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FAINTLY γ-CONTINUOUS FUNCTIONS

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Abstract. In this paper, we introduce the concepts of faintly γ-continuity and extremely γ-closed graph. And we study characterizations of such functions and relationships between faintly γ-continuity and extremely γ-closed graph.

1. Introduction

In [5], the author introduced the notions of γ-open sets and γ-continuity which are generalized notions of semiopen sets and semicontinuity, respectively. Latif investigated some properties of functions induced by γ-open sets in [2, 3]. The author introduced and investigated the notion of weakly γ-continuous functions which is a generalization of γ-continuity in [6]. In this paper, the notion of faintly γ-continuity is introduced, which is a generalization of weakly γ-continuity. The notion of extremely γ-closed graph is also introduced. And the relationships between faintly γ-continuity and extremely γ-closed graph are investigated.

Let X and Y be topological spaces on which no separation axioms are assumed unless explicit stated. Let S be a subset of X. The closure (resp. interior) of S will be denoted by $cl(S)$ (resp. $int(S)$). A subset S of X is called *semi-open set* [4] if $S \subseteq cl(int(S))$. The complement of a semi-open set is called *semi-closed set*.

A subset $M(x)$ of a space X is called a *semi-neighborhood* of a point $x \in X$ if there exists a semi-open set S such that $x \in S \subseteq M(x)$. In [1], Latif introduced the notion of semi-convergence of filters. And he investigated some characterizations related to semi-open continuous functions. Now we recall the concept of semi-convergence of filters. Let $S(x) = \{A \in SO(X) : x \in A\}$ and let $S_x = \{A \subseteq X$: there exists $\mu \subseteq S(x)$ such that μ is finite and $\cap\mu \subseteq A\}$. Then S_x is called the *semi-neighborhood filter* at x. For any filter F on X, we say that F semi-converges to x if and only if F

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is finer than the semi-neighborhood filter at x. A subset U of X is called a *γ-open set* [5] in X if whenever a filter F semi-converges to x and $x \in U$, $U \in F$.

The class of all γ-open sets in X will be denoted by $\gamma(X)$. We recall the following notions defined in [6]: X is said to be γ-T_2 if for every two distinct points x and y in X, there exist two disjoint γ-open sets U and V such that $x \in U$ and $y \in V$. The *γ-interior* of a set A in X, denoted by $I_\gamma(A)$, is the union of all γ-open sets contained in A. The *γ-closure* of a set A in X, denoted by $Cl_\gamma(A)$, $Cl_\gamma(A) = \{x \in X : A \cap U \neq \emptyset$ for all $U \in S_x\}$.

Theorem 1.1 ([5]). *Let (X, τ) be a topological space and $A \subseteq X$.*

(1) $I_\gamma(A) = \{x \in A : A \in S_x\}$.
(2) *A is γ-open set if and only if $A = I_\gamma(A)$.*
(3) $I_\gamma(A) \subseteq A \subseteq Cl_\gamma(A)$.
(4) *A is γ-closed if and only if $A = Cl_\gamma A$.*
(5) $I_\gamma(A) = X - Cl_\gamma(X - A)$.
(6) $Cl_\gamma(A) = X - I_\gamma(X - A)$.

Definition 1.2 ([5, 6]). Let $f : (X, \tau) \to (Y, \mu)$ be a function on two topological spaces. Then

(1) f is said to be *weakly γ-continuous* [6] if for each $x \in X$ and each open set V containing $f(x)$, there is a γ-open set U containing x such that $f(U) \subseteq cl(V)$.

(2) f is said to be *γ-continuous* [5] if the inverse image of each open set of Y is a γ-open set in X.

Latif [2] showed that f is γ-continuous if and only if for each $x \in X$ and each open set V containing $f(x)$, there is a γ-open set U containing x such that $f(U) \subseteq V$.

2. Faintly γ-continuity

We recall that a point $x \in X$ is called a *θ-cluster point* [8] of A if $A \cap cl(V) \neq \emptyset$ for every open set V containing x. The set of all θ-cluster points of A is called *θ-closure* of A and is denoted by $Cl_\theta(A)$. If $A = Cl_\theta(A)$, then A is called *θ-closed.* The complement of a θ-closed set is said to be *θ-open.* The union of $\{U \subseteq X : U \subseteq A$ for θ-open set U of $X\}$ is called the *θ-interior* of A and is denoted by $I_\theta(A)$.

Definition 2.1. Let (X, τ) and (Y, μ) be two topological spaces. Then $f : X \to Y$ is said to be *faintly γ-continuous* at $x \in X$ if for each θ-open subset V containing $f(x)$, there is a γ-open set U containing x such that $f(U) \subseteq V$. A function f :

$(X,\tau) \to (Y,\mu)$ is said to be *faintly γ-continuous* if it has the property at each point of X.

