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SECTIONAL ANALYTICITY IN SEQUENCE SPACES

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Abstract. The object of the present paper is to introduce Λ-dual and the concept of sectional analyticity (Abschinitts-anatytique or AA property) of an FK-space. The motivation for AA-property is that a sequence space having AK-property possess AA-property.

1. Introduction

A sequence whose k-th term is x_k is denoted by (x_k) or x. Let ω denote the set of all sequences. A sequence x is said to be an entire sequence if $|x_k|^{1/k} \to 0$ as $k \to \infty$. The set Γ of all entire sequences is an FK space [3] with seminorms $q_i = \sup\left\{\left|\sum_{k=1}^{\infty} x_k z^k\right| : |z| = i\right\}$ for $i = 1, 2, \ldots$. A sequence x is said to be an analytic sequence if $(|x_k|^{1/k})$ is bounded. Let Λ denote the set of all analytic sequences.

For each positive integer k, let δ^k stands for the sequence $(0, 0, \ldots, 0, 1, 0, \ldots)$ with 1 in the k-th place and zeros elsewhere. A sequence space X is said to be an AK space if $x^{[n]} \to x$ for each $x \in X$ where $x^{[n]} = (x_1, \ldots, x_n, 0, 0, \ldots)$. For a sequence space X its conjugate space is denoted by X'.

Let X be any sequence space. Then X^α is the Kothe-Toeplitz dual of X introduced in [7]. X^β is the space called the "g-dual" of X by Chillingworth in [1] and the β-dual of X by Kothe and others [8, p. 427]. For arbitrary sequences X and Y, X^Y is the space called $X \to Y$ by Goes [5, p. 137] and elsewhere. For $Y = bs$ and arbitrary X, X^γ corresponds to the γ-dual of X of Garling [4] and others. Let X be an FK space containing ϕ. Then the f-dual denoted by X^f is defined by [10] $X^f = \{[f(\delta^n)] : f \in X'\}$. An FK space X is called an integral space [2] if and only if $\Gamma \subset X$. The work presented in this paper is motivated by the following questions. "Are all integral spaces Λ-perfect?" and "Are all AA-space having AK-property?".

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In the sequel the following sequence spaces are required.

$c_0 =$ the BK space of all null sequences.

$c =$ the BK space of all convergent sequences.

$l =$ the BK space of all sequences (x_k) such that $\sum_{k=1}^{\infty} |x_k|$ converges.

$cs =$ the BK space of all sequences (x_k) such that $\sum\limits_{k=1}^{\infty} x_k$ converges.

$bs =$ the BK space of all sequences (x_k) such that $\sup\limits_n \left|\sum\limits_{k=1}^{\infty} x_k\right| < \infty$.

The rest of the paper is organized as follows:

In Section 2, we introduce the concepts of Λ-dual and Λ-perfect. We have also tried to find the Λ-dual and Λ-perfect space of X with $\Gamma \subseteq X \subseteq \Lambda$.

In Section 3, we introduce the concept of sectional analyticity and try to find the relation between f-dual and Λ-dual.

2. Analytical Dual of a Sequence Space X

Definition 2.1. Let X be an FK space. The Λ-dual of X (denoted by $X^{\wedge}$) and may be called analytic dual of X is defined as $X^{\wedge} = \{x \in \omega : xu \in \Lambda$ for every $u \in X$

Definition 2.2. An FK-space X is called a perfect space or a Λ-perfect space if $X^{\Lambda\Lambda} = X$.

Remark 2.3. The definitions also hold when X is a singleton or a sequence space instead of an FK space.

Lemma 2.4. *The Λ-dual of a sequence space has the following properties.*

(1) $X^{\wedge}$ *is linear subspace of* ω *for* $X \subset \omega$.

(2) $X \subset Y$ *implies* $X^{\wedge} \supset Y^{\wedge}$ *for every* $X,\ Y \subset \omega$.

(3) $X^{\wedge\wedge} = (X^{\wedge})^{\wedge} \supset X$ *for every* $X \subset \omega$.

Lemma 2.5. (i) $1^{\Lambda} = \Lambda$ *where* $1 = (1, 1, 1, \ldots)$.

(ii) $\phi^{\Lambda} = \omega$.

