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MINIMAL BASICALLY DISCONNECTED COVER OF WEAKLY P-SPACES AND THEIR PRODUCTS

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Abstract. In this paper, we introduce the concept of a weakly P-space which is a generalization of a P-space and prove that for any covering map $f : X \longrightarrow Y$, X is a weakly P-space if and only if Y is a weakly P-space. Using these, we investigate the minimal basically disconnected cover of weakly P-spaces and their products.

1. Introduction

All spaces in this paper are Tychonoff spaces and for any space X, βX denotes the Stone-Čech compactification of X.

In [7], Vermeer showed that every Tychonoff space X has the minimal basically disconnected cover $(\Lambda X, \Lambda_X)$ and that for any compact space X, ΛX is given by the Stone space $\mathrm{S}(\sigma Z(X)^{\#})$ of a Boolean algebra $\sigma Z(X)^{\#}$. In [1], Comfort, Hindman, and Negrepontis showed that if X is a P-space and Y is a countably locally weakly Lindelöf space, then $X \times Y$ is a basically disconnected space.

In this paper, we first introduce the concept of weakly P-spaces and show that for any covering map $f : X \longrightarrow Y$, X is a weakly P-space if and only if Y is a weakly P-space. Using this, we will show that if X is a weakly P-space, then ΛX is a P-space. For any space X, let S_X denote the subspace $\{\alpha | \alpha$ is a fixed $\sigma Z(X)^{\#}$-ultrafilter $\}$ of $\Lambda(\beta X)$ ([3]). For any spaces X, Y such that $\Lambda X = S_X$ and $\Lambda Y = S_Y$, we will show that there is a homeomorphism $h : S_X \times S_Y \longrightarrow S_{X\times Y}$ such that $\Lambda_X \times \Lambda_Y = g \circ h$, where the map $g : S_{X\times Y} \longrightarrow X \times Y$ is defined by $g(\delta) = \cap\delta$ and that the following are equivalent :

(1) $\Lambda X \times \Lambda Y = \Lambda(X \times Y)$,

(2) $S_{X\times Y} = \Lambda(X \times Y)$, and

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(3) $\Lambda X \times \Lambda Y$ is a basically disconnected space.

For the terminology, we refer to [2] and [5].

2. Minimal Basically Disconnected Cover of Weakly P-spaces

For any space X, the set $\mathrm{R}(X)$ of all regular closed sets in X, when partially ordered by inclusion, becomes a complete Boolean algebra, in which the join, meet and complementation operations are defined as follows :

$\bigvee\{A_i|i \in I\} = cl_X(int_X(\bigcup\{A_i|i \in I\}))$,

$\bigwedge\{A_i|i \in I\} = cl_X(\bigcap\{A_i|i \in I\})$ and

$A' = cl_X(X - A)$

and a sublattice of $R(X)$ is a subset of $R(X)$ that contains $\emptyset$, X and is closed under finite joins and meets.

We recall that a space X is called *a P-space* if every zero-set in X is open in X and that X is called *an almost P-space* if the empty set is the only zero-set Z in X with $int_X(Z) = \emptyset$. Similarly, for the set $RO(X)$ of all regular open sets in X, we can define a complete Boolean algebra $(RO(X), \subseteq)$.

Lemma 2.1 ([6]). *Let X be a compact space. Then X is an almost P-space if and only if for any increasing sequence (U_n) in $RO(X)$, $\bigcup\{U_n|n \in N\} \in RO(X)$.*

Note that A is a regular closed set in a space X if and only if $X - A$ is a regular open set in X. Using this, we have the following :

Corollary 2.2. *A compact space X is an almost P-space if and only if for any decreasing sequence (U_n) in $R(X)$, $\bigcap\{U_n|n \in N\} \in R(X)$.*

We introduce the concept of another generalization of P-spaces.

Definition 2.3. A space X is called *a weakly P-space* if for any decreasing sequence (U_n) in $R(X)$, $\bigcap\{U_n|n \in N\} \in R(X)$.

Proposition 2.4. *Let X be a space. Then the following are equivalent :*

(1) *X is a weakly P-space,*

(2) *for any decreasing sequence (U_n) in $R(X)$,*
$\bigwedge\{U_n|n \in N\}=\bigcap\{U_n|n \in N\}$, *and*

(3) *for any decreasing sequence (U_n) in $R(X)$ with $\bigwedge\{U_n|n \in N\} = \emptyset$, $\bigcap\{U_n|n \in N\} = \emptyset$*

Proof. (1) $\Rightarrow$ (2) Let (U_n) be a decreasing sequence in $R(X)$. Then clearly, $\bigwedge\{U_n|n \in N\} \subseteq \bigcap\{U_n|n \in N\}$. Since $\bigcap\{U_n|n \in N\} \in R(X)$, $\bigcap\{U_n|n \in N\} \subseteq \bigwedge\{U_n|n \in N\}$.

