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ON THE STABILITY OF A BI-JENSEN FUNCTIONAL EQUATION

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Abstract. In this paper, we investigate the generalized Hyers-Ulam stability of a bi-Jensen functional equation

$$4f(\frac{x+y}{2}, \frac{z+w}{2}) = f(x,z) + f(x,w) + f(y,z) + f(y,w).$$

Also, we establish improved results for the stability of a bi-Jensen equation on the punctured domain.

1. Introduction

In 1940, S.M. Ulam [15] raised a question concerning the stability of homomorphisms: Let G_1 be a group and let G_2 be a metric group with the metric $d(\cdot,\cdot)$. Given $\varepsilon > 0$, does there exists a $\delta > 0$ such that if a mapping $h : G_1 \to G_2$ satisfies the inequality

$$d(h(xy), h(x)h(y)) < \delta$$

for all $x, y \in G_1$ then there is a homomorphism $H : G_1 \to G_2$ with

$$d(h(x), H(x)) < \varepsilon$$

for all $x \in G_1$? The case of approximately additive mappings was solved by D.H.Hyers [4] under the assumption that G_1 and G_2 are Banach spaces. In 1978, Th.M.Rassias [14] gave a generalization of D.H. Hyers' result. Recently, P.Găvruta [3] also obtained a further generalization of the Hyers-Ulam theorem. Since then, a further generalization of the Hyers-Ulam theorem has been extensively investigated by a number of mathematicians[6,11,12].

Throughout this paper, let X and Y be real vector spaces. A mapping $g : X \to Y$

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is called a Jensen mapping if g satisfies the functional equation $2g(\frac{x+y}{2}) = g(x)+g(y)$. For a given mapping $f : X \times X \to Y$, define

$$Jf(x,y,z,w) = 4f(\frac{x+y}{2}, \frac{z+w}{2}) - f(x,z) - f(x,w) - f(y,z) - f(y,w)$$

for all $x, y, z, w \in X$. A mapping $f : X \times X \to Y$ is called a bi-Jensen mapping if f satisfies the equation $Jf(x,y,z,w) = 0$ and the functional equation $Jf = 0$ is called a bi-Jensen functional equation.

Kannappan [8] solved the equation

$$(*) \qquad g(x+y+z) + g(x) + g(y) + g(z) - g(x+y) - g(y+z) - g(z+x) = 0.$$

The stability of the equation (*) was proved in various methods [2,7,9]. Park and Bae [13] investigated the relation between the Cauchy-Jensen functional equation and the equation (*). Also they [1] obtained the general solution and the Hyers-Ulam stability of a bi-Jensen mapping. The authors[5] improved their stability results.

For a given mapping $g : X \to Y$, define

$$\begin{aligned} Dg(x,y,z) =& g(x+y+z) + g(x) + g(y) + g(z) \\ & - g(x+y) - g(y+z) - g(z+x) - g(0). \end{aligned}$$

When $X = Y = \mathbb{R}$, the function $f : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ given by $f(x,y) := axy+bx+cy+d$ is a solution of the functional equation $Jf(x,y,z,w) = 0$. In particular, letting $x = y$, we get a function $g : \mathbb{R} \to \mathbb{R}$ given by $g(x) := ax^2 + bx + c$ satisfying a equation $Dg(x,y,z) = 0$. In this paper, we investigate the relation between solution of the bi-Jensen functional equation and solution of the equation $Dg = 0$. We estabilish the generalized Hyers-Ulam stability of a bi-Jensen functional equation and obtain the stability in the sense of Th.M.Rassias as an application. Also we get improved results for the stability of a bi-Jensen equation on the punctured domain.

2. The relation between the solution of the bi-Jensen equation and the solution of the equation $Dg = 0$

Theorem 1. *Let $g : X \to Y$ be a mapping satisfying the functional equation $Dg(x,y,z) = 0$ and $f : X \times X \to Y$ a mapping defined by*

$$f(x,y) := \frac{1}{2}g(x+y) - \frac{1}{4}g(-x) - \frac{1}{4}g(-y) - \frac{1}{4}g(x) - \frac{1}{4}g(y) + \frac{3}{2}g(0)$$

for all $x, y \in X$. Then f is a bi-Jensen mapping and satisfies

$$f(x,x) = g(x)$$

for all $x \in X$.

Proof. By the assumptions of f and g, we get

$$f(x,x) = \frac{1}{2}g(2x) - \frac{1}{2}g(-x) - \frac{1}{2}g(x) + \frac{3}{2}g(0) + \frac{1}{2}Dg(x,x,-x) = g(x)$$

and

$$\begin{aligned}
&Jf(x,y,z,w)\\
&= 2g(\frac{x+y+z+w}{2}) - g(\frac{x+y}{2}) - g(\frac{-x-y}{2}) - g(\frac{z+w}{2}) - g(\frac{-z-w}{2})\\
&- \frac{1}{2}g(x+z) - \frac{1}{2}g(x+w) - \frac{1}{2}g(y+z) - \frac{1}{2}g(y+w)\\
&+ \frac{1}{2}(g(x)+g(-x)+g(z)+g(-z)+g(y)+g(-y)+g(w)+g(-w))\\
&= 2Dg(\frac{x}{2},\frac{y}{2},\frac{z+w}{2}) + 2Dg(\frac{x}{2},\frac{w}{2},\frac{z}{2}) + 2Dg(\frac{y}{2},\frac{w}{2},\frac{z}{2})\\
&+ \frac{1}{2}Dg(\frac{x+z}{2},\frac{x+z}{2},\frac{-x-z}{2}) + \frac{1}{2}Dg(\frac{x+w}{2},\frac{x+w}{2},\frac{-x-w}{2})\\
&+ \frac{1}{2}Dg(\frac{y+z}{2},\frac{y+z}{2},\frac{-y-z}{2}) + \frac{1}{2}Dg(\frac{y+w}{2},\frac{y+w}{2},\frac{-y-w}{2})\\
&- \frac{1}{2}Dg(\frac{x}{2},\frac{z}{2},\frac{-x-z}{2}) - \frac{1}{2}Dg(\frac{x}{2},\frac{w}{2},\frac{-x-w}{2}) - \frac{1}{2}Dg(\frac{y}{2},\frac{z}{2},\frac{-y-z}{2})\\
&- \frac{1}{2}Dg(\frac{y}{2},\frac{w}{2},\frac{-y-w}{2}) - Dg(\frac{x}{2},\frac{y}{2},\frac{-x-y}{2}) - Dg(\frac{z}{2},\frac{w}{2},\frac{-z-w}{2})\\
&- \frac{1}{2}Dg(\frac{x}{2},\frac{x}{2},\frac{-x}{2}) - \frac{1}{2}Dg(\frac{-x}{2},\frac{-x}{2},\frac{x}{2}) - \frac{1}{2}Dg(\frac{y}{2},\frac{y}{2},\frac{-y}{2})\\
&- \frac{1}{2}Dg(\frac{-y}{2},\frac{-y}{2},\frac{y}{2}) - \frac{1}{2}Dg(\frac{z}{2},\frac{z}{2},\frac{-z}{2}) - \frac{1}{2}Dg(\frac{-z}{2},\frac{-z}{2},\frac{z}{2})\\
&- \frac{1}{2}Dg(\frac{w}{2},\frac{w}{2},\frac{-w}{2}) - \frac{1}{2}Dg(\frac{-w}{2},\frac{-w}{2},\frac{w}{2}) = 0
\end{aligned}$$

