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STABILITY ANALYSIS FOR PREDATOR-PREY SYSTEMS

Seong-A Shim

Abstract. Various types of predator-prey systems are studied in terms of the stabilities of their steady-states. Necessary conditions for the existences of non-negative constant steady-states for those systems are obtained. The linearized stabilities of the non-negative constant steady-states for the predator-prey system with monotone response functions are analyzed. The predator-prey system with non-monotone response functions are also investigated for the linearized stabilities of the positive constant steady-states.

1. Introduction

In this paper we deal with a general predator-prey system in mathematical population dynamics as in the following form :

$$\begin{cases} u_t = uf(u) - v\,p(u) & \text{for } t \in (0,\infty), \\ v_t = v(-g(u) + c\,p(u) - r(v)) & \text{for } t \in (0,\infty), \\ u(0) = u_0 \geq 0, \quad v(0) = v_0 \geq 0, \end{cases} \tag{1.1}$$

where f, p, g are functions of u, the population density of the prey species, and r is a function of v, the population density of the predator species. Throughout this paper we impose conditions (i), (ii), (iii), and (iv) as described below on the functions f, p, g and r, respectively.

(i) $f(0) > 0,\ f'(0) \leq 0,\ f'(u) < 0$ for $u > 0,\ f(K) = 0$ for some $K > 0$,
(ii) $p(u) \geq 0$ for $u \geq 0$,
(iii) $g(0) > 0,\ g'(u) \leq 0$ for $u \geq 0,\ \lim_{u\to\infty} g(u) = g_\infty > 0$,
(iv) $r(0) = 0,\ r'(0) \leq 0,\ r'(u) > 0$ for $u > 0$.

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The function $f(u)$ represents the birth rate of the prey species, $p(u)$, the predator response to the prey density, $g(u)$, the death rate of the predator species, and $r(v)$, the competition within the predator species.

In the following examples of various types of predator-prey responses the constants a_i, b_i, $(i = 1, 2)$, c_1 are positive, and q, s, c_2 are nonnegative real numbers.

The classical Lotka-Volterra predator-prey model, as (1.2) below, uses $f(u) = a_1 - b_1 u$, $p(u) = c_1 u$, $g(u) = a_2$, $r(v) = c_2 v$ and $c = b_2/c_1$. This model is also called the Holling type I predator-prey model.

$$\begin{cases} u_t = u(a_1 - b_1 u - c_1 v) & \text{for } t \in (0, \infty), \\ v_t = v(-a_2 + b_2 u - c_2 v) & \text{for } t \in (0, \infty), \\ u(0) = u_0 \geq 0, \quad v(0) = v_0 \geq 0. \end{cases} \tag{1.2}$$

The Holling type II predator-prey reaction is represented by $f(u) = a_1 - b_1 u$, $p(u) = \frac{c_1 u}{1+qu}$, $g(u) = a_2$, and $c = b_2/c_1$. The general predator-prey system (1.1) is written as the following for the case of Holling type II reaction :

$$\begin{cases} u_t = u(a_1 - b_1 u - \frac{c_1 v}{1+qu}) & \text{for } t \in (0, \infty), \\ v_t = v(-a_2 + \frac{b_2 u}{1+qu}) & \text{for } t \in (0, \infty), \\ u(0) = u_0 \geq 0, \quad v(0) = v_0 \geq 0. \end{cases} \tag{1.3}$$

If $q = 0$, system (1.3) reduces to the predator-prey system (1.1) with Lotka-Volterra reaction.

