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CR MANIFOLDS OF ARBITRARY CODIMENSION WITH A CONTRACTION

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Abstract. Let (M, p) be a germ of a C^∞ CR manifold of CR dimension n and CR codimension d. Suppose (M, p) admits a C^∞ contraction at p. In this paper, we show that (M, p) is CR equivalent to a generic submanifold in $\mathbb{C}^{n+d}$ defined by a vector valued weighted homogeneous polynomial.

Introduction

Let M be a smooth manifold of real dimension $2n+d$. M is called a *CR manifold of CR dimension* n *and CR codimension* d if there exist a vector bundle $T^cM \subset TM$ of rank $2n$ and a bundle isomorphism $J : T^cM \to T^cM$ such that $J \circ J = -id$ and $[X, JY] + [JX, Y] = J\{[X, Y] - [JX, JY]\}$ for any local sections X and Y of T^cM. The last condition is the formal integrability of CR structure. The pair (T^cM, J) is called a *CR structure over* M. If $d = 1$, then M is called a CR manifold of *hypersurface type*.

A C^1 map f from a CR manifold M to another CR manifold $\widetilde{M}$ is called a *CR map* if $df(v) \in T^c\widetilde{M}$ and $df \circ J(v) = \widetilde{J} \circ df(v)$ for all $v \in T^cM$, where $(T^c\widetilde{M}, \widetilde{J})$ is the CR structure over $\widetilde{M}$. Let $p \in M$. A CR diffeomorphism f from M to itself is called a *contraction* at p if $f(p) = p$ and $\|df_p\| < 1$.

In [7], Kim and Yoccoz proved that if (M, p) is a germ of a C^∞ CR manifold of hypersurface type admitting a C^∞ contraction f at p, then (M, p) is CR equivalent to a real hypersurface in a complex space defined by a weighted homogeneous polynomial.

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In this paper we show that the same is true for CR manifolds of arbitrary CR codimension.

Theorem 1. *Let (M,p) be a germ of a C^∞ CR manifold with CR dimension n and CR codimension d. Suppose (M,p) admits a C^∞ contraction at p. Then there exists a C^∞ CR embedding $\Phi : (M,p) \to (\mathbb{C}^{n+d}, 0)$ such that*

$$\Phi(M) = \{(z,w) \in \mathbb{C}^n \times \mathbb{C}^d : \mathrm{Im}\ w = P(z,\bar{z},\mathrm{Re}\ w)\}$$

for some weighted homogeneous vector valued real polynomial $P(z,\bar{z},\mathrm{Re}\ w)$.

The main novelty of this paper is Theorem 3. With this theorem and approximation of $(M,0)$ by C^ω CR manifolds(Lemma 2), we can prove Theorem 1 by following the same argument in §3 of [7].

1. Preliminaries

Let M be a C^∞ CR manifold of CR dimension n, CR codimension d and let (T^cM, J) be the CR structure of M. Define subbundles $T^{1,0}M$ and $T^{0,1}M$ of the complexified tangent bundle $\mathbb{C}TM$ by

$$T^{1,0}_pM := \{v - \sqrt{-1}J(v) : v \in T^c_pM\}$$

and

$$T^{0,1}_pM := \{v + \sqrt{-1}J(v) : v \in T^c_pM\}.$$

Then $T^{1,0}M$ and $T^{0,1}M$ are complex vector bundles of dimension n over M and it holds that

$$\overline{T^{1,0}M} = T^{0,1}M$$

and

$$T^{1,0}M \cap T^{0,1}M = \{0\}.$$

A section of $T^{1,0}M$ is called a $(1,0)$ vector field and a section of $T^{0,1}M$ is called a $(0,1)$ vector field. Denote by $\Gamma(M, T^{1,0}M)$ the set of all smooth sections of $T^{1,0}M$. Then the integrability condition of the CR structure implies that

$$[L, \widetilde{L}] \in \Gamma(M, T^{1,0}M)$$

for any $L,\ \widetilde{L} \in \Gamma(M, T^{1,0}M)$.