for all $x, y, z, w \in X$.

Theorem 2. *Let* $f : X \times X \to Y$ *be a bi-Jensen mapping and let* $g : X \to Y$ *a mapping given by*

$$g(x) := f(x,x)$$

for all $x \in X$. *Then* g *satisfies*

$$Dg(x,y,z) = 0$$

and

$$f(x,y) + f(y,x) = g(x+y) - \frac{1}{2}(g(-x)+g(-y)+g(x)+g(y)) + 3g(0)$$

for all $x, y \in X$.

Proof. By the assumptions of f and g, we get

$$\begin{aligned}
4Dg(x,y,z) &= 4f(x+y+z, x+y+z) - 4f(x+y, x+y) - 4f(x+z, x+z) \\
&\quad - 4f(y+z, y+z) + 4f(x,x) + 4f(y,y) + 4f(z,z) - 4f(0,0) \\
&= Jf(2x+2y, 2z, x+y+z, x+y+z) + \frac{1}{2}Jf(2z, 2z, 2x+2y, 2z) \\
&+ \frac{1}{2}Jf(2x+2y, 2x+2y, 2x+2y, 2z) - Jf(2x+2y, 0, 2x+2y, 0) \\
&+ \frac{1}{4}Jf(4x, 4y, 2z, 2z) + \frac{1}{4}Jf(2z, 2z, 4x, 4y) - Jf(2y, 2z, y+z, y+z) \\
&- \frac{1}{2}Jf(2y, 2y, 2y, 2z) - \frac{1}{2}Jf(2z, 2z, 2y, 2z) - Jf(2x, 2z, x+z, x+z) \\
&- \frac{1}{2}Jf(2x, 2x, 2x, 2z) - \frac{1}{2}Jf(2z, 2z, 2x, 2z) - \frac{1}{4}Jf(4x, 0, 2z, 2z) \\
&- \frac{1}{4}Jf(0, 4y, 2z, 2z) - \frac{1}{4}Jf(2z, 2z, 4x, 0) - \frac{1}{4}Jf(2z, 2z, 0, 4y) \\
&- \frac{1}{4}Jf(4x, 4y, 0, 0) - \frac{1}{4}Jf(0, 0, 4x, 4y) + \frac{1}{4}Jf(4x, 0, 0, 0) \\
&+ \frac{1}{4}Jf(0, 0, 4x, 0) + \frac{1}{4}Jf(0, 4y, 0, 0) + \frac{1}{4}Jf(0, 0, 0, 4y) \\
&+ Jf(2x, 0, 2x, 0) + Jf(2y, 0, 2y, 0) + Jf(2z, 0, 2z, 0) = 0
\end{aligned}$$

and

$$\begin{aligned}
g(x+y) &- \frac{1}{2}g(-x) - \frac{1}{2}g(-y) - \frac{1}{2}g(x) - \frac{1}{2}g(y) + 3g(0) - f(x,y) - f(y,x) \\
&= f(x+y, x+y) - \frac{1}{2}f(-x,-x) - \frac{1}{2}f(x,x) \\
&- \frac{1}{2}f(y,y) - \frac{1}{2}f(-y,-y)) + 3f(0,0) - f(x,y) - f(y,x) \\
&= \frac{1}{4}Jf(2x, 2y, 2x, 2y) - \frac{1}{4}Jf(2x, 0, 0, 2y) - \frac{1}{4}Jf(0, 2y, 2x, 0) \\
&+ \frac{1}{4}Jf(-x, -x, -x, x) - \frac{1}{4}Jf(x, -x, x, x) + \frac{1}{2}Jf(x, -x, 0, 0) \\
&+ \frac{1}{4}Jf(-y, -y, -y, y) - \frac{1}{4}Jf(y, -y, y, y) + \frac{1}{2}Jf(y, -y, 0, 0) \\
&- \frac{1}{4}Jf(2x, 0, 2x, 0) + \frac{1}{4}Jf(2x, 0, 0, 0) + \frac{1}{4}Jf(0, 0, 2x, 0) \\
&- \frac{1}{4}Jf(2y, 0, 2y, 0) + \frac{1}{4}Jf(2y, 0, 0, 0) + \frac{1}{4}Jf(0, 0, 2y, 0) = 0
\end{aligned}$$

for all $x, y \in X$.

3. The stability of a bi-Jensen functional equation

In the rest of the paper, let X, Y be a normed space and a Banach space, respectively. We need the following lemma[5] to prove the main results.

Lemma 3. *Let $f : X \times X \to Y$ be a bi-Jensen mapping. Then the following equalities hold;*

$$f(x,y) = \frac{1}{4^n} f(2^n x, 2^n y) + (\frac{1}{2^n} - \frac{1}{4^n})(f(2^n x, 0) + (f(0, 2^n y)) + (1 - \frac{1}{2^n})^2 f(0,0),$$

$$f(x,y) = \frac{1}{4^n} f(2^n x, 2^n y) + (2^n - 1)(f(\frac{x}{2^n}, 0) + f(0, \frac{y}{2^n})) - (2^{n+1} - 3 + \frac{1}{4^n}) f(0,0)),$$

$$f(x,y) = 4^n f(\frac{x}{2^n}, \frac{y}{2^n}) + (2^n - 4^n)(f(\frac{x}{2^n}, 0) + f(0, \frac{y}{2^n})) + (2^n - 1)^2 f(0,0)),$$

$$f(x,y) = \frac{1}{2^n} f(2^n x, y) + \frac{1}{2^n}(1 - \frac{1}{2^n}) f(0, 2^n y) + (1 - \frac{1}{2^n})^2 f(0,0) \quad \textit{and}$$

$$f(x,y) = \frac{1}{2^n} f(2^n x, y) + \frac{1}{2^{n+1}}(1 - \frac{1}{2^n})(f(x, 2^n y) + f(-x, 2^n y)) + (1 - \frac{1}{2^n})^2 f(0,0)$$

for all $x, y \in X$ and $n \in \mathbb{N}$.