The Holling type III predator-prey reaction is represented by $f(u) = a_1 - b_1 u$, $p(u) = \frac{c_1 u^2}{1+su+qu^2}$, $g(u) = a_2$, and $c = b_2/c_1$. The function $p(u) = \frac{c_1 u^2}{1+su+qu^2}$ is a sigmoidal response function.

$$\begin{cases} u_t = u(a_1 - b_1 u - \frac{c_1 uv}{1+su+qu^2}) & \text{for } t \in (0, \infty), \\ v_t = v(-a_2 + \frac{b_2 u^2}{1+su+qu^2}) & \text{for } t \in (0, \infty), \\ u(0) = u_0 \geq 0, \quad v(0) = v_0 \geq 0. \end{cases} \tag{1.4}$$

The Holling type IV predator-prey reaction uses $f(u) = a_1 - b_1 u$, $p(u) = \frac{c_1 u}{1+qu^2}$, $g(u) = a_2$, and $c = b_2/c_1$. The function $p(u) = \frac{c_1 u}{1+su+qu^2}$, which is proposed by Andrews [1], is called the Monod-Haldane response function.

$$\begin{cases} u_t = u(a_1 - b_1 u - \frac{c_1 v}{1+su+qu^2}) & \text{for } t \in (0, \infty), \\ v_t = v(-a_2 + \frac{b_2 u}{1+su+qu^2}) & \text{for } t \in (0, \infty), \\ u(0) = u_0 \geq 0, \quad v(0) = v_0 \geq 0. \end{cases} \tag{1.5}$$

By adopting Holling-type functional responses in predator-prey systems we may represent the phenomenon whereby predation is decreased, or even prevented altogether, due to the increased ability of the prey to better defend or disguise themselves when their numbers are large enough. This type of functional response had first been introduced by Haldane [6] in enzymology. Results on Holling-type predator-prey systems are found in [1], [2], [3], [5], [6], [7], [8], [10], [11], [12], [13], [14], [15], [17], [18]. More explanations for the response functions of Holling type appear in [4], [8], [9], [10], [15] and references therein.

The response functions $p(u)$ in the Holling type I, II, III satisfy the following monotonicity condition.

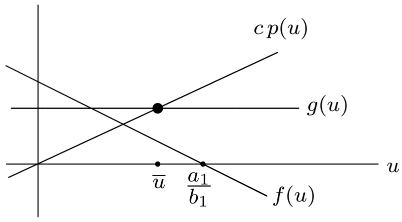
(v) $p(0) = 0,\ p'(0) \geq 0,\ p'(u) > 0$ for $u > 0$.

Condition (v) implies the assumption that the more prey, the higher consumption rate of the predator. In contrast the Holling type IV response function $p(u) = \frac{c_1 u}{1+qu^2}$ reflects some inhibitory effect by the high density of the prey. The graphs of the functions $c\,p(u)$, $f(u)$, $g(u)$ of Lotka-Volterra and Holling type are illustrated in Figures 1, 2, 3 and 4 below.

Traditionally in many predator-prey models, the predator response to prey density is assumed to be monotone increasing. However, there have come up experimental as well as observational evidences which indicate that this assumption may not be always true. We refer the readers to Rosenzweig [12] where he considered six different mathematical models of prey-predator or parasite-host interaction and showed that sufficient enrichment or increase of the prey carrying capacity can cause destabilization of an otherwise stable interior equilibrium.

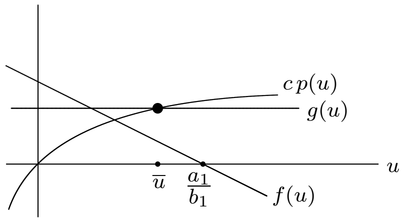
We first study in this paper the existences of non-negative constant steady-states of the general predator-prey system (1.1). Especially necessary conditions that guarantees the existence of the positive steady-state $(\overline{u}, \overline{v})$ are investigated for systems (1.3), (1.4) and (1.5) in Lemmas 1, 2 and 3, respectively. The necessary conditions are stated as some inequalities on the parameters a_i, b_i, c_i, $(i = 1, 2)$ and s, q. The linearized stabilities of the predator-prey system with monotone response functions and non-monotone response functions are studied by analyzing the eigenvalues of the community matrix A. Lemma 4 and 5 show the stability of the non-negative constant steady-states for the general predator-prey system (1.1). The stability analysis of the positive steady-state $(\overline{u}, \overline{v})$ is stated in Theorems 7 and 8 for Holling type III and IV, respectively. Example sets of the coefficients a_i, b_i, c_i, $(i = 1, 2)$ and s, q which guarantee the stability of the positive-steady state $(\overline{u}, \overline{v})$ are also given.

