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PROPERTIES OF SOLUTIONS OF LINEAR DIFFERENTIAL EQUATIONS IN THE COMPLEX PLANE

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Abstract. We research the properties of solutions of general higher order homogeneous linear differential equations and apply the hyper order to obtain more precise estimation for the growth of solutions of infinite order.

1. Introduction and Results

The complex differential equations in the complex plane $\mathbb{C}$ have been an active research area. Meanwhile, investigation of the complex differential equations in the unit disc $\Delta = \{z : |z| < 1\}$ also is paid attention toby CH. Pommerenke [5], J. Heittokangas [4], I. Chyzhykov, G. Gundersen and J. Heittokangas [2], etc. In this paper, we obtain some precise estimations of the order and the hyper order of solutions for some linear differential equations in Δ. We assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna's value distribution theory of meromorphic functions in $\mathbb{C}$ and in Δ. (e.g., see [3, 6]).

The order of a meromorphic function f in Δ is defined by

$$\sigma(f) = \limsup_{r\to 1^-} \frac{\log^+ T(r,f)}{\log \frac{1}{1-r}},$$

where $T(r, f)$ is the Nevanlinna characteristic function of $f(z)$. For an analytic function f in Δ, we also define

$$\sigma_M(f) = \limsup_{r\to 1^-} \frac{\log^+ \log^+ M(r,f)}{\log \frac{1}{1-r}},$$

where $M(r, f)$ is the maximum value of $|f(z)|$ on $|z| = r$.

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We define the hyper-order of f in Δ similarly as in the plane case

$$\sigma_2(f)=\limsup_{r\to 1^-}\frac{\log^+\log^+ T(r,f)}{\log\frac{1}{1-r}}.$$

If f is an analytic function in Δ, we define

$$\sigma_{M2}(f)=\limsup_{r\to 1^-}\frac{\log^+\log^+\log^+ M(r,f)}{\log\frac{1}{1-r}}.$$

Definition 1. A meromorphic function f in Δ is called *admissible*, [*non-admissible*, resp.] if and only if

$$\limsup_{r\to 1^-}\frac{T(r,f)}{\log\frac{1}{1-r}}=\infty,\ \left[\limsup_{r\to 1^-}\frac{T(r,f)}{\log\frac{1}{1-r}}<\infty\right].$$

Theorem 1. *Let A_j $(j=0,\dots,k-1)$ be analytic in Δ and A_0 be admissible. Suppose that $\sigma_M(A_j)<\sigma_M(A_0)=\mu<\infty$, for $j=1,\dots,k-1$; or $\sigma_M(A_0)=0, A_j$ are non-admissible for $j=1,\dots,k-1$. Then all solutions $f(\not\equiv 0)$ of the equation*

$$f^{(k)}+A_{k-1}(z)f^{(k-1)}+\cdots+A_0(z)f=0 \tag{1.1}$$

in Δ satisfy that $\sigma(f)=\infty$ and $\sigma_2(f)=\mu$.

Theorem 2. *Let A_j $(j=0,\dots,k-1)$ be analytic in Δ, $\sigma_M(A_j)\le\mu,(\mu>0$ is a constant). Suppose that $A_0(z)$ satisfies that there exists a set $H\subset[0,2\pi)$ with linear measure $mH>0$ such that for any $\varphi\in H$, there exist constants $\gamma=\gamma(\varphi)$ and $\delta=\delta(\varphi)$ satisfying $0\le\delta<\gamma$ and*

$$|A_0(z)|\ge\exp\left\{\frac{\gamma+o(1)}{(1-|z|)^\mu}\right\}, \tag{1.2}$$

$$|A_j(z)|\le\exp\left\{\frac{\delta+o(1)}{(1-|z|)^\mu}\right\},\ (j=1,\dots,k-1). \tag{1.3}$$

Then all solutions $f(\not\equiv 0)$ of the equation (1.1) *satisfy $\sigma_2(f)=\mu$.*