Assume that (M,p) is a germ of a C^∞ real submanifold of real codimension d in $\mathbb{C}^{n+d}$. (M,p) is said to be *generic* if M has a local defining function $\rho = (\rho_1,\ldots,\rho_d)$ near p such that $\partial\rho_1,\ldots,\partial\rho_d$ are $\mathbb{C}$-linearly independent. In this case, (M,p) inherits a CR structure from the complex structure of $\mathbb{C}^{n+d}$ with CR dimension n and CR codimension d.

The following lemma is proved in [1].

Lemma 1. *Let $(M,0)$ be a germ of a C^ω generic real submanifold in $\mathbb{C}^{n+d}$ with real codimension d. Then there exists a holomorphic map $\mathcal{Q} : \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}^d \to \mathbb{C}^d$ satisfying $\mathcal{Q}(z,0,\tau) \equiv \mathcal{Q}(0,\chi,\tau) \equiv \tau$ such that*

$$M = \{(z,w) \in \mathbb{C}^n \times \mathbb{C}^d : \ w = \mathcal{Q}(z,\bar{z},\bar{w})\}.$$

Now let (M,p) be a germ of a C^∞ (abstract) CR manifold of CR dimension n and CR codimension d. Choose local coordinates $(x,y,t) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^d$ centered at p such that

$$T^{1,0}_p M = span\left\{\frac{\partial}{\partial z_j},\ j = 1,\ldots,n\right\},$$

where $z_j = x_j + \sqrt{-1}y_j$. Then there exist C^∞ functions ${\xi_j}^k$ and ${\eta_j}^a$, $j,k = 1,\ldots,n$ and $a = 1,\ldots,d$ such that

$$L_j = \frac{\partial}{\partial z_j} + \sum_{k=1}^{n} {\xi_j}^k(x,y,t)\frac{\partial}{\partial \bar{z}_k} + \sum_{a=1}^{d} {\eta_j}^a(x,y,t)\frac{\partial}{\partial t_a},\ j = 1,\ldots,n$$

span $T^{1,0}M$. Let

$$L_j^{(m)} = \frac{\partial}{\partial z_j} + \sum_{k=1}^{n} \xi_j^{(m),k}(x,y,t)\frac{\partial}{\partial \bar{z}_k} + \sum_{a=1}^{d} \eta_j^{(m),a}(x,y,t)\frac{\partial}{\partial t_a},\ j = 1,\ldots,n,$$

where $\xi_j^{(m),k}$, $\eta_j^{(m),a}$ are m-th order Taylor polynomials of ${\xi_j}^k$ and ${\eta_j}^a$ at 0, respectively. In [1], it is proved that if (M,p) is a C^ω CR manifold, then there exists a C^ω CR embedding $\Phi : (M,p) \to (\mathbb{C}^{n+d},0)$ such that $\Phi(M)$ is generic. By this fact, we can proved the following.

Lemma 2. *Let $(M,0)$ be a germ of a C^∞ CR manifold of CR dimension n and CR codimension d. Then for any positive integer m, there exists a C^∞ embedding $\Phi : (M,p) \to (\mathbb{C}^{n+d},0)$ such that*

$$\Phi(M) = \{(z,w) \in \mathbb{C}^n \times \mathbb{C}^d : w = \mathcal{Q}(z,\bar{z},\bar{w})\}$$

for some holomorphic map $\mathcal{Q} : \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}^d \to \mathbb{C}^d$ satisfying $\mathcal{Q}(z,0,\tau) \equiv \mathcal{Q}(0,\chi,\tau) \equiv \tau$ and that

$$\Phi_*(L)/T^{1,0}\Phi(M) \in o(m)$$

for all $L \in \Gamma(M, T^{1,0}M)$, where $T^{1,0}\Phi(M)$ is the $(1,0)$ vector bundle over $\Phi(M)$ induced by the complex structure of $\mathbb{C}^{n+d}$.

2. Weighted Homogeneous Generic CR Manifolds

Let $f : (\mathbb{C}^N, 0) \to (\mathbb{C}^N, 0)$ be a local biholomorphic map at 0 such that $\|df_0\| < 1$ and let $df_0 = L$. Write

$$L = D + A,$$

where D is diagonal, A is nilpotent and $DA = AD$.