Now we have the stability of a bi-Jensen mapping in the following theorem.

Theorem 4. *Let $\varphi : X \times X \times X \times X \to [0, \infty)$ be a function satisfying*

$$\hat{\varphi}(x,y,z,w) := \sum_{j=1}^{\infty} \frac{1}{2^j} \varphi(2^j x, 2^j y, 2^j z, 2^j w) < \infty \tag{1}$$

for all $x, y, z, w \in X$ and let $f : X \times X \to Y$ be a mapping such that

$$\|Jf(x,y,z,w)\| \le \varphi(x,y,z,w) \tag{2}$$

for all $x, y, z, w \in X$. Then there exists a unique bi-Jensen mapping $F : X \times X \to Y$ such that

$$\begin{aligned} \|f(x,y) - F(x,y)\| &\le \tilde{\varphi}(x,0,y,0) + \tilde{\varphi}(x,0,0,0) + \tilde{\varphi}(0,0,y,0) \\ &\quad + \frac{1}{2}\hat{\varphi}(x,0,0,0) + \frac{1}{2}\hat{\varphi}(0,0,y,0) \end{aligned} \tag{3}$$

for all $x, y \in X$ with $F(0,0) = f(0,0)$, where $\tilde{\varphi} : X \times X \times X \times X \to [0, \infty)$ is the map given by

$$\tilde{\varphi}(x,y,z,w) := \sum_{j=1}^{\infty} \frac{1}{4^j} \varphi(2^j x, 2^j y, 2^j z, 2^j w) < \infty$$

for all $x, y, z, w \in X$. The mapping $F : X \times X \to Y$ is given by

$$F(x,y) := \lim_{j \to \infty} [\frac{1}{4^j} f(2^j x, 2^j y) + \frac{1}{2^j} f(2^j x, 0) + \frac{1}{2^j} f(0, 2^j y)] + f(0,0)$$

for all $x, y \in X$.

Proof. Since

$$
\begin{aligned}
&\|\frac{1}{4^j}(f(2^jx,2^jy)-f(2^jx,0)-f(0,2^jy)+f(0,0)) \\
&\quad -\frac{1}{4^{j+1}}(f(2^{j+1}x,2^{j+1}y)-f(2^{j+1}x,0)-f(0,2^{j+1}y)+f(0,0))\| \\
&\quad =\frac{1}{4^{j+1}}\|Jf(2^{j+1}x,0,2^{j+1}y,0)-Jf(2^{j+1}x,0,0,0)-Jf(0,0,2^{j+1}y,0)\| \\
&\quad \leq \frac{1}{4^{j+1}}(\varphi(2^{j+1}x,0,2^{j+1}y,0)+\varphi(2^{j+1}x,0,0,0)+\varphi(0,0,2^{j+1}y,0)),
\end{aligned} \tag{4}
$$

$$
\|\frac{1}{2^j}f(2^jx,0)-f(0,0)-\frac{1}{2^{j+1}}(f(2^{j+1}x,0)-f(0,0))\| \leq \frac{1}{2^{j+2}}\varphi(2^{j+1}x,0,0,0) \tag{5}
$$

and

$$
\|\frac{1}{2^j}f(0,2^jy)-f(0,0)-\frac{1}{2^{j+1}}(f(0,2^{j+1}y)-f(0,0))\| \leq \frac{1}{2^{j+2}}\varphi(0,0,\frac{1}{2^{j+1}}y,0) \tag{6}
$$

for all $x, y \in X$ and $j \in \mathbb{N}$, we get

$$
\begin{aligned}
&\|\frac{1}{4^l}(f(2^lx,2^ly)-f(2^lx,0)-f(0,2^ly)+f(0,0)) \\
&\quad -\frac{1}{4^m}(f(2^mx,2^my)-f(2^mx,0)-f(0,2^my)+f(0,0))\| \\
&\quad \leq \sum_{j=l}^{m-1}\frac{1}{4^{j+1}}(\varphi(2^{j+1}x,0,2^{j+1}y,0)+\varphi(2^{j+1}x,0,0,0)+\varphi(0,0,2^{j+1}y,0)),
\end{aligned} \tag{7}
$$

$$
\|\frac{1}{2^l}f(2^lx,0)-f(0,0)-\frac{1}{2^m}(f(2^mx,0)-f(0,0))\| \leq \sum_{j=l}^{m-1}\frac{1}{2^{j+2}}\varphi(2^{j+1}x,0,0,0) \tag{8}
$$

and

$$
\|\frac{1}{2^l}f(0,2^ly)-f(0,0)-\frac{1}{2^m}(f(0,2^my)-f(0,0))\| \leq \sum_{j=l}^{m-1}\frac{1}{2^{j+2}}\varphi(0,0,2^{j+1}y,0) \tag{9}
$$

for given integers l, m $(0 \leq l < m)$ and for all $x, y \in X$. By (1), the sequences $\{\frac{1}{4^j}(f(2^jx,2^jy)-f(2^jx,0)-f(0,2^jy)+f(0,0))\}$, $\{\frac{1}{2^j}(f(2^jx,0)-f(0,0)\}$ and $\{\frac{1}{2^j}(f(0,2^jy)-f(0,0)\}$ are Cauchy sequences for all $x, y \in X$. Since Y is complete, the sequences $\{\frac{1}{4^j}(f(2^jx,2^jy)-f(2^jx,0)-f(0,2^jy)+f(0,0))\}$, $\{\frac{1}{2^j}(f(2^jx,0)-f(0,0)\}$ and $\{\frac{1}{2^j}(f(0,2^jy)-f(0,0)\}$ converge for all $x, y \in X$. Define F_1, F_2, F_3 :