(2) $\Rightarrow$ (3) It is trivial.

(3) $\Rightarrow$ (1) Let (U_n) be a decreasing sequence in $R(X)$. Clearly, $\bigwedge\{U_n|n \in N\} \subseteq \bigcap\{U_n|n \in N\}$. Let $x \notin \bigwedge\{U_n|n \in N\}$. Then there is a regular closed neighborhood V of x in X such that $V \cap int_X(\bigcap\{U_n|n \in N\}) = \emptyset$ and hence $int_X(\bigcap\{V \wedge U_n|n \in N\}) = \emptyset$. Since $(V \wedge U_n)$ is a decreasing sequence in $R(X)$, $\bigcap\{V \wedge U_n|n \in N\} = \emptyset$. For any $n \in N$,

$$\begin{aligned} V \wedge U_n &= cl_X(int_X(V) \cap int_X(U_n)) \\ &\supseteq int_X(V) \cap cl_X(int_X(U_n)) \\ &= int_X(V) \cap U_n. \end{aligned}$$

Hence $int_X(V) \cap (\bigcap\{U_n|n \in N\}) = \emptyset$ and so $x \notin \bigcap\{U_n|n \in N\}$. Thus $\bigcap\{U_n|n \in N\} \subseteq \bigwedge\{U_n|n \in N\}$. □

Corollary 2.5. (1) *If X is a weakly P-space, then X is an almost P-space.*

(2) *A locally compact space X is a weakly P-space if and only if X is an almost P-space.*

Recall that a space X is called *a basically disconnected space* if every cozero-set in X is C^*-embedded in X, equivalently, for any zero-set Z in X, $int_X(Z)$ is closed in X.

Let X be a weakly P-space and Z a zero-set in X. Then there is a continuous map $f : X \longrightarrow \mathbb{R}$ such that $f^{-1}(0) = Z$, where $\mathbb{R}$ is the space with the usual topology. For any $n \in N$, let $U_n = cl_X(int_X(f^{-1}([0, \frac{1}{n}])))$. Then (U_n) is a decreasing sequence in $R(X)$ such that $Z = \bigcap\{U_n|n \in N\}$. Since X is a weakly P-space, Z is a regular closed set in X. Using this, we have the following :

Proposition 2.6. *Every basically disconnected weakly P-space is a P-space.*

Definition 2.7. Let X be a space. Then a pair (Y, f) is called *a cover of X* if $f : Y \longrightarrow X$ is a covering map, that is, an onto, continuous, closed and compact map.

Suppose that $f : X \longrightarrow Y$ is a covering map. Then the map $\overline{f} : R(X) \longrightarrow R(Y)$, defined by $\overline{f}(A) = f(A)$, is a Boolean isomorphism ([5]).

Proposition 2.8. *Let $f : X \longrightarrow Y$ be a covering map. Then X is a weakly P-space if and only if Y is a weakly P-space.*

Proof. (⇒) Let (U_n) be a decreasing sequence in $R(Y)$ such that $\bigwedge\{U_n|n \in N\} = \emptyset$. Then $(cl_X(int_X(f^{-1}(U_n))))$ is a decreasing sequence in $R(X)$. Since X is a weakly P-space, $\bigcap\{cl_X(int_X(f^{-1}(U_n)))|n \in N\} = \emptyset$. Suppose that $\bigcap\{U_n|n \in N\} \neq \emptyset$. Pick $y \in \bigcap\{U_n|n \in N\}$. Since $f^{-1}(y)$ is a compact subset of X, there is a $k \in N$ such that $f^{-1}(y) \cap cl_X(int_X(f^{-1}(U_k))) = \emptyset$ and so $y \notin U_k$. This is a contradiction.

(⇐) Let (H_n) be a decreasing sequence in $R(X)$ with $\bigwedge\{H_n|n \in N\} = \emptyset$. Then $(f(H_n))$ is a decreasing sequence in $R(Y)$ such that $\bigwedge\{f(H_n)|n \in N\} = \emptyset$. Since Y is a weakly P-space, $\bigcap\{f(H_n)|n \in N\} = \emptyset$ and so $\bigcap\{H_n|n \in N\} = \emptyset$. □

Definition 2.9. Let X be a space.