Definition 1. A holomorphic polynomial map G: $(\mathbb{C}^N, 0) \to (\mathbb{C}^N, 0)$ is said to satisfy the *resonance condition with respect to* f, if $G \circ D = D \circ G$.

The next theorem gives a normalization for holomorphic contractions. See [3] as a reference.

Theorem 2. (Poincaré-Dulac) *Suppose that* f *is a local biholomorphic map fixing* 0 *such that* $\|df(0)\| < 1$. *Then there exists a local biholomorphic map* h *fixing* 0 *such that* $dh(0) = id$ *and that* $h \circ f \circ h^{-1}$*satisfies the resonance condition with respect to* f.

Let

$$D = diag(\lambda_1, \dots, \lambda_N).$$

Assume that

$$\lambda = \max_j(|\lambda_j|, j = 1, \dots, N).$$

Define m_j, $j = 1, \dots, N$, by

$$|\lambda_j| = \lambda^{m_j}.$$

For $\varepsilon > 0$, define $S_\varepsilon : \mathbb{C}^N \to \mathbb{C}^N$ by

$$S_\varepsilon(z_1, \dots, z_N) = (\varepsilon^{m_1} z_1, \dots, \varepsilon^{m_N} z_N).$$

Definition 2. A polynomial P defined in $\mathbb{C}^N$ is said to have *weight* ω *with respect to* f if

$$P \circ S_\varepsilon = \varepsilon^\omega \widetilde{P} + o(\varepsilon^\omega)$$

as $\varepsilon \to 0$ for some non-zero polynomial $\widetilde{P}$. The zero polynomial is understood as having *weight* ∞. *We denote by* $wt_f(P)$ *the weight of* P *with respect to* f.

If a polynomial map G satisfies $G \circ D = D \circ G$, then one can easily see that $G \circ S_\varepsilon = S_\varepsilon \circ G$. Hence we have the following lemma.

Lemma 3. *Suppose* $G\colon (\mathbb{C}^N,0) \to (\mathbb{C}^N,0)$ *satisfies the resonance condition with respect to* f*. If* $dG(0)$ *is invertible, then* G *preserves the weight with respect to* f*, i.e., for any polynomial* P*, it holds that*

$$wt_f(P) = wt_f(P \circ G).$$

In this section we show the following.

Theorem 3. *Let* $(M,0)$ *be a germ of a* C^ω *generic submanifold in* $\mathbb{C}^{n+d}$ *with real codimension* d*. Assume that* $(M,0)$ *admits a* C^ω *CR contraction at* 0*. Then* $(M,0)$ *is biholomorphically equivalent to a real submanifold defined by*

$$w = \mathcal{Q}(z,\bar z,\bar w)$$

for some weighted homogeneous $\mathbb{C}^d$*-valued polynomial* $\mathcal{Q}$ *such that*

$$(z,0,\tau) \equiv \mathcal{Q}(0,\chi,\tau) \equiv \tau.$$

Proof. Assume that

$$T_0^{1,0}M = span\left\{\frac{\partial}{\partial z_j}, j=1,\ldots,n\right\}.$$

After a linear change of coordinates, we may assume that M is defined by

$$w = \mathcal{Q}(z,\bar z,\bar w)$$

for some vector valued holomorphic function $\mathcal{Q}(z,\chi,\tau)$ such that

$$\mathcal{Q}(z,\chi,\tau) = \tau + o(1).$$

Now let f be a C^ω CR contraction at 0. Since M and f are real analytic, f extends holomorphically to a neighborhood of 0. Then by Poincaré-Dulac Theorem, we can choose a local biholomorphic map $h : (\mathbb{C}^{n+d},0) \to (\mathbb{C}^{n+d},0)$ with $h = id + o(1)$ such that $h \circ f \circ h^{-1}$ satisfies the resonance condition with respect to f. Hence we may assume that f itself satisfies the resonance condition with respect to f.

