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ON THE WEAK NATURAL NUMBER OBJECT OF THE WEAK TOPOS FUZ

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Abstract. Category *Fuz* of fuzzy sets has a similar function to the Category *Set*. But it forms a weak topos. We study a natural number object and a weak natural number object in the weak topos *Fuz*. Also we study the weak natural number object in Fuz^C.

1. Introduction

Category *Fuz* of fuzzy sets has a similar function to the topos *Set*. *Fuz* has finite products, middle object, equalizers, exponentials and weak subobject classifier. But *Fuz* is not a topos, it forms a weak topos. There are some comparisons between weak topos *Fuz* and topos *Set*. A natural number object in a topos means an object together with morphisms. An important characterization of natural number objects in a topos was given by P. Freyd. Natural number object applied to define the order structure and retains a certain amount of Booleanness. In this paper, first we show that *Fuz* has no nontrivial natural number object. So we define a weak natural number object in a weak topos. And we show that there exists a weak natural number object in the weak topos *Fuz* and Fuz^C.

2. Preliminaries

In this section, we state some definitions and properties which will serve as the basic tools for the arguments used to prove our results.

Definition 2.1. An *elementary topos* is a category $\mathcal{E}$ that satisfies the following;

(T1) $\mathcal{E}$ is finitely complete,

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(T2) $\mathcal{E}$ has exponentiation,

(T3) $\mathcal{E}$ has a subobject classifier.

(T2) means that for every object A in $\mathcal{E}$, the endofunctor $(-) \times A$ has its right adjoint $(-)^A$. Hence for every object A in $\mathcal{E}$, there exists an object B^A, and a morphism $ev_A : B^A \times A \to B$, called the evaluation map of A, such that for any Y and $f : Y \times A \to B$ in $\mathcal{E}$, there exists a unique morphism g such that $ev_A \circ (g \times id) = f$;

$$\begin{array}{ccc} Y \times A & \xrightarrow{f} & B \\ {\scriptstyle g\times id}\downarrow & & \downarrow{\scriptstyle id} \\ B^A \times A & \xrightarrow[ev_A]{} & B \end{array}$$

And subobject classifier in (T3) is an $\mathcal{E}$-object Ω, together with a morphism $\top : \mathbf{1} \to \Omega$ such that for any monomorphism $h : D \to C$, there is a unique morphism $\chi_h : C \to \Omega$, called the character of $h : D \to C$ which makes the following diagram a pull-back;

$$\begin{array}{ccc} D & \xrightarrow{!} & \mathbf{1} \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle \top} \\ C & \xrightarrow[\chi_h]{} & \Omega \end{array}$$

Example 2.2. Category *Set* is a topos. $\{*\}$ is a terminal object. $\Omega = \{0, 1\}$ and $\top : \{*\} \to \Omega$ with $\top(*) = 1$ is a subobject classifier. If we define

$\chi_h = 1$ if $c = h(d)$ for some $d \in D$,

$\chi_h = 0$ otherwise

then χ_h is a characteristic function of D .

Category *Fuz* of fuzzy sets is a category whose object is (A, α_A) where A is an object and $\alpha_A : A \to I$ is a morphism with $I = (0, 1]$ in *Set* and morphism from (A, α_A) to (B, α_B) is a function $f : A \to B$ in *Set* such that $\alpha_A(a) \leqq \alpha_B \circ f(a)$ [3].

Definition 2.3. We say that an object (I, α_I) is a *middle object* of *Fuz* if there exists a unique morphism $f : A \to I$ such that $\alpha_A(a) = \alpha_I \circ f(a)$ for all (A, α_A) and $a \in A$.

Definition 2.4. We say that $((J, \alpha_J), i)$ is a *weak subobject classifier* of *Fuz* if there exists a unique morphism $\alpha_f : (A, \alpha_A) \to (J, \alpha_J)$ for all monomorphism $f : (B, \alpha_B) \to (A, \alpha_A)$ where $J = [0, 1]$ and $\alpha_J(j) = 1$ for all $j \in J$ such that $\alpha_f(a) \leqq$

$\alpha_A(a)$ and the following diagram

$$\begin{array}{ccc} (B,\alpha_B) & \xrightarrow{\alpha_B} & (I,\alpha_I) \\ f\downarrow & & \downarrow i \\ (A,\alpha_A) & \xrightarrow[\alpha_f]{} & (J,\alpha_J) \end{array}$$

is a pull-back.