Theorem 2.2. *Every weakly γ-continuous function is faintly γ-continuous.*

Proof. Let $f : (X,\tau) \to (Y,\mu)$ be a weakly γ-continuous function on topological spaces X, Y. For $x \in X$, let V be a θ-open set containing $f(x)$. Then there exists an open set G such that $f(x) \subseteq G \subseteq cl(G) \subseteq V$. By hypothesis, there exists a γ-open set U containing x such that $f(U) \subseteq cl(G) \subseteq V$. Hence f is faintly γ-continuous. $\square$

In Theorem 2.2, the converse is not true in general as shown in the next example.

Example 2.3. Let $X = \{a,b,c,d\}$ and $\tau = \{\emptyset, \{d\}, \{a,b\}, \{a,b,d\}, X\}$ a topology on X. Then $\gamma(X) = \{\emptyset, \{c\}, \{d\}, \{c,d\}, \{a,b\}, \{a,b,c\}, \{a,b,d\}, X\}$.

Consider a function $f : (X,\tau) \to (X,\tau)$ defined as follows $f(a) = d$, $f(b) = a$ and $f(c) = f(d) = b$. Then since X is the only nonempty θ-open set in (X,τ), obviously f is faintly γ-continuous. But f is not weakly γ-continuous at a because of $cl(\{d\}) = \{c,d\}$, $f(\{a,b\}) = \{a,d\}$ and $f(\{a,b,c\}) = f(\{a,b,d\}) = \{a,b,d\}$. Hence f is not weakly γ-continuous.

The following implications are obtained:

continuous $\Rightarrow$ semicontinuous $\Rightarrow$ γ-continuous
$\Rightarrow$ weakly γ-continuous $\Rightarrow$ faintly γ-continuous

Theorem 2.4. *Let $f : (X,\tau) \to (Y,\mu)$ be a function on topological spaces (X,τ) and (Y,μ). Then the following statements are equivalent:*

(1) *f is faintly γ-continuous.*
(2) *$f^{-1}(V)$ is a γ-open set for every θ-open subset V of Y.*
(3) *$f^{-1}(K)$ is a γ-closed set for every θ-closed set K of Y.*

Proof. (1) $\Rightarrow$ (2) Let V be a θ-open set in Y and $x \in f^{-1}(V)$. Then there exists a γ-open set U of X containing x such that $f(U) \subseteq V$. Since $x \in U \subseteq f^{-1}(V)$, by the definition of γ-interior, $x \in I_\gamma(f^{-1}(V))$ and so $f^{-1}(V) \subseteq I_\gamma(f^{-1}(V))$. Hence, $f^{-1}(V)$ is a γ-open set.

(2) $\Rightarrow$ (1) Let $x \in X$ and V any θ-open set in Y containing $f(x)$. By (2), we know that $f^{-1}(V)$ is a γ-open set containing x. Set $U = f^{-1}(V)$, then it satisfies $f(U) \subseteq V$. Hence f is faintly γ-continuous.

(2) $\Leftrightarrow$ (3) It is obvious. $\square$

Theorem 2.5. *Let $f : (X, \tau) \to (Y, \mu)$ be a function on topological spaces (X, τ) and (Y, μ). Then if f is faintly γ-continuous and if Y is regular, then it is γ-continuous.*

Proof. Let V be any open set in Y. Then since Y is regular, V is θ-open. From Theorem 2.4 (2), f is γ-continuous. □

Corollary 2.6. *Let $f : (X, \tau) \to (Y, \mu)$ be a function on topological spaces (X, τ) and (Y, μ). Then if f is faintly γ-continuous and if Y is regular, then it is weakly γ-continuous.*

A topological space X is said to be θ-T_2 [7] if each distinct $x, y \in X$, there exist θ-open sets U, V containing x and y, respectively, such that $U \cap V = \emptyset$.

Theorem 2.7. *Let $f : (X, \tau) \to (Y, \mu)$ be a function on topological spaces (X, τ) and (Y, μ). Then if f is faintly γ-continuous injection and if Y is θ-T_2, then X is γ-T_2.*

Proof. Let x_1 and x_2 be two distinct elements in X, then $f(x_1) \neq f(x_2)$. There exist two θ-open sets U and V in Y containing $f(x_1)$, $f(x_2)$, respectively, such that $U \cap V = \emptyset$. Since f is faintly γ-continuous, there exist γ-open sets U_1, V_2 containing x_1, x_2, respectively, such that $f(U_1) \subseteq U$, $f(V_2) \subseteq V$. So $U_1 \cap V_2 = \emptyset$ and hence X is γ-T_2. □

A subset K of a topological space (X, τ) is said to be *θ-compact relative to* (X, τ) [8] if every cover of K by θ-open subsets of X has a finite subcover. A topological space (X, τ) is said to be *θ-compact* if the set X is θ-compact relative to (X, τ).