$X \times X \to Y$ by

$$F_1(x,y) := \lim_{j\to\infty} \frac{1}{4^j}(f(2^j x, 2^j y) - f(2^j x, 0) - f(0, 2^j y))$$
$$F_2(x,y) := \lim_{j\to\infty} \frac{1}{2^j} f(2^j x, 0)$$
$$F_3(x,y) := \lim_{j\to\infty} \frac{1}{2^j} f(0, 2^j y)$$

for all $x, y \in X$. Putting $l = 0$ and taking $m \to \infty$ in (7), (8) and (9), one can obtain the inequalities

$$\|f(x,y) - f(x,0) - f(0,y) + f(0,0)) - F_1(x,y)\| \leq \tilde{\varphi}(x,0,y,0) + \tilde{\varphi}(x,0,0,0) + \tilde{\varphi}(0,0,y,0), \tag{10}$$

$$\|f(x,0) - f(0,0) - F_2(x,0)\| \leq \frac{1}{2}\hat{\varphi}(x,0,0,0) \quad \text{and} \tag{11}$$

$$\|f(0,y) - f(0,0) - F_3(0,y)\| \leq \frac{1}{2}\hat{\varphi}(0,0,y,0) \tag{12}$$

for all $x, y \in X$. Since $\lim_{n\to\infty} \frac{1}{4^n} f(2^n x, 0) = \lim_{n\to\infty} \frac{1}{2^n} F_2(x,0) = 0$ and $\lim_{n\to\infty} \frac{1}{4^n} f(0, 2^n y) = \lim_{n\to\infty} \frac{1}{2^n} F_3(0,y) = 0$, we get

$$F_1(x,y) := \lim_{j\to\infty} \frac{1}{4^j} f(2^j x, 2^j y)$$

for all $x, y \in X$. Since

$$JF_1(x,y,z,w) = \lim_{n\to\infty} \frac{1}{4^n} Jf(2^n x, 2^n y, 2^n z, 2^n w) = 0$$
$$JF_2(x,y,z,w) = \lim_{n\to\infty} \frac{1}{2^n} Jf(2^n x, 2^n y, 0, 0) = 0$$
$$JF_3(x,y,z,w) = \lim_{n\to\infty} \frac{1}{2^n} Jf(0, 0, 2^n z, 2^n w) = 0$$

for all $x, y, z, w \in X$ and all $n \in \mathbb{N}$, F is a bi-Jensen mapping satisfying (3), where $F : X \times X \to Y$ is given by

$$F(x,y) = F_1(x,y) + F_2(x,y) + F_3(x,y) + f(0,0)$$

for all $x, y \in X$. Now, let $F' : X \times X \to Y$ be another bi-Jensen mapping satisfying (3) with $F'(0,0) = f(0,0)$. Using Lemma 3 and $\tilde{\varphi}(x,y,z,w) \le \hat{\varphi}(x,y,z,w)$, we have

$$\begin{aligned}
&\|F(x,y) - F'(x,y)\| \\
&\quad \le \|\frac{1}{4^n}(F-F')(2^n x, 2^n y) + (\frac{1}{2^n} - \frac{1}{4^n})((F-F')(2^n x, 0) + (F-F')(0, 2^n y))\| \\
&\quad \le \frac{1}{4^n}(\|(F-f)(2^n x, 2^n y)\| + \|(f-F')(2^n x, 2^n y)\|) + \frac{1}{2^n}\|(F-f)(2^n x, 0)\| \\
&\quad + \frac{1}{2^n}\|(f-F')(2^n x, 0)\| + \frac{1}{2^n}\|(F-f)(0, 2^n y)\| + \frac{1}{2^n}\|(f-F')(0, 2^n y)\| \\
&\quad \le \frac{1}{4^n}(2\tilde{\varphi}(2^n x, 0, 2^n y, 0) + 3\hat{\varphi}(2^n x, 0, 0, 0) + 3\hat{\varphi}(0, 0, 2^n y, 0)) \\
&\quad + \frac{1}{2^n}(5\hat{\varphi}(2^n x, 0, 0, 0) + 5\hat{\varphi}(0, 0, 2^n y, 0)) + 6\hat{\varphi}(0,0,0,0))
\end{aligned}$$

for all $n \in \mathbb{N}$ and $x, y \in X$. As $n \to \infty$, we may conclude that $F(x,y) = F'(x,y)$ for all $x, y \in X$. Thus such a bi-Jensen mapping $F : X \times X \to Y$ is unique.

Now we have another stability of a bi-Jensen mapping for the several cases in the applications.

Theorem 5. *Let $\varphi : X \times X \times X \times X \to [0, \infty)$ be a function satisfying*

$$\tilde{\varphi}(x,y,z,w) := \sum_{j=0}^{\infty} 4^j \varphi(\frac{x}{2^j}, \frac{y}{2^j}, \frac{z}{2^j}, \frac{w}{2^j}) < \infty$$

for all $x, y, z, w \in X$ and let $f : X \times X \to Y$ be a mapping such that the inequality (2) holds for all $x, y, z, w \in X$. Then there exists a unique bi-Jensen mapping $F : X \times X \to Y$ satisfying the inequality (3) for all $x, y \in X$, where $\hat{\varphi} : X \times X \times X \times X \to [0, \infty)$ is the map given by

$$\hat{\varphi}(x,y,z,w) := \sum_{j=0}^{\infty} 2^j \varphi(\frac{x}{2^j}, \frac{y}{2^j}, \frac{z}{2^j}, \frac{w}{2^j})$$

for all $x, y, z, w \in X$. The mapping $F : X \times X \to Y$ is given by

$$\begin{aligned}
F(x,y) &:= \lim_{j\to\infty} 4^j (f(\frac{x}{2^j}, \frac{y}{2^j}) - f(\frac{x}{2^j}, 0) - f(0, \frac{y}{2^j}) + f(0,0)) \\
&+ \lim_{j\to\infty} 2^j (f(\frac{x}{2^j}, 0) - f(0,0)) + \lim_{j\to\infty} 2^j (f(0, \frac{y}{2^j}) - f(0,0)) + f(0,0)
\end{aligned}$$

for all $x, y \in X$.