(iii) *The Λ-dual of* $\chi = \{u \in \omega : [n!|u_n|]^{1/n} \to 0$ *as* $n \to \infty\}$ *is* $S_\infty = \{u \in \omega : (|u_n|/n!)^{1/n})$ *is bounded*$\}$ *suppose if* $x \in S_\infty$ *then* $|x_n u_n|^{1/n} \to 0$ *as* $n \to \infty$ *for all* $u \in X$. *So the sequence* $(|x_n u_n|^{1/n})$ *is bounded and hence* $S_\infty \subset \chi^{\Lambda}$ *on the other hand suppose* $x \notin S_\infty$. *Then there exists an increasing sequence* $n_1 < n_2 < \cdots$ *such that*

$$\left[\frac{|x_{n_k}|}{n_k!}\right]^{\frac{1}{n_k}} > k.$$

Define $u = (u_n)$ *by*

$$u_n = \begin{cases} \frac{1}{(n!)^n}, & \text{for } n = n_k \\ 0, & \text{other wise} \end{cases}$$

Then $[n!|u_n|]^{1/n} = \left[\frac{1}{(n!)^{n-1}}\right]^{\frac{1}{n}} = \frac{1}{(n-1)!} \quad \frac{(n!)^{\frac{1}{n}}}{n} \to 0$ *as* $n \to \infty$ *thus* u *is an element of* χ.

But $|x_{n_k} u_{n_k}|^{1/n_k} = \left[\frac{|x_{n_k}|}{n_k!}\right]^{\frac{1}{n_k}} > k$.

This contradicts the fact that $x \in \chi^\Lambda$ *and hence the* Λ*-dual of* χ *is* S_∞.

(iv) *The* Λ*-dual of* $R = \{x : (n!|x_n|)$ *is bounded*$\}$ *is* S_∞.

$$\text{Now} \quad |x_k u_k| = \left[\frac{|x_k|}{k!}\right] k!|u_k| \le ||u|| \frac{|x_k|}{k!}, \quad x = (x_k) \in S_\infty \text{ ([9])}.$$

Therefore $(|x_k u_k|^{1/k})$ is bounded and x is an element of $R^\wedge$. On the other hand if $x \in R^\wedge$ then $(|x_k u_k|^{1/k})$ is bounded for all $x \in R$.

Therefore $([|x_k/k!]^{1/k})$ is bounded for a particular $(1/k!) \in R$. Hence the Λ-dual of R is S_∞.

Theorem 2.6. *Suppose* $\Gamma \subseteq X \subseteq \Lambda$. *Then* $X^\wedge = \Lambda$.

Proof. Step (i): We first claim that $\Gamma^\wedge = \Lambda$. If $x \in \Lambda$ then $(|x_k|^{1/k})$ is bounded. For any $u \in \Gamma$ and x$\in \Lambda$, u x$\in \Lambda$ therefore x$\in \Gamma^\wedge$.

On the other hand suppose $x \notin \Lambda$ then there would exist an increasing sequence of positive integers $n_1 < n_2 < \ldots < n_k < \ldots$ such that $|x_{n_k}|^{1/n_k} > p^{n_k}$ where $p > 1$ is an integer. Construct a sequence $u = (u_n)$ as follows.

$$u_n = \begin{cases} \frac{k^n}{p^{n_k}}, & \text{if } n = n_k \ (k = 1, 2, \ldots) \\ 0, & \text{otherwise} \end{cases}$$

Obviously $u \in \Gamma$.

But $|x_{n_k} u_{n_k}|^{1/n_k} > k$, so that $(|x_n u_n|^{1/n})$ is unbounded which is a contradiction to the fact that $x \in \Gamma^\wedge$.

Thus $\Gamma^\wedge = \Lambda$.

Step (ii): We show that $\Lambda^\wedge = \Lambda$.

$N \subset \Lambda$ implies $\Lambda^\wedge \subset \Gamma^\wedge = \Lambda$ (by step (i)). That is $\Lambda^\wedge \subset \Lambda$. Also we have $\Lambda \subset \Lambda^\wedge$. Hence $\Lambda \subset \Lambda^\wedge$.

Step (iii): We show that $X^\wedge = \Lambda$.

$N \subseteq X \subseteq \Lambda$ implies $X^\wedge \subseteq \Gamma^\wedge$. Then by step (i) we have $X^\wedge \subseteq \Lambda$. Also $X \subseteq \Lambda$ implies $\Lambda^\wedge \subseteq X^\wedge$. Then by step (ii) we have $\Lambda \subseteq X^\wedge$. Thus $X^\wedge = \Gamma$. □

Corollary 2.7. *The only Λ perfect space X with $\Gamma \subseteq X \subseteq \Lambda$ is Λ.*

Proof. Let X be such that $X^{\wedge\wedge} = X$. Since $\Gamma \subseteq X$ we have $X^{\wedge} \subseteq \Gamma^{\wedge} = \Lambda$ (by step (i) of 2.6). By applying step (ii) of 2.6, $\Lambda = \Lambda^{\wedge} \subseteq X^{\wedge\wedge} = X$. Also by hypothesis $X \subseteq \Lambda$. □

3. Sectional Analyticity

Definition 3.1. Let X and Y be FK spaces containing ϕ. Then A^{+} is defined as $A^{+}(X) = \{z \in \omega : (z_k f(\delta^k) \in \Lambda \text{ for all } f \in X'\}$ and we put $A = A^{+} \cap X$.