(1) A cover (Y, f) of X is called *a basically disconnected cover of* X if Y is a basically disconnected space.
(2) A basically disconnected cover (Y, f) of X is called *a minimal basically disconnected cover of* X if for any basically disconnected cover (Z, g) of X, there is a covering map $h : Z \longrightarrow Y$ such that $f \circ h = g$.

Vermeer([7]) showed that every space X has a minimal basically disconnected cover $(\Lambda X, \Lambda_X)$.

By Proposition 2.6. and Proposition 2.8., we have the following :

Proposition 2.10. *If X is a weakly P-space, then ΛX is a P-space.*

3. Products of Covers

A lattice L is called *σ-complete* if every countable subset of L has join and meet.

Let L be a complete Boolean algebra and M a sublattice of L. Then there is the smallest σ-complete Boolean subalgebra of L containing M, denoted by σM. For any space X, let $Z(X)$ denote the set of all zero-sets and $Z(X)^{\#} = \{cl_X(int_X(A))|A \in Z(X)\}$. Then $Z(X)^{\#}$ is a sublattice of $R(X)$ and so there a σ-complete Boolean subalgebra $\sigma Z(X)^{\#}$ of $R(X)$ containing $Z(X)^{\#}$.

Let X be a space. A $\sigma Z(X)^{\#}$-filter α is said to be *fixed* if $\cap\alpha \neq \emptyset$. Let S_X denote the subspace $\{\alpha|\alpha$ is a fixed $\sigma Z(X)^{\#}$-ultrafilter$\}$ of $S(\sigma Z(X)^{\#})$. Then $\{\lambda_A \mid A \in \sigma Z(X)^{\#}\}$ is a base for S_X and also a closed base for S_X, where $\lambda_A = \{\alpha \in S_X|A \in \alpha\}$.

Recall that a space X is called *weakly Lindelöf* if for any open cover $\mathcal{U}$ of X, there is a countable subset $\mathcal{V}$ of $\mathcal{U}$ such that $\bigcup\mathcal{V}$ is dene in X and that a space X is called *locally weakly Lindelöf* if every element of X has a weakly Lindelöf neighborhood in X.

Definition 3.1. A space X is called *a countably locally weakly Lindelöf space* if for any countable set $\{\mathcal{U}_n | n \in N\}$ of open covers of X and for any $x \in X$, there is a neighborhood G of x in X such that for any $n \in N$, there is a countable subset $\mathcal{V}_n$ of $\mathcal{U}_n$ such that $G \subseteq cl_X(\cup \mathcal{V}_n)$.

Every locally weakly Lindelöf space is a countably locally weakly Lindelöf space but the converse need not be true ([1]).

Lemma 3.2 ([3, 4, 7]). *Let X be a space.*

(1) *If X is a compact space, then $S(\sigma Z(X)^{\#}) = \Lambda X$ and $\Lambda_X(\alpha) = \cap\alpha$.*
(2) *$\Lambda(\beta X) = S(\sigma Z(X)^{\#})$.*
(3) *If X is a countably locally weakly Lindelöf space, then $\Lambda X = S_X$ and $\Lambda_X(\alpha) = \cap\alpha$.*
(4) *$\Lambda_{\beta X}^{-1}(X)$ is a basically disconnected space if and only if $\Lambda X = S_X$.*

Lemma 3.3 ([5]). *Let $f : X \longrightarrow Y$ be a continuous map and S a dense subspace of X such that $f|_S : S \longrightarrow f(S)$ is a perfect map. Then $f(X - S) \subseteq Y - f(S)$.*

By the fact that for any space X, $\sigma Z(X)^{\#}$, $\sigma Z(X)^{\#} \times Y$ and $\sigma(Z(X)^{\#} \times Y)$ are Boolean isomorphic, we have the following :

Proposition 3.4. *Let X, Y be spaces. Then we have the following :*

(1) *$\sigma Z(X)^{\#} \times Y \subseteq \sigma Z(X \times Y)^{\#}$,*
(2) *$\sigma Z(X)^{\#} \times \sigma Z(Y)^{\#} \subseteq \sigma Z(X \times Y)^{\#}$, and*
(3) *for any $A \in \sigma Z(X)^{\#}$ and $B \in \sigma Z(Y)^{\#}$ such that $(A \times Y) \wedge (B \times Y) = \emptyset$, $\lambda_{A\times Y} \cap \lambda_{A\times Y} = \emptyset$.*