2. Lemmas for Proofs of Theorems

Lemma 1 ([2]). *Let f be a meromorphic function in Δ. Let $\alpha\in(1,\infty)$ and $\beta\in(0,1)$ be constants, and k and j be integers satisfying $k>j\ge 0$. Assume that $f^{(j)}\not\equiv 0$. Let $\{a_m\}$ denote the sequence of all the zeros and poles of $f^{(j)}$ listed according to multiplicities and ordered by increasing moduli, and let $n_j(r)$ denote the counting function of the points $\{a_m\}$. Then the following three statements hold:*

(1) *If* $\{a_m\}$ *is a finite sequence, then there exist constants* $R \in (0,1)$ *and* $C \in (0,\infty)$, *such that for all* z *satisfying* $R < |z| < 1$, *we have (with* $r = |z|$*)*

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \leq C\left[\frac{(T(1-\beta(1-r),f)) - \log(1-r)}{(1-r)^2}\right]^{k-j}. \tag{2.1}$$

(2) *If* $\{a_m\}$ *is an infinite sequence, then there exists an infinite sequence of discs* $D_i = \{z : |z - c_i| < R_i\} \subset \Delta\backslash\{0\}$ $(i = 1, 2, \ldots)$, *such that*

$$\sum_{i=1}^{\infty} \frac{R_i}{1-|c_i|} < \infty, \tag{2.2}$$

and there exist constants $R \in (0,1)$ *and* $C \in (0,\infty)$, *such that for all* z *satisfying* $z \notin \bigcup_{i=1}^{\infty} D_i$ *and* $R < |z| < 1$, *we have (with* $r = |z|$*)*

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \leq C\left[\frac{T(1-\beta(1-r),f) - \log(1-r)}{(1-r)^2} + W(r)\right]^{k-j}, \tag{2.3}$$

where

$$W(r) = \frac{n_j(1-\beta(1-r))}{1-r}\left(\log\frac{1}{1-r}\right)^{\alpha}\log^+ n_j(1-\beta(1-r)). \tag{2.4}$$

(3) *There exist a set* $E \subset [0,2\pi)$ *which has linear measure zero, and a constant* $C > 0$, *such that if* $\theta \in [0,2\pi)\backslash E$, *then there is a constant* $R = R(\theta) \in (0,1)$ *such that for all* z *satisfying* $\arg z = \theta$ *and* $R \leq |z| < 1$, *either* (2.1) *or* (2.3) *holds, depending on whether* $\{a_m\}$ *is a finite or infinite sequence, respectively.*

Lemma 2 ([4])**.** *Let* f *be an admissible meromorphic function in* Δ *and let* $k \in \mathbb{N}\bigcup\{0\}, a \in \hat{\mathbb{C}}$. *Set a set* $E \subset (0,1)$ *such that* $\int_E \frac{1}{1-r}dr = \alpha < \infty$. *Set* $b = e^{-(\alpha+1)}$, $s(r) = 1 - b(1-r)$ *and* $u(r) = (1-r)(1-b)$. *Then there exists* $R \in (0,1)$ *such that*

$$n(r, \frac{1}{f^{(k)} - a}) \leq \frac{k+3}{u(r)} T(s(r), f) \tag{2.5}$$

for all $r \in (R,1)$.

Lemma 3. *Let* f *be an admissible meromorphic function in* Δ *and let* $\beta \in (0,1)$ *be a constant, and* k *and* j *be integers satisfying* $k > j \geq 0$. *Assume that* $f^{(j)} \not\equiv 0$. *Then the following statements hold:*

(1) *There exists a set* $E \subset (0,1)$ *such that* $\int_E \frac{1}{1-r}dr < \infty$, *and there exist constants* $R \in (0,1)$ *and* $C \in (0,\infty)$, *such that for all* z *satisfying* $|z| = r \notin E$ *and*

$R \le |z| < 1$, we have

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \le C\left[\frac{(T(1-\beta(1-r),f))^{1+\varepsilon}}{(1-r)^{2+\varepsilon}}\right]^{k-j} \tag{2.6}$$

where ε is any constant satisfying $0<\varepsilon<1$.