Definition 2.5. A weak topos is a category $\mathcal{E}$ that satisfies the following;

(WT1) $\mathcal{E}$ has equalizer, finite product and exponentiation.

(WT2) $\mathcal{E}$ has a middle object.

(WT3) $\mathcal{E}$ has a weak subobject classifier.

Proposition 2.6. *Category Fuz is a weak topos.*

For the proof see Yuan and Lee [4].

Definition 2.7. A natural number object in a topos $\mathcal{E}$ means an object N together with morphisms

$$\mathbf{1} \xrightarrow{0} N \xrightarrow{s} N,$$

where $\mathbf{1}$ is a terminal object in a topos, such that for any diagram

$$\mathbf{1} \xrightarrow{a} A \xrightarrow{f} A,$$

there exists a unique morphism $h : N \to A$ such that

$$\begin{array}{ccccc} \mathbf{1} & \xrightarrow{0} & N & \xrightarrow{s} & N \\ id\downarrow & & \downarrow h & & \downarrow h \\ \mathbf{1} & \xrightarrow[a]{} & A & \xrightarrow[f]{} & A \end{array}$$

commutes.

Definition 2.8. A weak natural number object in a weak topos *Fuz* means an object N together with morphisms

$$I \xrightarrow{0} N \xrightarrow{s} N,$$

where I is the middle object in the weak topos *Fuz*, such that for any diagram with a normal object A

$$I \xrightarrow{a} A \xrightarrow{f} A$$

there exists a unique morphism $h : N \to A$ such that

$$\begin{array}{ccccc} I & \xrightarrow{0} & N & \xrightarrow{s} & N \\ {\scriptstyle id}\downarrow & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle h} \\ I & \xrightarrow[a]{} & A & \xrightarrow[f]{} & A \end{array}$$

commutes.

3. Main Parts

Proposition 3.1. *Fuz has no nontrivial natural number object.*

Proof. In *Fuz*, the terminal object $\mathbf{1}$ is a singleton set $(\{*\}, \alpha_{\{*\}})$ with $\alpha_{\{*\}}(*) = 1 \in I$. Assume that there exists a natural number object in *Fuz*. That is, there exists an object (N, α_N) together with morphisms

$$\mathbf{1} = (\{*\}, \alpha_{\{*\}}) \xrightarrow{0} (N, \alpha_N) \xrightarrow{s} (N, \alpha_N)$$

such that for any diagram

$$\mathbf{1} = (\{*\}, \alpha_{\{*\}}) \xrightarrow[a]{} (A, \alpha_A) \xrightarrow[f]{} (A, \alpha_A),$$

there exists a unique morphism $h : (N, \alpha_N) \to (A, \alpha_A)$ such that

$$\begin{array}{ccccc} \mathbf{1} = (\{*\}, \alpha_{\{*\}}) & \xrightarrow{0} & (N, \alpha_N) & \xrightarrow{s} & (N, \alpha_N) \\ {\scriptstyle id}\downarrow & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle h} \\ \mathbf{1} = (\{*\}, \alpha_{\{*\}}) & \xrightarrow[a]{} & (A, \alpha_A) & \xrightarrow[f]{} & (A, \alpha_A) \end{array}$$

commutes.

We need a condition that $\alpha_N \circ 0(*) \geqq \alpha_{\{*\}}(*)$, so that we have $\alpha_N \circ 0(*) = \alpha_N(0) = 1$ for $0 \in N$. Since $s : (N, \alpha_N) \to (N, \alpha_N)$ is a morphism in *Fuz*, where $s(n) = n+1$, it satisfy that $\alpha_N \circ s(0) \geqq \alpha_N(0)$. That is, $\alpha_N(1) \geqq \alpha_N(0)$. So we get $\alpha_N(1) = 1$. Inductively we get $\alpha_N(n) = 1$ for all $n \in N$. Also, we need a condition that $\alpha_A \circ h \geqq \alpha_N$, so that we have $\alpha_A(a) = 1$ for all $a \in A$. □