This paper consists of four sections : Section 1. Existences of non-negative constant steady-states. In Section 2 we investigate the linearized stabilities of the



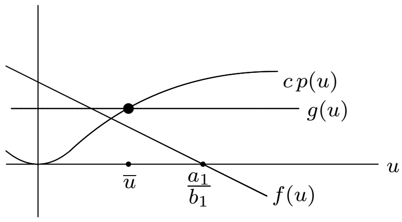
$g(\frac{a_1}{b_1}) < cp(\frac{a_1}{b_1})$, that is, $\frac{a_2}{b_2} < \frac{a_1}{b_1}$

Figure 1. The graphs of the functions $cp(u) = b_2 u$, $f(u) = a_1 - b_1 u$, $g(u) = a_2$ in system (1.1) with Lotka-Volterra reaction.



$g(\frac{a_1}{b_1}) < cp(\frac{a_1}{b_1})$

Figure 2. The graphs of the functions $cp(u) = \frac{b_2 u}{1+qu}$, $f(u) = a_1 - b_1 u$, $g(u) = a_2$ in system (1.1) with Holling type II reaction.



$g(\frac{a_1}{b_1}) < cp(\frac{a_1}{b_1})$

Figure 3. The graphs of the functions $cp(u) = \frac{b_2 u^2}{1+su+qu^2}$, $f(u) = a_1 - b_1 u$, $g(u) = a_2$ in system (1.1) with Holling type III reaction.

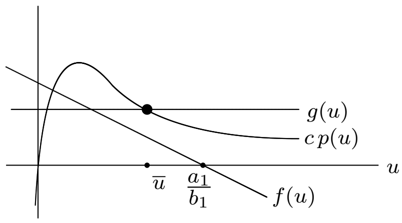


Figure 4. The graphs of the functions $cp(u) = \frac{b_2 u}{1+su+qu^2}$, $f(u) = a_1 - b_1 u$, $g(u) = a_2$ in system (1.1) with Holling type IV reaction.

general predator-prey system. In Section 3 we present the linearized stabilities of the predator-prey system with monotone response functions which are Holling type I, II and III. Section 4 has the results on the linearized stabilities of Holling type IV predator-prey system.

2. Existences of Non-negative Constant Steady-states

The general predator-prey system (1.1) possesses the non-negative steady states $(0, 0)$ and $(K, 0)$ in the (u, v)-phase plane where the constant $K > 0$ satisfies the condition $f(K) = 0$. In systems (1.3), (1.4), (1.5), $f(u) = a_b - b_1 u$, and thus $K = \frac{a_1}{b_1}$. The following condition is a necessary condition for the general predator-prey system (1.1) to have a positive steady-state $(\overline{u}, \overline{v})$:

$$g(\overline{u}) - c\,p(\overline{u}) = 0 \qquad \text{for some } \overline{u} \in (0, K), \tag{2.1}$$

where $\overline{v}$ is obtained solving the equation $\overline{u}f(\overline{u}) - \overline{v}p(\overline{u}) = 0$.

In this section we examine condition (2.1) in Lemmas 1, 2, 3 for systems (1.3), (1.4), (1.5), respectively, to investigate the existence of the positive steady-state $(\overline{u}, \overline{v})$.

Lemma 1. *Assume that* $\frac{a_2}{b_2} < \frac{a_1}{b_1}$ *and* $0 \le q < \frac{b_2}{a_2} - \frac{b_1}{a_1}$ *for system* (1.3)*, the predator-prey system with Holling type II reaction. Then system* (1.3) *has a unique positive steady-state* $(\overline{u}, \overline{v})$*, where*

$$\overline{u} = \frac{a_2}{a_2 q + b_2}, \qquad \overline{v} = \tfrac{1}{c_1}(1 + q\overline{u})(a_1 - b_1\overline{u}) = \frac{b_2(a_1 b_2 - a_2 b_1 - a_1 a_2 q)}{c_1(b_2 - a_2 q)^2}. \tag{2.2}$$