(2) *There exist a set $E \subset [0,2\pi)$ which has linear measure zero, and a constant $C>0$, such that if $\theta \in [0,2\pi)\backslash E$, then there is a constant $R=R(\theta)\in(0,1)$ such that for all z satisfying $\arg z=\theta$ and $R\le|z|<1$, the inequality (2.6) holds.*

Proof. (1) By Lemma 1(1) and (2) , we only need prove that Lemma 3 holds for the case that $f^{(j)}$ has infinitely many zeros and poles. Setting $E=\bigcup_{i=1}^{\infty}(|c_i|-R_i,|c_i|+R_i)$, we have $E\subset\Delta$. By (2.2), there exists an integer I satisfying that as $i>I$, $\frac{R_i}{1-|c_i|}<\frac{1}{2}$. From this and (2.2), we deduce that

$$\int_E \frac{1}{1-r}dr=\sum_{i=1}^{\infty}\int_{|c_i|-R_i}^{|c_i|+R_i}\frac{1}{1-r}dr\le\sum_{i=1}^{\infty}\frac{2R_i}{1-|c_i|-R_i}$$

$$\le\sum_{i=1}^{I}\frac{\frac{2R_i}{1-|c_i|}}{1-\frac{R_i}{1-|c_i|}}+\sum_{i=I+1}^{\infty}\frac{4R_i}{1-|c_i|}<\infty. \tag{2.7}$$

Set $\int_E\frac{1}{1-r}dr=\alpha<\infty$. By Lemma 2, if we take $\beta_1=b=e^{-(\alpha+1)}$, then $0<\beta_1<\frac{1}{2}$,

$$s(r)=1-\beta_1(1-r),\ s(1-\beta_1(1-r))=1-\beta_1^2(1-r); \tag{2.8}$$

$$u(r)=(1-r)(1-\beta_1),\ u(1-\beta_1(1-r))=(1-r)\beta_1(1-\beta_1)\ge(1-r)\frac{\beta_1}{2}. \tag{2.9}$$

By Lemma 2, (2.8) and (2.9), we deduce that

$$n_j(1-\beta_1(1-r))\le 2\frac{j+3}{u(1-\beta_1(1-r))}T(s(1-\beta_1(1-r)),f)$$

$$\le\frac{4(j+3)}{\beta_1(1-r)}T(1-\beta_1^2(1-r),f). \tag{2.10}$$

Since for any given ε $(0<\varepsilon<1)$ and M $(0<M<\infty)$, as a sufficiently large positive real number x,

$$(\log x)^M\le x^{\varepsilon},$$

we deduce that

$$\log n_j(1-\beta_1(1-r))\le\left(\frac{T(1-\beta_1^2(1-r),f)}{1-r}\right)^{\frac{\varepsilon}{2}}+O(1); \tag{2.11}$$

$$\left(\log\frac{1}{1-r}\right)^{\alpha} \le \left(\frac{1}{1-r}\right)^{\frac{\varepsilon}{2}}; \tag{2.12}$$

$$\left(\log\frac{1}{1-r}\right) \le \left(\frac{1}{1-r}\right)^{\frac{\varepsilon}{2}}. \tag{2.13}$$

By Lemma 1 and (2.10)-(2.12), we deduce that

(2.14)

$$W(r) \le \frac{4(j+3)T(1-\beta_1^2(1-r),f)}{\beta_1(1-r)^2}\left(\frac{1}{1-r}\right)^{\frac{\varepsilon}{2}}\left[\left(\frac{T(1-\beta_1^2(1-r),f)}{1-r}\right)^{\frac{\varepsilon}{2}} + O(1)\right].$$

