g(r) f(r, c_0, 0) dr d\tau ds.
$$

If $\mu = 1$, then $c_0 = 0$. Otherwise, for $\mu \in [0, 1)$, if $|c_0| > B$, in view of (3.3), one gets

$$
0 \le \mu c_0^2 = c_0 (1 - \mu) \frac{1}{\omega} \cdot \int_0^\infty g(s) \int_0^s \frac{1}{p(\tau)} \int_\tau^\infty g(r) f(r, c_0, 0) dr d\tau ds < 0,
$$

which is a contradiction. Thus, $U_3 \subset \{x \in \ker L : \|x\|_X \le B\}$ is bounded.

Let U be a bounded open subset of X such that $U = \{x \in X : \|x\|_X < \max\{E_1, E_2, B\} + 1\}$. Clearly, $\bigcup_{i=1}^{3} \overline{U}_i \subset U$. So, the first two conditions in Theorem 2.1 are satisfied. To this end, it remains to show that condition (C3) holds, since L is a Fredholm operator of index zero and N is L-compact from Lemma 2.1-2.2.

Consider the homotopy $H : (\ker L \cap \overline{U}) \times [0,1] \to X$ defined by

$$H(x,\mu) = -\mu x + (1-\mu)JQNx.$$

Since $H(x,\mu) \neq 0$ for $x \in \ker L \cap \partial U$. By the homotopy invariance of degree, we get

$$\begin{aligned} \deg\{JQN\mid_{\overline{U}\cap\ker L}, U\cap\ker L, 0\} &= \deg\{H(\cdot,0), U\cap\ker L, 0\} \\ &= \deg\{H(\cdot,1), U\cap\ker L, 0\} \\ &= \deg\{-I, U\cap\ker L, 0\} \neq 0. \end{aligned}$$

Then Theorem2.1 yields that $Lx = Nx$ has at least one solution in dom$L \cap \overline{U}$. The proof is completed. □

Remark 3.1. When the second part of condition (H3) holds, we choose $\tilde{U}_3 = \{x \in \ker L : \mu x + (1-\mu)JQNx = 0, \mu \in [0,1]\}$ and take homomorphism $\tilde{H}(x,\mu) = \mu x + (1-\mu)JQNx$. By a similar argument, we can complete the proof.

Theorem 3.2. *If f is a g–Carathéodory function, assume that the conditions* (H1), (H2) *and* (H3) *in Theorem* 3.1 *hold with* (3.2) *in* (H2) *replaced by*

$$|f(t,u,v)| \leq \alpha(t)|u| + \beta(t)|v| + \gamma(t)|v|^{\sigma} + \rho(t). \tag{3.9}$$

Then the BVP (1.1)-(1.2) *has at least one solution provided*

$$\max\left\{\alpha_1\left\|\frac{1}{p}\right\|_{L^1}, \frac{\beta_1\|\frac{1}{p}\|_{\infty}}{1-\alpha_1\|\frac{1}{p}\|_{L^1}}\right\} < 1. \tag{3.10}$$

To illustrate our main results, we see the following example.

Example 3.1 Consider

$$\begin{cases} 3e^t(e^t x'(t))' + f(t,x(t),x'(t)) = 0, \\ x(0) = \displaystyle\int_0^{\infty} e^{-s}x(s)ds, \ \lim_{t\to\infty} 3e^t x'(t) = 0. \end{cases} \tag{3.11}$$

Corresponding to the BVP (1.1)-(1.2), we have $p(t) = 3e^t$, $g(t) = e^{-t}$. Obviously, the conditions (A1) and (A2) are satisfied. Taking $f(t,u,v) = te^{-2t}$, it is easy to verify that the assumptions (H1)-(H3) hold. Let $\alpha(t) = te^{-2t}$, $\beta(t) = 0$, then $\alpha_1 = \frac{1}{9}$, $\beta_1 = 0$. Since $\|\frac{1}{p}\|_{\infty} = \frac{1}{3}$, $\|\frac{1}{p}\|_1 = \frac{1}{3}$, (3.5) and (3.10) are satisfied.

Thus, thanks to theorems 3.1 and 3.2, the BVP (3.11) has at least one solution.

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