Corollary 3.2. *In Fuz, there exists an object (N, α_N), where $\alpha_N(n) = 1$ for all $n \in N$, with morphisms*

$$\mathbf{1} = (\{*\}, \alpha_{\{*\}}) \xrightarrow{0} (N, \alpha_N) \xrightarrow{s} (N, \alpha_N)$$

such that for any diagram

$$\mathbf{1} = (\{*\}, \alpha_{\{*\}}) \xrightarrow{a} (A, \alpha_A) \xrightarrow{f} (A, \alpha_A),$$

where $\mathbf{1} = (\{*\}, \alpha_{\{*\}})$ *is a terminal object and* $\alpha_A(a) = 1$ *for all* $a \in A$*, there exists a unique* $h : (N, \alpha_N) \to (A, \alpha_A)$ *such that*

$$\begin{array}{ccccc}
\mathbf{1} = (\{*\}, \alpha_{\{*\}}) & \xrightarrow{0} & (N, \alpha_N) & \xrightarrow{s} & (N, \alpha_N) \\
id \downarrow & & \downarrow h & & \downarrow h \\
\mathbf{1} = (\{*\}, \alpha_{\{*\}}) & \xrightarrow[a]{} & (A, \alpha_A) & \xrightarrow[f]{} & (A, \alpha_A)
\end{array}$$

commutes.

Lemma 3.3. *Fuz has finite products.*

Proof. Let $(A, \alpha_A), (B, \alpha_B)$ be two objects in *Fuz*. Consider $((A \times B, \alpha_{A\times B}), p_A, p_B)$ where $A\times B$ is the cartesian product of a pair (A, B) of the topos *Set* with $\alpha_{A\times B}$=min $\{\alpha_A, \alpha_B\}$ and projection morphisms $p_A : (A \times B, \alpha_{A\times B}) \to (A, \alpha_A), p_B : (A \times B, \alpha_{A\times B}) \to (B, \alpha_B)$ satisfying $\alpha_B \circ p_B \geqq \alpha_{A\times B}$ and $\alpha_A \circ p_A \geqq \alpha_{A\times B}$. Then, for any morphisms $f : (X, \alpha_X) \to (A, \alpha_A)$ and $g : (X, \alpha_X) \to (B, \alpha_B)$, there exists a unique morphism $< f, g >: (X, \alpha_X) \to (A\times B, \alpha_{A\times B})$ such that $p_A\circ < f, g >= f$ and $p_B\circ < f, g >= g$. Since $\alpha_A f(x) \geqq \alpha_X(x)$, $\alpha_B g(x) \geqq \alpha_X(x)$ and $\alpha_{A\times B}(f(x), g(x)) =$ min $\{\alpha_A f(x), \alpha_B g(x)\}$, we have that $\alpha_{A\times B}(f(x), g(x)) \geqq \alpha_X(x)$, so $\alpha_{A\times B} \circ (f, g) \geqq \alpha_X$. Thus $< f, g >: (X, \alpha_X) \to (A \times B, \alpha_{A\times B})$ is a morphism in *Fuz*. □

Theorem 3.4. *Fuz has a weak natural number object.*

Proof. Let (N, α_N) be an object with $\alpha_N(n) = 1$ for all $n \in N$. Then by Lemma 3.3, there exists an object $((N \times I), \alpha_{N\times I})$, where (I, α_I) is the middle object in *Fuz*. Consider the object $((N \times I), \alpha_{N\times I})$ with morphisms

$$I \xrightarrow{0} N \times I \xrightarrow{s'} N \times I$$

defined by $0(i) = (0, i)$ and $s'(n, i) = (n + 1, i)$. Then it satisfy that $\alpha_{N\times I} \circ 0 \geqq \alpha_I$ and $\alpha_{N\times I} \circ s' \geqq \alpha_{N\times I}$.

For any normal object (A, α_A), we define a morphism $a : I \to A$ with $a(i) = a$ for all $i \in (0, \alpha_A(a)]$ and $a(i) = c$ for all $i \in (\alpha_A(a), 1]$, where $\alpha_A(c) = 1$. Then $\alpha_A a(i) \geqq \alpha_I(i)$ making $a : I \to A$ a morphism in *Fuz*.