Proof. Since the inequalities (4), (5) and (6) hold for all negative integer j, we can use the similar method in Theorem 4 to get the sequences $\{4^j(f(\frac{x}{2^j}, \frac{y}{2^j}) - f(\frac{x}{2^j}, 0) - f(0, \frac{y}{2^j}) + f(0,0))\}$, $\{2^j(f(\frac{x}{2^j}, 0) - f(0,0))\}$ and $\{2^j(f(0, \frac{y}{2^j}) - f(0,0))\}$ converge for all $x, y \in X$. Define $F_1, F_2, F_3 : X \times X \to Y$ by

$$F_1(x,y) := \lim_{j\to\infty} 4^j (f(\frac{x}{2^j}, \frac{y}{2^j}) - f(\frac{x}{2^j}, 0) - f(0, \frac{y}{2^j}) + f(0,0)),$$
$$F_2(x,y) := \lim_{j\to\infty} 2^j (f(\frac{x}{2^j}, 0) - f(0,0)),$$
$$F_3(x,y) := \lim_{j\to\infty} 2^j (f(0, \frac{y}{2^j}) - f(0,0))$$

for all $x, y \in X$. By the similar method in Theorem 4, we obtain the inequalities (10), (11) and (12). Since

$$\begin{aligned} JF_1(x,y,z,w) &= \lim_{n\to\infty} 4^n (Jf(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}, \frac{w}{2^n}) \\ &\quad - Jf(\frac{x}{2^n}, \frac{y}{2^n}, 0, 0) - Jf(0, 0, \frac{z}{2^n}, \frac{w}{2^n})) = 0, \\ JF_2(x,y,z,w) &= \lim_{n\to\infty} 2^n Jf(\frac{x}{2^n}, \frac{y}{2^n}, 0, 0) = 0 \quad \text{and} \\ JF_3(x,y,z,w) &= \lim_{n\to\infty} 2^n Jf(0, 0, \frac{z}{2^n}, \frac{w}{2^n}) = 0 \end{aligned}$$

for all $x, y, z, w \in X$, F is a bi-Jensen mapping satisfying (3), where $F : X \times X \to Y$ is given by

$$F(x,y) = F_1(x,y) + F_2(x,y) + F_3(x,y) + f(0,0)$$

for all $x, y \in X$. Now, let $F' : X \times X \to Y$ be another bi-Jensen mapping satisfying (3). Using Lemma 3, $\hat{\varphi}(x,y,z,w) \le \tilde{\varphi}(x,y,z,w)$, $\varphi(0,0,0,0) = 0$ and $F(0,0) = f(0,0) = F'(0,0)$, we have

$$\begin{aligned} &\|F(x,y) - F'(x,y)\| \\ &\qquad \le \|4^n (F - F')(\frac{x}{2^n}, \frac{y}{2^n}) + (2^n - 4^n)((F - F')(\frac{x}{2^n}, 0) + (F - F')(0, \frac{y}{2^n}))\| \\ &\qquad \le 4^n \|F(\frac{x}{2^n}, \frac{y}{2^n}) - f(\frac{x}{2^n}, \frac{y}{2^n})\| + 4^n \|f(\frac{x}{2^n}, \frac{y}{2^n}) - F'(\frac{x}{2^n}, \frac{y}{2^n})\| \\ &\qquad + 4^n \|F(\frac{x}{2^n}, 0) - f(\frac{x}{2^n}, 0)\| + 4^n \|f(\frac{x}{2^n}, 0) - F'(\frac{x}{2^n}, 0)\| \\ &\qquad + 4^n \|F(0, \frac{y}{2^n}) - f(0, \frac{y}{2^n})\| + 4^n \|f(0, \frac{y}{2^n}) - F'(0, \frac{y}{2^n})\| \\ &\qquad \le 2(4^n \tilde{\varphi}(\frac{x}{2^n}, 0, \frac{y}{2^n}, 0) + 4^{n+1} \tilde{\varphi}(\frac{x}{2^n}, 0, 0, 0)) + 4^{n+1} \tilde{\varphi}(0, 0, \frac{y}{2^n}, 0)) \end{aligned}$$

for all $n \in \mathbb{N}$ and $x, y \in X$. As $n \to \infty$, we may conclude that $F(x,y) = F'(x,y)$ for all $x, y \in X$. Thus such a bi-Jensen mapping $F : X \times X \to Y$ is unique.

Theorem 6. *Let $\varphi : X \times X \times X \times X \to [0, \infty)$ be a function satisfying*

$$\tilde{\varphi}(x, y, z, w) := \sum_{j=1}^{\infty} \frac{1}{4^j} \varphi(2^j x, 2^j y, 2^j z, 2^j w) < \infty,$$
$$\hat{\varphi}(x, y, z, w) := \sum_{j=0}^{\infty} 2^j \varphi(\frac{x}{2^j}, \frac{y}{2^j}, \frac{z}{2^j}, \frac{w}{2^j}) < \infty$$

for all $x, y, z, w \in X$ and let $f : X \times X \to Y$ be a mapping such that the inequality (2) holds for all $x, y, z, w \in X$. Then there exists a bi-Jensen mapping $F : X \times X \to Y$ satisfying the inequality (3) for all $x, y \in X$. The mapping $F : X \times X \to Y$ is given by

$$F(x, y) := \lim_{j\to\infty} \frac{1}{4^j}(f(2^j x, 2^j y) - f(2^j x, 0) - f(0, 2^j y)) + f(0, 0)$$
$$+ \lim_{j\to\infty} 2^j (f(\frac{x}{2^j}, 0) - f(0, 0)) + \lim_{j\to\infty} 2^j (f(0, \frac{y}{2^j}) - f(0, 0)) + f(0, 0)$$

for all $x, y \in X$.

Proof. Let F_1 be as in Theorem 4 and let F_2 and F_3 be as in Theorem 5. Since

$$JF_1(x, y, z, w) = \lim_{n\to\infty} \frac{1}{4^n}(Jf(2^n x, 2^n y, 2^n z, 2^n w)$$
$$- Jf(2^n x, 2^n y, 0, 0) - Jf(0, 0, 2^n z, 2^n w)) = 0,$$

F is a bi-Jensen mapping satisfying (3), where F is defined by

$$F(x, y) = F_1(x, y) + F_2(x, y) + F_3(x, y) + f(0, 0)$$

for all $x, y \in X$.

As an application of Theorems 4, 5 and 6, we obtain the stability of a bi-Jensen mapping in the sense of Th.M.Rassias(See [5]).