Lemma 3.2. *Let X and Y be a be FK spaces containing ϕ. Then $A^{+}(X) \subset A^{+}(Y)$ wherever $X \subset Y$.*

Proof. Let $Z \in A^{+}(X)$. Then $(Z_n f(\delta^n)) \in \Lambda$ for all $f \in X'$. Accordingly $(z_n g(\delta^n)) \in \Lambda$ for all $g \in Y'$ since $g|X \in X'$. This shows that $z \in A^{+}(Y)$. Hence $A^{+}(x) \subset A^{+}(Y)$. □

Definition 3.3. Let X be an FK space containing ϕ. Then X is said to have AA. Property (Abschnitts analytique) or sectional analyticity if and only if $X = A$.

Example 3.4. The space c_0 has both AK and AA properties. The space c_0 has AK [10]. It is enough to prove that c_0 has AA-property. For that we have to show that $c_0 \subset A^{+}$, $f \in c_0'$. Then $f(z) = \sum\limits_{k=1}^{\infty} a_k z_k$ where $(a_k) \in l$. Therefore $f(\delta^k) = a_k$ for all k. But $l \subset \Lambda = \mathrm{c}^{\wedge}$. Hence $(z_k f(\delta^k)) \in \Lambda$ and so $z \in A^{+}$. Hence $c_0 \subset A^{+}$. Therefore $A = A^{+} \cap c_0 = c_0$.

Lemma 3.5. *Let X be an FK space containing ϕ. Let $z \in \omega$. Then $z \in A^{+}$ if and only if $z^{-1}X \supset \Gamma$.*

Proof. Let $f \in (z^{-1}X)'$. Then by Theorem 4.4.10 of [10] $f(x) = \alpha x + g(zx)$ where $\alpha \in \phi$, $g \in X'$ and $\alpha x = \sum\limits_{k=1}^{\infty} \alpha_k x_k$. Consequently $f(\delta^k) = \alpha_k + g(z\delta^k)$. That is $f(\delta^k) = \alpha_k + z_k g(\delta^k)$. Hence if $z \in A^{+}$, then $(z_k f(\delta^k)) \in A$ and so $(f(\delta^k)) \in \Lambda$ for all $f \in (z^{-1}X)'$. That is $(z^{-1}X)^f \subset \Lambda$. But $\Lambda = \Gamma^f$. Since Γ has AD by Theorem 8.6.1 of [10], $\Gamma \subset z^{-1}X$. The reverse implication follows similarly. □

Theorem 3.6. *Let X be an FK space containing ϕ. Then $z \in X^{f\wedge}$ if and only if $z^{-1}X \supset \Gamma$.*

Proof. First we note that by definition $z \in A^+$ if and only if $z\ u \in A$ for every $u \in X^f$. Hence $A^+ = X^{f\Lambda}$. By the Lemma 3.5, $z \in A^+$ if and only if $z^{-1}X \supset \Gamma$. Hence $z \in X^{f\Lambda}$ if and only if $z^{-1}X \supset \Gamma$. □

Theorem 3.7. *Let X be an FK space containing ϕ. If X has AA, then $X^f \subset X^\wedge$.*

Proof. Suppose that X has AA. Then $X = A = A^+ \cap X$. So that $X \subset A^+ = X^{f\wedge}$. Hence $X^\wedge \supset Xf^{\wedge\wedge}$. Therefore $X^\wedge \supset X^f$. □

Theorem 3.8. *Let X be an FK space $\supset \phi$. If X has AK then X has AA.*

Proof. Suppose X has AK. Then we have $X^\beta = X^f$. This implies $X \subset X^{\beta\beta} = X^{f\beta}$. Also we have $X \subset X^{f\beta} \subset X^{f\wedge}$. That is $X \subset X^{f\wedge}$. This means that $X \subset A^+$ consequently $A = X$. Hence X has AA property. □

Remark 3.9. The converse of Theorem 3.8 need not be true. Consider the space c, $A^+(c) = c^{f\wedge} = l^\wedge = \Lambda$. Now $A = A^+ \cap c = \Lambda \cap c = c$. Therefore c has AA. But c does not posses AK-Property.

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