If $q = 0$*, which is the case of the Lotka-Volterra reaction, then* $(\overline{u}, \overline{v})$ *is given by*

$$(\overline{u}, \overline{v}) = \left(\frac{a_2}{b_2}, \ \frac{a_1 b_2 - a_2 b_1}{b_2 c_1}\right).$$

Proof. A steady-state $(\overline{u}, \overline{v})$ for the system (1.3) with Holling type II response is obtained by solving the equations

$$(a_1 - b_1 u)(1 + qu) - c_1 v = 0, \tag{2.3}$$

and

$$-a_2 + \frac{b_2 u}{1 + qu} = 0. \tag{2.4}$$

From (2.4) we have that

$$\overline{u} = \frac{a_2}{b_2 - a_2 q}. \tag{2.5}$$

Since a_2, $b_2 > 0$ and $q \geq 0$ in systems (1.3) with Holling type II response we notice that $\overline{u} > 0$ if and only if

$$0 \leq q < \frac{b_2}{a_2}.$$

And in order to have $\overline{v} > 0$ it must hold from equation (2.3) that $\overline{u} < \frac{a_1}{b_1}$, or equivalently

$$0 \leq q < \frac{b_2}{a_2} - \frac{b_1}{a_1},$$

which may hold provided that

$$\frac{b_1}{a_1} < \frac{b_2}{a_2}, \quad \text{that is,} \quad \frac{a_2}{b_2} < \frac{a_1}{b_1}.$$

Hence we conclude that system (1.3) has a unique positive steady-state $(\overline{u}, \overline{v})$ if the positive constants a_i, b_i $(i = 1, 2)$, and q satisfy that

$$\frac{a_2}{b_2} < \frac{a_1}{b_1} \quad \text{and} \quad 0 \leq q < \frac{b_2}{a_2} - \frac{b_1}{a_1}.$$

From (2.3) and (2.5), it is obtained that

$$\begin{aligned}
\overline{v} &= \tfrac{1}{c_1}(1 + q\overline{u})(a_1 - b_1\overline{u}) \\
&= \tfrac{1}{c_1}(1 + \tfrac{a_2 q}{b_2 - a_2 q})(a_1 - \tfrac{a_2 b_1}{b_2 - a_2 q}) \\
&= \tfrac{1}{c_1} \cdot \tfrac{b_2}{b_2 - a_2 q}(a_1 - \tfrac{a_2 b_1}{b_2 - a_2 q}) \\
&= \tfrac{b_2(a_1 b_2 - a_2 b_1 - a_1 a_2 q)}{c_1 (b_2 - a_2 q)^2}.
\end{aligned}$$

And in the case $q = 0$ for system (1.3), which reduces to system (1.1) with the Lotka-Volterra reaction, the positive steady-state $(\overline{u}, \overline{v})$ exists as

$$\overline{u} = \frac{a_2}{b_2} \quad \text{and} \quad \overline{v} = \frac{a_1 b_2 - a_2 b_1}{b_2 c_1},$$

if the positive constants a_i, b_i $(i = 1, 2)$ satisfy that

$$\frac{a_2}{b_2} < \frac{a_1}{b_1}.$$

□

Lemma 2. *Assume that* $q < \frac{b_2}{a_2}$, $\left(\frac{b_1}{a_1}\right)^2 < \left(\frac{b_2}{a_2} - q\right)$, *and* $s < \left(\frac{b_2}{a_2} - q\right)\left(\frac{a_1}{b_1}\right) - \left(\frac{b_1}{a_1}\right)$ *for system* (1.4)*, the predator-prey system with Holling type III reaction. Then system* (1.4) *has a unique positive steady-state* $(\overline{u}, \overline{v})$*, where*

$$\overline{u} = \frac{s + \sqrt{s^2 + 4(\frac{b_2}{a_2} - q)}}{2(\frac{b_2}{a_2} - q)}, \qquad \overline{v} = \frac{b_2 \overline{u}}{a_2 c_1}(a_1 - b_1 \overline{u}). \tag{2.6}$$