Theorem 3.5. *Let X, Y be spaces such that $\Lambda X = S_X$ and $\Lambda Y = S_Y$. Then there is a homeomorphism $h : S_X \times S_Y \longrightarrow S_{X\times Y}$ such that $\Lambda_X \times \Lambda_Y = g \circ h$, where the map $g : S_{X\times Y} \longrightarrow X \times Y$ is defined by $g(\delta) = \cap\delta$.*

Proof. Since $S_{X\times Y} = \Lambda_{\beta(X\times Y)}^{-1}(X \times Y)$ and $\Lambda_X \times \Lambda_Y$ is a covering map, there is a continuous map $h : S_X \times S_Y \longrightarrow S_{X\times Y}$ such that $\Lambda_X \times \Lambda_Y = g \circ h$.

Suppose that $(\alpha_1, \alpha_2) \neq (\gamma_1, \gamma_2)$. We may assume that $\alpha_1 \neq \gamma_1$. Then there are $A \in \alpha_1$ and $B \in \gamma_1$ such that $A \wedge B = \emptyset$. Then $(A \times Y) \wedge (B \times Y) = \emptyset$ and by Proposition 3.4., $\lambda_{A\times Y} \cap \lambda_{B\times Y} = \emptyset$. Note that $h((\alpha_1, \alpha_2)) \in \lambda_{A\times Y}$ and $h((\gamma_1, \gamma_2)) \in \lambda_{B\times Y}$. Hence h is one-to-one.

We will show that the co-restriction $\bar{h} : S_X \times S_Y \longrightarrow h(S_X \times S_Y)$ of h with respect to $h(S_X \times S_Y)$ is a closed map. Since $\bar{h}$ is one-to-one and onto, $\{\lambda_P \times$

$\lambda_Q | P \in \sigma Z(X)^{\#}, Q \in \sigma Z(Y)^{\#}\}$ is a base for $S_X \times S_Y$ and for $P \in \sigma Z(X)^{\#}, Q \in \sigma Z(Y)^{\#}$, $\lambda_P \times \lambda_Q$ is closed and open in $S_X \times S_Y$, it is enough to show that for any $P \in \sigma Z(X)^{\#}, Q \in \sigma Z(Y)^{\#}$, $\bar{h}(\lambda_P \times \lambda_Q)$ is closed in $h(S_X \times S_Y)$. Take any $C \in \sigma Z(X)^{\#}, D \in \sigma Z(Y)^{\#}$ and $t \in h(S_X \times S_Y) - h(\lambda_C \times \lambda_D)$. Since $\bar{h}$ is one-to-one and onto, there is a unique (α, β) in $S_X \times S_Y$ such that $\bar{h}((\alpha, \beta)) = t$. Then $(\alpha, \beta) \notin \lambda_C \times \lambda_D$ and so there are $G \in \alpha$ and $H \in \beta$ such that $(\lambda_G \times \lambda_H) \cap (\lambda_C \times \lambda_D) = \emptyset$. Hence $(G \times H) \wedge (C \times D) = \emptyset$. By Proposition 3.4., $\lambda_{G \times H} \cap \lambda_{C \times D} = \emptyset$. Since $\bar{h}((\alpha, \beta)) = t \in \lambda_{G \times H}$ and $\bar{h}(\lambda_C \times \lambda_D) \subseteq \lambda_{C \times D}$, $t \notin cl_{X \times Y}(\bar{h}(\lambda_C \times \lambda_D))$. Hence $\bar{h}$ is a closed map. Thus $\bar{h}$ is a dense embedding. Since $g : S_{X \times Y} \longrightarrow X \times Y$ is a covering map, by Lemma 3.3.,

$$
\begin{aligned}
& g(S_{X \times Y} - h(S_X \times S_Y)) \\
& \subseteq X \times Y - g(h(S_X \times S_Y)) \\
& = X \times Y - (\Lambda_X \times \Lambda_Y)(S_X \times S_Y) = \emptyset.
\end{aligned}
$$

Hence $h(S_X \times S_Y) = S_{X \times Y}$ and so h is a homeomorphism. □

Theorem 3.6. *Let X, Y be spaces such that $\Lambda X = S_X$ and $\Lambda Y = S_Y$. Then the following are equivalent :*

(1) $\Lambda X \times \Lambda Y = \Lambda(X \times Y)$
(2) $S_{X \times Y} = \Lambda(X \times Y)$, *and*
(3) $\Lambda X \times \Lambda Y$ *is a basically disconnected space.*

Proof. By the above theorem, (1) ⇒ (2) and (2) ⇒ (3) hold.