Let λ_j, $j=1,\ldots,n$, be the eigenvalues of df_0 restricted to $T_0^{1,0}M$ and let μ_a, $a=1,\ldots,d$, be the eigenvalues of df_0 restricted to $\mathbb{C}T_0M/(T_0^{1,0}M + T_0^{0,1}M)$. Assume that

$$|\lambda_1| \le \cdots \le |\lambda_n|$$

and

$$|\mu_1| \le \cdots \le |\mu_d|.$$

Since f preserves $T^{1,0}M$, we may assume that

$$df_0\left(\frac{\partial}{\partial z_j}\right) = \lambda_j \frac{\partial}{\partial z_j} \mod \frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_{j-1}}.$$

Since f preserves M and M is defined by $\mathcal{Q}$ satisfying $\mathcal{Q}(z,\chi,\tau) = \tau + o(1)$, we have

$$df_0\left(\frac{\partial}{\partial w_a}\right) \in span\left\{\frac{\partial}{\partial w_1}, \dots, \frac{\partial}{\partial w_d}\right\}.$$

Therefore we may assume that

$$df_0\left(\frac{\partial}{\partial w_a}\right) = \mu_a \frac{\partial}{\partial w_a} \mod \frac{\partial}{\partial w_1}, \dots, \frac{\partial}{\partial w_{a-1}}. \tag{2.1}$$

Let $\mathcal{Q} = (\mathcal{Q}_1, \dots, \mathcal{Q}_d)$. Write

$$\mathcal{Q}_a = \mathcal{Q}_{a,-} + \mathcal{Q}_{a,0} + \mathcal{Q}_{a,+}, \quad a = 1, \dots, d,$$

where $\mathcal{Q}_{a,-}$, $\mathcal{Q}_{a,0}$, $\mathcal{Q}_{a,+}$ consist of monomials with weight $< wt_f(w_a)$, $= wt_f(w_a)$ and $> wt_f(w_a)$, respectively. We will show that for each a, it holds that

$$\mathcal{Q}_{a,-} \equiv \mathcal{Q}_{a,+} \equiv 0$$

and hence M is defined by

$$w = \mathcal{Q}_0(z, \bar{z}, \bar{w}),$$

where $\mathcal{Q}_0 := (\mathcal{Q}_{1,0}, \dots, \mathcal{Q}_{d,0})$.

Since f satisfies the resonance condition with respect to f, we can apply Lemma 3. Therefore the manifold defined by

$$w = \mathcal{Q}_-(z, \bar{z}, \bar{w})$$

is invariant under f, where $\mathcal{Q}_- := (\mathcal{Q}_{1,-}, \dots, \mathcal{Q}_{d,-})$. Suppose that $\mathcal{Q}_{a,-} \not\equiv 0$ for some a. Let ℓ_0 be the smallest degree of non-trivial terms in $\mathcal{Q}_{a,-}$, $a = 1, \dots, d$. Write

$$\mathcal{Q}_{a,-} = \mathcal{Q}_{a,-}^{(\ell_0)} + o(\ell_0).$$

Let $\mathcal{Q}_-^{(\ell_0)} := (\mathcal{Q}_{1,-}^{(\ell_0)}, \dots, \mathcal{Q}_{d,-}^{(\ell_0)})$. Then real submanifold defined by

$$w = \mathcal{Q}_-^{(\ell_0)}(z, \bar{z}, \bar{w})$$

is invariant under df_0. Now suppose $\mathcal{Q}_{1,-}^{(\ell_0)} \not\equiv 0$. Since we assumed (2.1), this implies that by considering lexicographic ordering, there exists a nontrivial monomial $\alpha(z, \bar{z}, \bar{w})$ in $\mathcal{Q}_{1,-}^{(\ell_0)}(z, \bar{z}, \bar{w})$ such that

$$\alpha \circ D = \mu_1 \cdot \alpha.$$

But this means that $\mathcal{Q}_{1,-}^{(\ell_0)}$ contains a nontrivial term of weight $wt_f(w_1)$, which is a contradiction. Hence we conclude that