Corollary 7. *Let $0 \le p(\neq 1, 2)$ and $\theta > 0$. Let $f : X \times X \to Y$ be a mapping such that*

$$\|Jf(x, y, z, w))\| \le \theta(\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p)$$

for all $x, y, z, w \in X$. Then there exists a bi-Jensen mapping $F : X \times X \to Y$ such that

$$\|f(x, y) - F(x, y)\| \le (\frac{2^p}{|2(2 - 2^p)|} + \frac{2 \cdot 2^p}{|4 - 2^p|})\theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$ with $F(0, 0) = f(0, 0)$.

4. The stability of a bi-Jensen functional equation

We can easily check the following lemma. **Lemma 8.** *Let $f : X \times X \to Y$ be a mapping and let $f' : X \times X \to Y$ be the map defined by*

$$f'(x, y) = f(x, y) - f(x, 0) - f(0, y) + f(0, 0)$$

for all $x, y \in X$. Then

$$f(x, 0) - f(0, 0) - \frac{f(2x, 0) - f(0, 0)}{2} = B_1(x, y),$$

$$f(0, y) - f(0, 0) - \frac{f(0, 2y) - f(0, 0)}{2} = B_2(x, y),$$

$$f'(x, y) - \frac{f'(2x, 2y)}{4} = B_3(x, y)$$

for all $x, y, z, w \in X\backslash\{0\}$, where

$$\begin{aligned} B_1(x, y) = &\frac{1}{8}[Jf(x, x, y, -y) - Jf(x, -x, y - y) \\ &- Jf(3x, x, y, -y) + Jf(3x, -x, y, -y)], \\ B_2(x, y) = &\frac{1}{8}[Jf(x, -x, y, y) - Jf(x, -x, y, -y) \\ &- Jf(x, -x, 3y, y) + Jf(x, -x, 3y, -y)], \\ B_3(x, y) = &\frac{1}{16}[-Jf(3x, x, 2y, 2y) + Jf(3x, -x, 2y, 2y) - Jf(x, -x, 2y, 2y) \\ &- 2Jf(x, x, 3y, y) + 2Jf(x, x, \ 3y, -y) - 2Jf(x, x, y, -y)] \\ &- \frac{1}{2}B_1(x, y) - B_2(x, y). \end{aligned}$$

We need the following lemmas which is easily verified by the same method in the proof of Lemma 3.3 in [10] to prove the main theorem.

Lemma 9. *Let a set $A(\subset X)$ satisfy the following condition : for every $x \neq 0$, there exists a positive integer n_x such that $nx \notin A$ for all $|n| \geq n_x$ and $nx \in A$ for all $|n| < n_x$. If $F : X \times X \to Y$ satisfies the equality*

$$JF(x, y, z, w) = 0$$

for all $x, y, z, w \in X\backslash A$, then there exists a unique bi-Jensen map $F' : X \times X \to Y$ satisfying the equality

$$F'(x, y) = F(x, y)$$

for all $x, y \in X\backslash A$.

Lemma 10. *Let A, F be as in Lemma 9. Then the map $F : X \times X \to Y$ is a bi-Jensen mapping.*

Proof. Let $A_x = \{n \in \mathbb{N} | nx \notin A\}$ for each $x \neq 0$. By Lemma 9, there exists a unique bi-Jensen map F' satisfying the equality

$$F'(x, y) = F(x, y)$$

for all $x, y \in X \backslash A$. Choose $n \in A_x \cap A_y$ for the case $x, y \neq 0$, then

$$\begin{aligned} F(x,y) - F'(x,y) &= \frac{1}{4}[JF((n+2)x, -nx, (n+2)y, -ny) \\ &\quad - JF'((n+2)x, -nx, (n+2)y, -ny)] = 0, \\ F(0,y) - F'(0,y) &= \frac{1}{4}[JF(ny, -ny, (n+2)y, -ny) \\ &\quad - JF'(ny, -ny, (n+2)y, -ny)] = 0, \\ F(x,0) - F'(x,0) &= \frac{1}{4}[JF((n+2)x, -nx, nx, -nx) \\ &\quad - JF'((n+2)x, -nx, nx, -nx)] = 0, \\ F(0,0) - F'(0,0) &= \frac{1}{4}[JF(nx, -nx, nx, -nx) - JF'(nx, -nx, nx, -nx)] = 0. \end{aligned}$$

Hence $F(x, y) = F'(x, y)$ for all $x, y \in X$ as we desired.

From Lemma 3, we get the following lemma.

Lemma 11. *Let $f : X \times X \to Y$ be a bi-Jensen mapping. Then the equality*

$$\begin{aligned} f(x,y) = \frac{f(2^n x, 2^n y)}{2^n} &+ \frac{1}{2}(\frac{1}{2^n} - \frac{1}{4^n})(f(2^n x, -2^n y) \\ &+ f(-2^n x, 2^n y)) + (1 - \frac{1}{2^n})^2 f(0,0) \end{aligned}$$

holds for all $x, y \in X$ and $n \in \mathbb{N}$.

Theorem 12. *Let A be as in Lemma 9. Let $\varphi : X \times X \times X \times X \to [0, \infty)$ be a function satisfying*

$$\sum_{i=0}^{\infty} \frac{1}{2^i} \varphi(2^i x, 2^i y, 2^i z, 2^i w) < \infty \tag{13}$$

for all $x, y, z, w \in X$. If $f : X \times X \to Y$ satisfies

$$\|Jf(x, y, z, w)\| \leq \varphi(x, y, z, w) \tag{14}$$

for all $x, y, z, w \in X \backslash A$, then there exists a unique bi-Jensen mapping $F : X \times X \to Y$ such that

$$\|f(x, y) - F(x, y)\| \leq \hat{\varphi}(x, y) \tag{15}$$

for all $x, y \in X \backslash A$ *with* $F(0,0) = f(0,0)$, *where*

$$\hat{\varphi}(x,y) = \sum_{j=0}^{\infty}[\frac{\varphi_1(2^j x, 2^j y)}{2^j} + \frac{\varphi_2(2^j x, 2^j y)}{2^j} + \frac{\varphi_3(2^j x, 2^j y)}{4^j}],$$

$$\varphi_1(x,y) = \frac{1}{8}[\varphi(x,x,y,-y) + \varphi(x,-x,y-y) + \varphi(3x,x,y,-y) + \varphi(3x,-x,y,-y)],$$

$$\varphi_2(x,y) = \frac{1}{8}[\varphi(x,-x,y,y) + \varphi(x,-x,y,-y) + \varphi(x,-x,3y,y) + \varphi(x,-x,3y,-y)],$$

$$\varphi_3(x,y) = \frac{1}{16}[\varphi(3x,x,2y,2y) + \varphi(3x,-x,2y,2y) + \varphi(x,-x,2y,2y) + 2\varphi(x,x,3y,y) + 2\varphi(x,x,\ 3y,-y) + 2\varphi(x,x,y,-y)] + \frac{1}{2}\varphi_1(x,y) + \varphi_2(x,y).$$

The mapping $F : X \times X \to Y$ *is given by*

$$F(x,y) := \lim_{j\to\infty} \frac{1}{4^j} f(2^j x, 2^j y) + \lim_{j\to\infty} \frac{1}{2^j} f(2^j x, 0) + \lim_{j\to\infty} \frac{1}{2^j} f(0, 2^j y) + f(0,0)$$

for all $x, y \in X$.