(3) ⇒ (1) Since $\Lambda X \times \Lambda Y$ is a basically disconnected space, there is a covering map $f : \Lambda X \times \Lambda Y \longrightarrow \Lambda(X \times Y)$ such that $\Lambda_{X \times Y} \circ f = \Lambda_X \times \Lambda_Y$.

We will show that f is one-to-one.

Suppose that $(\alpha, \beta) \neq (\gamma, \delta)$ in $\Lambda X \times \Lambda Y$. We may assume that $\alpha \neq \gamma$. Then there is a $A \in \alpha$ and $B \in \gamma$ such that $A \wedge B = \emptyset$. Since $A \times Y, B \times Y \in \sigma Z(X \times Y)^{\#}$ and $(\Lambda(X \times Y), \Lambda_{X \times Y})$ is a minimal basically disconnected cover of $X \times Y$,

$$
\begin{aligned}
\emptyset & = cl_{\Lambda(X \times Y)}(\Lambda^{-1}_{X \times Y}(int_{X \times Y}(A \times Y))) \cap cl_{\Lambda(X \times Y)}(\Lambda^{-1}_{X \times Y}(int_{X \times Y}(B \times Y))) \\
& = cl_{\Lambda(X \times Y)}(\Lambda^{-1}_{X \times Y}(int_{X \times Y}((A \cap B) \times Y))).
\end{aligned}
$$

Since

$$
\begin{aligned}
\Lambda_{X \times Y}(f(\lambda_A \times \Lambda Y)) & = (\Lambda_X \times \Lambda_Y)(\lambda_A \times \Lambda Y) = A \times Y \\
& = \Lambda_{X \times Y}(cl_{\Lambda(X \times Y)}(\Lambda^{-1}_{X \times Y}(int_{X \times Y}(A \times Y))))
\end{aligned}
$$

and $\Lambda_{X \times Y}$ is a covering map, $f(\lambda_A \times \Lambda Y) = cl_{\Lambda(X \times Y)}(\Lambda^{-1}_{X \times Y}(int_{X \times Y}(A \times Y)))$ and similarly, $f(\lambda_B \times \Lambda Y) = cl_{\Lambda(X \times Y)}(\Lambda^{-1}_{X \times Y}(int_{X \times Y}(B \times Y)))$. Hence $f(\lambda_A \times \Lambda Y) \cap$

$f(\lambda_B \times \Lambda Y) = \emptyset$. Since $(\alpha, \beta) \in \lambda_A \times \Lambda Y$ and $(\gamma, \delta) \in \lambda_B \times \Lambda Y$, $f(\alpha, \beta) \neq f(\gamma, \delta)$. Hence f is one-to-one. Thus f is a homeomorphism. □

Corollary 3.7. (1) *If X is a countably locally weakly Lindelöf, weakly P-space and Y is a locally weakly Lindelöf, then $\Lambda X \times \Lambda Y = \Lambda(X \times Y)$.*

(2) *If $X \times Y$ is a countably locally weakly Lindelöf space, then $\Lambda X \times \Lambda Y = \Lambda(X \times Y)$.*

(3) *If X and Y is a locally compact space, then $\Lambda X \times \Lambda Y = \Lambda(X \times Y)$.*

References

1. W.W. Comfort, N. Hindman & S. Negrepontis: F'-space and their products with P-spaces. *Pacific J. Math.* **28** (1969).
2. L. Gillman & M. Jerison: *Rings of continuous functions.* Van Nostrand, Princeton, New York, 1960.
3. C.I. Kim: Minimal covers and filter spaces. *Topol. and its Appl.* **72** (1996), 31-37.
4. ______: Minimal basically disconnected covers of product spaces. *Commum. Korean Math.* **21** (2006), 347-353.
5. S.J. Porter & R.G. Woods: *Extensions and Absolutes of Hausdorff Spaces.* Springer, Berlin, 1988.
6. A.K. Veksler: P'-points, P'-sets, P'-spaces. *Soviet. Math. Dokl.* **4** (1973), 1443-1450.
7. J. Vermeer: The smallest basically disconnected preimage of a space. *Topol. Appl.* **17** (1984), 217-232.

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