From (2.3), (2.13) and (2.14), setting $\beta = \beta_1^2$, we get that

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \le C\left[\frac{(T(1-\beta(1-r),f))^{1+\varepsilon}}{(1-r)^{2+\varepsilon}}\right]^{k-j}.$$

(2) Using the similar reasoning as in (1), we can get (2). □

Lemma 4 ([1])**.** *Let A_j $(j=1,\dots,k-1)$ be analytic functions in Δ with $\sigma_M(A_j) \le \sigma$. Suppose that $f(\not\equiv 0)$ is a solution of the equation* (1.1)*. Then we have $\sigma_2(f) \le \sigma$.*

3. Proof of Theorem 1

We suppose that $\sigma_M(f) = 0$, then $A_j (j = 1,\dots,k-1)$ are non-admissible. By Lemma 4, we have $\sigma_2(f) = 0$. Now we prove that $\sigma(f) = \infty$. Assume $\sigma(f) < \infty$. Then we have

$$m(r,\frac{f^{(j)}}{f}) = O\left(\log\frac{1}{1-r}\right),\ (j=1,\dots,k). \tag{3.1}$$

Since A_j $(j=1,\dots,k-1)$ are non-admissible, we have that

$$\limsup_{r\to 1^-}\frac{m(r,A_j)}{\log\frac{1}{1-r}} < \infty\ (j=1,\dots,k-1). \tag{3.2}$$

By the equation (1.1), we get that

$$-A_0 = \frac{f^{(k)}}{f} + A_{k-1}\frac{f^{(k-1)}}{f} + \cdots + A_1\frac{f'}{f}. \tag{3.3}$$

By (3.1) and (3.3), we get that

$$m(r,A_0) \le \sum_{j=1}^{k} m\left(r,\frac{f^{(j)}}{f}\right) + \sum_{j=1}^{k-1} m(r,A_j) \le M\left(\log\frac{1}{1-r}\right) + \sum_{j=1}^{k-1} m(r,A_j), \tag{3.4}$$

where M $(0 < M < \infty)$ is some constant. By (3.2) and (3.4), we get that

$$\frac{m(r, A_0)}{\log \frac{1}{1-r}} \leq \frac{M(\log \frac{1}{1-r}) + \sum_{j=1}^{k-1} m(r, A_j)}{\log \frac{1}{1-r}} < \infty. \tag{3.5}$$

Thus, (3.5) contradicts the hypothesis that A_0 is admissible. Hence, we have $\sigma(f) = \infty$.

And also we assume that $\sigma_M(A_j) < \sigma_M(A_0) = \mu < \infty$, for $j = 1, \ldots, k-1$. By Lemma 4, we have $\sigma_2(f) \leq \mu$. Now we prove to $\sigma_2(f) \geq \mu$. By Lemma 3(1), there exists a set $E \subset (0, 1)$ such that $\int_E \frac{1}{1-r} dr < \infty$ and there exist constants $R_1 \in (0, 1)$ and $C \in (0, \infty)$ such that for all z satisfying $|z| = r \notin E$ and $R_1 \leq |z| < 1$, we have

$$\left| \frac{f^{(j)}(z)}{f(z)} \right| \leq C \left[\frac{\left(T(1 - \frac{1}{2}(1-r), f)\right)^2}{(1-r)^3} \right]^j, \ (j = 1, \ldots, k). \tag{3.6}$$

Set

$$\max\{\sigma_M(A_j);\ j = 1, \ldots, k-1\} = \delta < \mu. \tag{3.7}$$

For any given ε $(0 < 3\varepsilon < \mu - \delta)$, there exists a constant $R_2 (R_1 \leq R_2 < 1)$, such that for all z satisfying $|z| = r \in [R_2, 1)$, we have

$$|A_j(z)| \leq \exp\left\{ \frac{1}{(1-r)^{\delta+\varepsilon}} \right\}, \ (j = 1, \ldots, k-1). \tag{3.8}$$