$$\mathcal{Q}_{1,-}^{(\ell_0)} \equiv 0.$$

By induction on a, $a = 1, \ldots, d$ and by the same argument, we can show that

$$\mathcal{Q}_{a,-}^{(\ell_0)} \equiv 0, \ \forall a.$$

Similarly, we can prove that

$$\mathcal{Q}_{a,+} \equiv 0, \ \forall a.$$

Since $\mathcal{Q}_0$ is a weighted homogeneous polynomial map such that $\mathcal{Q}_0(z, \chi, \tau) = \tau + o(1)$, after a holomorphic change of coordinates preserving weighted homogeneity of $\mathcal{Q}_0$, we can remove all harmonic terms in $\mathcal{Q}(z, \bar{z}, \bar{w})$. Therefore can show that M is defined by

$$w = \mathcal{Q}(z, \bar{z}, \bar{w})$$

for some new weighted homogeneous polynomial map $\mathcal{Q}$ such that $\mathcal{Q}(z, 0, \tau) = \mathcal{Q}(0, \chi, \tau) = \tau$. □

3. Proof of Theorem 1

The proof presented in this section is a modification of the proof in §3 of [7].

Let (M, p) be a germ of a C^∞ CR manifold of CR dimension n and CR codimension d and let f be a C^∞ contraction at p. By Lemma 2, we can show that for any positive integer m, there exists a C^∞ embedding $\Phi : (M, p) \to (\mathbb{C}^{n+d}, 0)$ such that

$$\Phi(M) = \{(z, w) \in \mathbb{C}^n \times \mathbb{C}^d : w = \mathcal{Q}(z, \bar{z}, \bar{w})\}$$

for some holomorphic map $\mathcal{Q}(z, \chi, \tau)$ satisfying $\mathcal{Q}(z, \chi, \tau) = \tau + o(1)$ and

$$\Phi_*(L_j)/T^{1,0}\Phi(M) \in o(m), \ j = 1, \ldots, n$$

for a basis $\{L_j\}_{j=1,\ldots,n}$ of $(1, 0)$ vector fields of M.

Write $\widetilde{M} := \Phi(M)$. Consider

$$\widetilde{f} := \Phi \circ f \circ \Phi^{-1} : \widetilde{M} \to \widetilde{M}.$$

Since f is a CR map, by taking $m > 1$, we can show that $d\widetilde{f}_0$ is an $(n+d)$ by $(n+d)$ complex matrix. Hence we can extend $\widetilde{f}$ as a local C^∞ diffeomorphism of $\mathbb{C}^{n+d}$ at 0 such that $\|d\widetilde{f}_0\| < 1$. Then by the Normalization theorem for real contractions([7]), we can choose a local C^∞ diffeomorphism h of $\mathbb{C}^{n+d}$ at 0 such that $h^{-1} \circ \widetilde{f} \circ h$ has formal power series satisfying the resonance condition with respect to $\widetilde{f}$.

By following the same argument in §3 of [7] using Theorem 3, we can choose a C^ω generic submanifold $\widehat{M}$ defined by

$$\widehat{M} = (\{(z,w) \in \mathbb{C}^n \times \mathbb{C}^d : w = \mathcal{Q}_0(z,\bar{z},\bar{w})\}$$

for a weighted homogeneous polynomial map $\mathcal{Q}_0$ with $\mathcal{Q}_0(z,0,\tau) = \mathcal{Q}_0(0,\chi,\tau) = \tau$ and a local C^∞ diffeomorphism $\Psi : (\mathbb{C}^{n+d},0) \to (\mathbb{C}^{n+d},0)$ with $\Psi = id + o(m)$ such that

$$\Psi(\widetilde{M}) = \widehat{M}.$$

Assume that on a small neighborhood U of 0 in M, it holds that

$$\|f(x)\| \leq \lambda \|x\|$$

for all $x \in U$. Since f is a contraction at 0, we may assume that $\lambda < 1$. Choose m large enough so that on U, it holds that

$$\|df^{-1}\| \; \lambda^m \leq \frac{1}{2}.$$

Then by following the same argument in Lemma 3.1 of [7], we can prove the following lemma, which will complete the proof.

Lemma 4. *The map* $\Psi \circ \Phi : (M,0) \to (\widehat{M},0)$ *is a CR diffeomorphism.*

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