Then for any diagram,

$$I \xrightarrow{a} A \xrightarrow{f} A$$

there exists a unique morphism $h : N \times I \to A$ defined by $h(0, i) = a(i)$ and $f \circ h(n, i) = h(n + 1, i)$ such that

$$\begin{array}{ccccc} I & \xrightarrow{0} & N \times I & \xrightarrow{s'} & N \times I \\ {\scriptstyle id}\downarrow & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle h} \\ I & \xrightarrow[a]{} & A & \xrightarrow[f]{} & A \end{array}$$

commutes.

Then $\alpha_A a(i) \geqq \alpha_I(i)$ and $h(0,i) = a(i)$ imply $\alpha_A h(0,i) \geqq \alpha_I(i)$,
so $\alpha_A h(0,i) \geqq \alpha_{N\times I}(0,i)$.

And $\alpha_A \circ f \geqq \alpha_A$ implies $\alpha_A \circ f \circ h(0,i) \geqq \alpha_A \circ h(0,i) \geqq \alpha_I(i)$.

So we have that $\alpha_A h(1,i) \geqq \alpha_I(i)$. It implies that $\alpha_A h(1,i) \geqq \alpha_{N\times I}(1,i)$. Inductively we show that $\alpha_A h(n,i) \geqq \alpha_{N\times I}(n,i)$.

If there exists an another morphism $k : N \times I \to A$ such that $k(0,i) = a(i)$ and $f \circ k(n,i) = k \circ s'(n,i) = k(n+1,i)$.

Then we have that $f \circ k(0,i) = k \circ s'(0,i)$ and $f \circ h(0,i) = h \circ s'(0,i)$.

Also we have $k \circ 0(i) = a(i)$ and $h \circ 0(i) = a(i)$. So $f \circ k(0,i) = k(1,i)$ and $f \circ h(0,i) = h(1,i)$.

This imply $k(1,i) = f \circ a(i) = h(1,i)$.

Inductively, $h : N \times I \to A$ is the unique morphism in *Fuz*. □

Theorem 3.5. *For any small category C, Fuz^C has a weak natural number object.*

Proof. Consider a constant functor $W : C \to Fuz$ having $W(a) = N \times I$ for all $a \in C$ and $W(f) = id_{N\times I}$ for all $f \in C$. Also consider a constant natural transformation $s' : W \to W$ having $s'_a(n,i) = (n+1,i)$ and a constant natural transformation $0 : J \to W$ having $0_a(i) = (0,i)$ where $J : C \to Fuz$ is a functor defined by $J(a) = I$ for all $a \in C$ and $J(f) = id$. Then for any diagram

$$J \xrightarrow{k} K \xrightarrow{f} K$$

there exists a unique morphism $h : W \to K$ defined by $h_a(0,i) = k_a(i)$ and $f \circ h(n,i) = h(n+1,i)$, that is, the following diagram

$$\begin{array}{ccccc} J & \xrightarrow{0} & W & \xrightarrow{s'} & W \\ {\scriptstyle id}\downarrow & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle h} \\ J & \xrightarrow[k]{} & K & \xrightarrow[f]{} & K \end{array}$$

commutes.

Since $k : J \to K$ is a natural transformation, the square

$$\begin{array}{ccc} J(a) & \xrightarrow{J(\alpha)=id} & J(b) \\ {\scriptstyle k_a}\downarrow & & \downarrow{\scriptstyle k_b} \\ K(a) & \xrightarrow[K(\alpha)]{} & K(b) \end{array}$$

commutes. And we get $k_a(i) = h_a 0_a(i) = h_a(0, i)$. These imply $K(\alpha)h_a(0, i) = h_b(0, i)$. We assume that $K(\alpha)h_a(n, i) = h_b(n, i)$. Since $f : K \to K$ is a natural transformation, we have $K(\alpha) \circ f_a \circ h_a(n, i) = f_b \circ h_b(n, i)$. By definition of h, this implies $K(\alpha)h_a(n+1, i) = h_b(n+1, i)$. By induction, we get $K(\alpha)h_a(n, i) = h_b(n, i)$ for any $n \in N$. So $h : W \to K$ is a natural transformation. □

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