Proof. A steady-state $(\overline{u}, \overline{v})$ for the system (1.4) with Holling type III response is obtained by solving the equations

$$(a_1 - b_1 u)(1 + su + qu^2) - c_1 uv = 0, \tag{2.7}$$

and

$$-a_2 + \frac{b_2 u^2}{1 + su + qu^2} = 0,$$

which reduces to the equation

$$(a_2 q - b_2)u^2 + a_2 su + a_2 = 0. \tag{2.8}$$

Since a_2, $b_2 > 0$ and s, $q \geq 0$ in systems (1.4) with Holling type III response we notice that equation (2.8) possesses a single positive root $\overline{u}$ if and only if

$$a_2 q - b_2 < 0,$$

where

$$\overline{u} = \frac{s + \sqrt{s^2 + 4(\frac{b_2}{a_2} - q)}}{2(\frac{b_2}{a_2} - q)}$$

And in order to have $\overline{v} > 0$ it must hold from equation (2.7) that $\overline{u} < \frac{a_1}{b_1}$, or equivalently

$$(a_2 q - b_2)\left(\frac{a_1}{b_1}\right)^2 + a_2 s\left(\frac{a_1}{b_1}\right) + a_2 < 0. \tag{2.9}$$

By dividing both side of the inequality in (2.9) by a_2 we have that

$$\left(q - \frac{b_2}{a_2}\right)\left(\frac{a_1}{b_1}\right)^2 + s\left(\frac{a_1}{b_1}\right) + 1 < 0,$$

that is

$$0 < s\left(\frac{a_1}{b_1}\right) < \left(\frac{b_2}{a_2} - q\right)\left(\frac{a_1}{b_1}\right)^2 - 1,$$

which may hold provided that

$$\left(\frac{b_2}{a_2} - q\right)\left(\frac{a_1}{b_1}\right)^2 > 1.$$

Hence we conclude that system (1.4) has a unique positive steady-state $(\overline{u}, \overline{v})$ if the positive constants a_i, b_i $(i = 1, 2)$, s, and q satisfy that

$$q < \frac{b_2}{a_2}, \quad \left(\frac{b_1}{a_1}\right)^2 < \left(\frac{b_2}{a_2} - q\right), \quad \text{and} \quad s < \left(\frac{b_2}{a_2} - q\right)\left(\frac{a_1}{b_1}\right) - \left(\frac{b_1}{a_1}\right).$$

From (2.7) and (2.8), it is obtained that

$$\begin{aligned}\overline{v} &= \tfrac{1}{c_1\overline{u}}(1+s\overline{u}+q\overline{u}^2)(a_1-b_1\overline{u})\\ &= \tfrac{b_2\overline{u}}{a_2c_1}(a_1-b_1\overline{u}).\end{aligned}$$

□

Lemma 3. *Assume that* $s < \frac{b_2}{a_2}$, $(b_2-a_2s)^2-4a_2^2q>0$, $\frac{b_1}{a_1} < \frac{b_2}{a_2}-s$, *and* $q < \left(\frac{b_2}{a_2}-s\right)\frac{b_1}{a_1}-\left(\frac{b_1}{a_1}\right)^2$ *for system* (1.5)*, the predator-prey system with Holling type IV reaction. Then system* (1.5) *has a unique positive steady-state* $(\overline{u},\overline{v})$*, where*

$$\overline{u} = \frac{(b_2-a_2s)-\sqrt{(b_2-a_2s)^2-4a_2^2q}}{2a_2q}, \qquad \overline{v} = \frac{b_2\overline{u}}{a_2c_1}(a_1-b_1\overline{u}). \tag{2.10}$$

Assume that $s<\frac{b_2}{a_2}$, $(b_2-a_2s)^2-4a_2^2q=0$, *and* $\frac{b_2}{a_2} < s+\frac{2a_1}{b_1}q$ *for system* (1.5)*, the predator-prey system with Holling type IV reaction. Then system* (1.5) *has a unique positive steady-state* $(\overline{u},\overline{v})$*, where*

$$\overline{u} = \frac{(b_2-a_2s)}{2a_2q}, \qquad \overline{v} = \frac{b_2\overline{u}}{a_2c_1}(a_1-b_1\overline{u}). \tag{2.11}$$