Theorem 2.8. *Let $f : (X, \tau) \to (Y, \mu)$ be a function on topological spaces (X, τ) and (Y, μ). If f is faintly γ-continuous and K is γ-compact, then $f(K)$ is θ-compact relative to (Y, μ).*

Proof. Let $\{V_i : i \in J\}$ be a cover of $f(K)$ by θ-open subsets of Y. For each $x \in X$, there exists $i(x) \in J$ such that $f(x) = y \in V_{i(x)}$. Since f is faintly γ-continuous, there exists a γ-open set $U(x)$ containing x such that $f(U(x)) \subseteq V_{i(x)}$. The family $\{U(x) : x \in K\}$ is a cover of K by γ-open sets in X. Since K is γ-compact, there is a finite subcover $\{U(x_1), U(x_2), \cdots, U(x_n) : x_j \in K, j = j_1, j_2, \cdots, j_n\}$ such that $K \subseteq \cup U(x_j)$. Then

$$f(K) \subseteq f(\cup U(x_j)) = \cup f(U(x_j) \subseteq \cup V_{i(x_j)}.$$

Hence $f(K)$ is θ-compact relative to (Y, μ). □

Definition 2.9. Let $f:(X,\tau)\to(Y,\mu)$ be a function on topological spaces (X,τ) and (Y,μ). We call f has an *extremely γ-closed graph* if for each $(x,y)\notin G(f)$, there exist a γ-open set U and a θ-open set V containing x and y, respectively, such that $(U\times V)\cap G(f)=\emptyset$.

Lemma 2.10. Let $f:(X,\tau)\to(Y,\mu)$ be a function on topological spaces (X,τ) and (Y,μ). Then f has an extremely γ-closed graph if and only if for each $(x,y)\notin G(f)$, there exist a γ-open set U and a θ-open set V containing x and y, respectively, such that $f(U)\cap V=\emptyset$.

Proof. Obvious. □

Theorem 2.11. *Let $f:(X,\tau)\to(Y,\mu)$ be a function on topological spaces (X,τ) and (Y,μ). If f is faintly γ-continuous and Y is θ-T_2, then f has an extremely γ-closed graph.*

Proof. Let $(x,z)\notin G(f)$. Then $z\neq f(x)$ and since Y is θ-T_2, there exist two θ-open sets U and V containing z and $f(x)$, respectively, such that $U\cap V=\emptyset$. Since f is faintly γ-continuous, there exists a γ-open set H containing x such that $f(H)\subseteq V$. It implies $f(H)\cap U=\emptyset$. Hence f has an extremely γ-closed graph. □

Theorem 2.12. *Let $f:(X,\tau)\to(Y,\mu)$ be a function on topological spaces (X,τ) and (Y,μ). If f is a faintly γ-continuous injection with an extremely γ-closed graph, then X is γ-T_2.*

Proof. Let x_1 and x_2 be two distinct elements in X, then $f(x_1)\neq f(x_2)$. This implies that $(x_1,f(x_2))\in(X\times Y)-G(f)$. Since f has an extremely γ-closed graph, there exist a γ-open set U and a θ-open set V containing x_1 and $f(x_2)$, respectively, such that $f(U)\cap V=\emptyset$. Since f is faintly γ-continuous, there exists a γ-open set W containing x_2 such that $f(W)\subseteq V$. It implies $f(W)\cap f(U)=\emptyset$. Therefore $W\cap U=\emptyset$ and so X is a γ-T_2 space. □

Theorem 2.13. *Let $f:(X,\tau)\to(Y,\mu)$ be a function on topological spaces (X,τ) and (Y,μ). If f has an extremely γ-closed graph and K is γ-compact in (X,τ), then $f(K)$ is θ-closed in (Y,μ).*

Proof. Let $y\notin f(K)$. Then $(x,y)\notin G(f)$ for each $x\in K$. Since $G(f)$ is extremely γ-closed, there exist a γ-open set U_x containing x and a θ-open set V_x containing y such that $f(U_x)\cap V_x=\emptyset$. Since K is γ-compact and $\{U_x : x\in K\}$ is a cover of K by γ-open sets, there exists a finite subset K_0 of K such that $K\subseteq\cup\{U_x : x\in K_0\}$.

Put $V = \cap\{V_x : x \in K_0\}$. Then V is a θ-open set containing y. Thus

$$f(K) \cap V \subseteq \cup_{x\in K_0} f(U_x) \cap V \subseteq \cup_{x\in K_0}(f(U_x) \cap V) = \emptyset.$$

This implies $y \notin Cl_\theta(f(K))$. Hence $f(K)$ is θ-closed in Y. □

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