Proof. Let f', B_1, B_2, B_3 be as in Lemma 8. Using Lemma 8 and (14), we get

$$\|\frac{f(2^n x,0) - f(0,0)}{2^n} - \frac{f(2^{n+1}x,0) - f(0,0)}{2^{n+1}}\| = \|\frac{B_1(2^n x, 2^n y)}{2^n}\| \le \frac{\varphi_1(2^n x, 2^n y)}{2^n},$$

$$\|\frac{f(0,2^n y) - f(0,0)}{2^n} - \frac{f(0,2^{n+1}y) - f(0,0)}{2^{n+1}}\| = \|\frac{B_2(2^n x, 2^n y)}{2^n}\| \le \frac{\varphi_2(2^n x, 2^n y)}{2^n},$$

$$\|\frac{f'(2^n x, 2^n y)}{4^n} - \frac{f'(2^{n+1}x, 2^{n+1}y)}{4^{n+1}}\| = \|\frac{B_3(2^n x, 2^n y)}{4^n}\| \le \frac{\varphi_3(2^n x, 2^n y)}{4^n}$$

for all $x, y \in X\backslash\{0\}$. For given integers l, m $(0 \le l < m)$, the inequalities

$$\|\frac{f(2^l x,0) - f(0,0)}{2^l} - \frac{f(2^m x,0) - f(0,0)}{2^m}\| \le \sum_{j=l}^{m-1} \frac{\varphi_1(2^j x, 2^j y)}{2^j}, \tag{16}$$

$$\|\frac{f(0,2^l y) - f(0,0)}{2^l} - \frac{f(0,2^m y) - f(0,0)}{2^m}\| \le \sum_{j=l}^{m-1} \frac{\varphi_2(2^j x, 2^j y)}{2^j}, \tag{17}$$

$$\|\frac{f'(2^l x, 2^l y)}{4^l} - \frac{f'(2^m x, 2^m y)}{4^m}\| \le \sum_{j=l}^{m-1} \frac{\varphi_3(2^j x, 2^j y)}{4^j} \tag{18}$$

hold for all $x, y \in X\backslash A$. By the above inequalities and (13), the sequences $\{\frac{f(2^n x,0)-f(0,0)}{2^n}\}$, $\{\frac{f(0,2^n x)-f(0,0)}{2^n}\}$, $\{\frac{f'(2^n x,2^n y)}{4^n}\}$ are Cauchy sequences for all $x, y \in X$. Since Y is complete, the sequences $\{\frac{f(2^n x,0)-f(0,0)}{2^n}\}$, $\{\frac{f'(2^n x,2^n y)}{4^n}\}$, $\{\frac{f(0,2^n x)-f(0,0)}{2^n}\}$

converge for all $x, y \in X \backslash A$. Define $F_1, F_2, F_3 : X \times X \to Y$ by

$$\begin{aligned}
F_1(x,y) &:= \lim_{j\to\infty} \frac{1}{2^j} f(2^j x, 0)\\
F_2(x,y) &:= \lim_{j\to\infty} \frac{1}{2^j} f(0, 2^j y)\\
F_3(x,y) &:= \lim_{j\to\infty} \frac{1}{4^j} (f(2^j x, 2^j y) - f(2^j x, 0) - f(0, 2^j y)) = \lim_{j\to\infty} \frac{f(2^j x, 2^j y)}{4^j}
\end{aligned}$$

for all $x, y \in X$. Putting $l = 0$ and taking $m \to \infty$ in (16), (17) and (18), one can obtain the inequalities

$$\begin{aligned}
&\|f(x,0) - f(0,0) - F_1(x,y)\| \leq \sum_{j=0}^{\infty} \frac{\varphi_1(2^j x, 2^j y)}{2^j},\\
&\|f(0,y) - f(0,0) - F_2(x,y)\| \leq \sum_{j=0}^{\infty} \frac{\varphi_2(2^j x, 2^j y)}{2^j} \quad \text{and}\\
&\|f'(x,y) - F_3(x,y)\| \leq \sum_{j=0}^{\infty} \frac{\varphi_3(2^j x, 2^j y)}{4^j}
\end{aligned}$$

for all $x, y \in X \backslash A$. Using (14) and the definitions of F_1, F_2, F_3, we have

$$\begin{aligned}
JF_1(x,y,z,w) &= \lim_{n\to\infty} \frac{1}{2^{n+1}} (2Jf(2^n x, 2^n y, 2^n z, -2^n z)\\
&\quad - Jf(2^n x, 2^n x, 2^n z, -2^n z) - Jf(2^n y, 2^n y, 2^n z, -2^n z)) = 0,\\
JF_2(x,y,z,w) &= \lim_{n\to\infty} \frac{1}{2^{n+1}} (2Jf(2^n x, -2^n x, 2^n z, 2^n w)\\
&\quad - Jf(2^n x, -2^n x, 2^n z, 2^n z) - Jf(x, -x, 2^n w, 2^n w)) = 0,\\
JF_3(x,y,z,w) &= \lim_{n\to\infty} \frac{Jf(2^n x, 2^n y, 2^n z, 2^n w)}{4^n} = 0
\end{aligned}$$

for all $x, y, z, w \in X \backslash A$. Since

$$\begin{aligned}
\|f(x,y) - F(x,y)\| \leq \|f'(x,y) - F_3(x,y)\| &+ \|f(x,0) - f(0,0) - F_1(x,y)\|\\
&+ \|f(0,y) - f(0,0) - F_2(x,y)\|,
\end{aligned}$$