Since $\sigma_M(A_0) = \mu$, we can choose a sequence of points $\{z_n\}$ satisfying $|z_n| = r_n \in [R_2, 1) \backslash E$. Thus we have $|A_0(z_n)| = M(r_n, A_0)$, and

$$M(r_n, A_0) \geq \exp\left\{ \frac{1}{(1-r_n)^{\mu-\varepsilon}} \right\}. \tag{3.9}$$

By (3.3), (3.6)-(3.9), we deduce that

$$\exp\left\{ \frac{1}{(1-r_n)^{\mu-\varepsilon}} \right\} \leq kC \exp\left\{ \frac{1}{(1-r_n)^{\delta+\varepsilon}} \right\} \left[\frac{(T(1 - \frac{1}{2}(1-r_n), f))^2}{(1-r_n)^3} \right]^k. \tag{3.10}$$

Since $3\varepsilon < \mu - \delta$, by (3.10), we get that

$$\exp\left\{ \frac{1}{(1-r_n)^{\mu-2\varepsilon}} \right\} \leq \left[\frac{(T(1 - \frac{1}{2}(1-r_n), f))^2}{(1-r_n)^3} \right]^k. \tag{3.11}$$

Since ε is arbitrary, by (3.11), we get that $\sigma_2(f) \geq \mu$. Theorem 1 is proved. □

4. Proof of Theorem 2

Assume $f(z)(\not\equiv 0)$ is a solution of the equation (1.1). By the equation (1.1), we get that

$$-A_0(z) = \frac{f^{(k)}}{f} + A_{k-1}(z)\frac{f^{(k-1)}}{f} + \cdots + A_1(z)\frac{f'}{f}. \tag{4.1}$$

By Lemma 3(2), there exist a set $E \subset [0, 2\pi)$ with linear measure zero, and a constant $C > 0$ such that if $\arg z = \varphi \in [0, 2\pi)\backslash E$, then there exists a constant $R = R(\varphi) \in (0, 1)$, such that for all z satisfying $\arg z = \varphi$ and $R \leq |z| < 1$, the following inequality holds:

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \leq C\left[\frac{(T(1-\frac{1}{2}(1-r), f))^2}{(1-r)^3}\right]^j, \ (j = 1, \ldots, k). \tag{4.2}$$

Since the linear measure of H, $mH > 0$ and the linear measure of E, $mE = 0$, we can choose a ray $\arg z = \varphi_0 \in H\backslash E$. By the condition of Theorem 2, we know that there exist constants $\gamma = \gamma(\varphi_0)$ and $\delta = \delta(\varphi_0)$ satisfying $0 \leq \delta < \gamma$ and

$$|A_0(z)| \geq \exp\left\{\frac{\gamma + o(1)}{(1-|z|)^\mu}\right\}, \tag{4.3}$$

$$|A_j(z)| \leq \exp\left\{\frac{\delta + o(1)}{(1-|z|)^\mu}\right\}, \ (j = 1, \ldots, k-1). \tag{4.4}$$

By (4.1)-(4.4), we deduce that for all z satisfying $\arg z = \varphi_0$ and $|z| = r \in [R, 1)$, we have

$$\exp\left\{\frac{\gamma + o(1)}{(1-r)^\mu}\right\} \leq |A_0(z)| \leq kC\exp\left\{\frac{\delta + o(1)}{(1-r)^\mu}\right\}\left[\frac{(T(1-\frac{1}{2}(1-r), f))^2}{(1-r)^3}\right]^k. \tag{4.5}$$

By (4.5), we get that

$$\exp\left\{\frac{\gamma - \delta + o(1)}{(1-r)^\mu}\right\} \leq kC\left[\frac{(T(1-\frac{1}{2}(1-r), f))^2}{(1-r)^3}\right]^k. \tag{4.6}$$

By (4.6), we get that $\sigma_2(f) \geq \mu$. By Lemma 4, we have $\sigma_2(f) \leq \mu$. Therefore $\sigma_2(f) = \mu$. □

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