F is a bi-Jensen mapping satisfying (15) by Lemma 10, where

$$F(x,y) = F_1(x,y) + F_2(x,y) + F_3(x,y) + f(0,0)$$

for all $x, y \in X$. Now, let $F' : X \times X \to Y$ be another bi-Jensen mapping satisfying (15) with $F(0,0) = F'(0,0)$. By lemma 11, we have

$$\begin{aligned}\|F(x,y) - F'(x,y)\| &= \|\frac{(F - F')(2^n x, 2^n y)}{2^n} \\ &\quad + \frac{1}{2}(\frac{1}{2^n} - \frac{1}{4^n})[(F - F')(2^n x, -2^n y) + (F - F')(-2^n x, 2^n y)]\| \\ &\le \frac{\|(F - f)(2^n x, 2^n y)\| + \|(f - F')(2^n x, 2^n y)\|}{2^n} \\ &\quad + \frac{\|(F - f)(2^n x, -2^n y)\| + \|(f - F')(2^n x, -2^n y)\|}{2^{n+1}} \\ &\quad + \frac{\|(F - f)(-2^n x, 2^n y)\| + \|(f - F')(-2^n x, 2^n y)\|}{2^{n+1}} \\ &\le \frac{1}{2^n}[2\hat{\varphi}(2^n x, 2^n y) + \hat{\varphi}(2^n x, -2^n y) + \hat{\varphi}(-2^n x, 2^n y)]\end{aligned}$$

for all $n \in \mathbb{N}$ and $x, y \in X \backslash A$. As $n \to \infty$, we may conclude that $F(x,y) = F'(x,y)$ for all $x, y \in X \backslash A$. By Lemma 9, $F(x,y) = F'(x,y)$ for all $x, y \in X$ as we desired.

Corollary 13. *Let $B = \{x \in X | \|x\| \le 1\}$. If a mapping $f : X \times X \to Y$ satisfies the inequality*

$$\|Jf(x, y, z, w)\| \le \theta(\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p)$$

for all $x, y, z, w \in X \backslash B$ with fixed real numbers $p < 1$ and $\theta > 0$, then there exists a unique bi-Jensen mapping $F : X \times X \to Y$ such that

$$\begin{aligned}&\|f(x,y) - F(x,y)\| \\ &\qquad \le (\frac{3^p + 7}{2(2 - 2^p)} + \frac{19 + 2 \cdot 3^p}{2(4 - 2^p)})\theta\|x\|^p + (\frac{3^p + 7}{2(2 - 2^p)} + \frac{14 + 3 \cdot 2^p + 4 \cdot 3^p}{2(4 - 2^p)})\|y\|^p\end{aligned}$$

for all $x, y \in X \backslash B$ with $F(0,0) = f(0,0)$.

Now we prove the superstability in the following theorem.

Theorem 14. *Let A be as in Theorem 12 and let $\varphi : X \times X \times X \times X \to [0, \infty)$ a function such that*

$$\lim_{(x,y,z,w) \to \infty} \varphi(x, y, z, w) = 0$$

for all $x, y, z, w \in X$. Let $f : X \times X \to Y$ be a mapping satisfying (14) for all $x, y, z, w \in X \backslash A$. Then f is a bi-Jensen mapping.

Proof. Let $\hat{\varphi}, F$ be as in Theorem 12. Using (14), (15) and the equality

$$\begin{aligned} f(x,y) - F(x,y) = \frac{1}{4}[&Jf((k+2)x, -kx, (k+2)y, -ky) + (f-F)(-kx, -ky) \\ &+ (f-F)((k+2)x, -ky) + (f-F)((k+2)x, (k+2)y) \\ &+ (f-F)(-kx, (k+2)y) - JF((k+2)x, -kx, (k+2)y, -ky)] \end{aligned}$$

for all $x, y \neq 0$ and $k \in \mathbb{N}$, we get

$$\begin{aligned} \|f(x,y) - F(x,y)\| \le \frac{1}{4}[&\varphi((k+2)x, -kx, (k+2)y, -ky) + \hat{\varphi}(-kx, -ky) \\ &+ \hat{\varphi}((k+2)x, -ky) + \hat{\varphi}((k+2)x, (k+2)y) + \hat{\varphi}(-kx, (k+2)y)] \end{aligned}$$

for all $x, y \neq 0$ with $kx, ky \notin A$. Since

$$\begin{aligned} \lim_{k\to\infty} [&\varphi((k+2)x, -kx, (k+2)y, -ky) + \hat{\varphi}(-kx, -ky) \\ &+ \hat{\varphi}((k+2)x, -ky) + \hat{\varphi}((k+2)x, (k+2)y) + \hat{\varphi}(-kx, (k+2)y)] = 0, \end{aligned}$$

we have

$$f(x,y) = F(x,y)$$

for all $x, y \neq 0$. Similarly, using the inequalities

$$\begin{aligned} f(x,0) - F(x,0) = \frac{1}{4}[&Jf((k+2)x, -kx, ky, -ky) + (f-F)(-kx, -ky) \\ &+ (f-F)((k+2)x, -ky) + (f-F)((k+2)x, ky) \\ &+ (f-F)(-kx, ky) - JF((k+2)x, -kx, ky, -ky)], \\ f(0,y) - F(0,y) = \frac{1}{4}[&Jf(kx, -kx, (k+2)y, -ky) + (f-F)(-kx, -ky) \\ &+ (f-F)(kx, -ky) + (f-F)(kx, (k+2)y) \\ &+ (f-F)(-kx, (k+2)y) - JF(kx, -kx, (k+2)y, -ky)], \\ f(0,0) - F(0,0) = \frac{1}{4}[&Jf(kx, -kx, ky, -ky) + (f-F)(-kx, -ky) \\ &+ (f-F)(kx, -ky) + (f-F)(kx, ky) \\ &+ (f-F)(-kx, ky) - JF(kx, -kx, ky, -ky)] \end{aligned}$$

for all $x, y \neq 0$ and $k \in \mathbb{N}$, we easily get

$$f(x,0) = F(x,0), \quad f(0,y) = F(0,y), \quad f(0,0) = F(0,0)$$

for all $x, y \neq 0$ as we desired.

Corollary 15 *Let $p < 0$ and let $f : X \times X \to Y$ be as in Corollary 13. Then f is a bi-Jensen mapping. Proof.* Apply Theorem 14.

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