Proof. A steady-state $(\overline{u},\overline{v})$ for the system (1.5) with Holling type IV response is obtained by solving the equations

$$(a_1-b_1u)(1+su+qu^2)-c_1v=0, \tag{2.12}$$

and

$$-a_2+\frac{b_2u}{1+su+qu^2}=0,$$

which reduces to the equation

$$a_2qu^2-(b_2-a_2s)u+a_2=0. \tag{2.13}$$

Since a_2, $b_2>0$ and s, $q\geq 0$ in systems (1.1) with Holling type IV response we notice that equation (2.13) possesses two distinct positive roots $\overline{u}_1$ and $\overline{u}_2$ if and only if

$$b_2-a_2s>0 \quad \text{and} \quad (b_2-a_2s)^2-4a_2^2q>0,$$

and a single positive root $\overline{u}$ if and only if

$$b_2-a_2s>0 \quad \text{and} \quad (b_2-a_2s)^2-4a_2^2q=0.$$

From (2.12) and (2.13), it is obtained that

$$\begin{aligned}\overline{v} &= \tfrac{1}{c_1}(1+s\overline{u}+q\overline{u}^2)(a_1-b_1\overline{u})\\ &= \tfrac{b_2\overline{u}}{a_2c_1}(a_1-b_1\overline{u}).\end{aligned}$$

In order to have $\overline{v} > 0$ it must hold from equation (2.12) that $\overline{u} < \frac{a_1}{b_1}$. In the case that $(b_2 - a_2 s)^2 - 4a_2^2 q > 0$ we have that $\overline{u} = \frac{(b_2 - a_2 s) - \sqrt{(b_2 - a_2 s)^2 - 4a_2^2 q}}{2a_2 q} < \frac{a_1}{b_1}$ if

$$a_2 q \left(\frac{a_1}{b_1}\right)^2 - (b_2 - a_2 s)\left(\frac{a_1}{b_1}\right) + a_2 < 0. \tag{2.14}$$

By dividing both side of the inequality in (2.14) by a_2 we have that

$$q\left(\frac{a_1}{b_1}\right)^2 - \left(\frac{b_2}{a_2} - s\right)\left(\frac{a_1}{b_1}\right) + 1 < 0,$$

that is

$$0 < q\left(\frac{a_1}{b_1}\right)^2 < \left(\frac{b_2}{a_2} - s\right)\left(\frac{a_1}{b_1}\right) - 1,$$

which may hold provided that

$$\left(\frac{b_2}{a_2} - s\right)\left(\frac{a_1}{b_1}\right) > 1, \quad \text{that is,} \quad \left(\frac{b_2}{a_2} - s\right) > \left(\frac{b_1}{a_1}\right).$$

In the case that $(b_2 - a_2 s)^2 - 4a_2^2 q = 0$ we have that $\overline{u} = \frac{(b_2 - a_2 s)}{2a_2 q} < \frac{a_1}{b_1}$ if

$$\frac{b_2}{a_2} < s + \frac{2a_1}{b_1} q. \tag{2.15}$$

Hence we conclude that system (1.5) has a unique positive steady-state $(\overline{u}, \overline{v}) = (\frac{(b_2 - a_2 s) - \sqrt{(b_2 - a_2 s)^2 - 4a_2^2 q}}{2a_2 q}, \frac{b_2 \overline{u}}{a_2 c_1}(a_1 - b_1 \overline{u}))$ if the positive constants a_i, b_i $(i = 1, 2)$, s, and q satisfy that

$$s < \frac{b_2}{a_2}, \quad (b_2 - a_2 s)^2 - 4a_2^2 q > 0, \quad \left(\frac{b_1}{a_1}\right) < \left(\frac{b_2}{a_2} - s\right),$$

and

$$q < \left(\frac{b_2}{a_2} - s\right)\left(\frac{b_1}{a_1}\right) - \left(\frac{b_1}{a_1}\right)^2,$$

and system (1.5) has a unique positive steady-state $(\overline{u}, \overline{v}) = (\frac{(b_2 - a_2 s)}{2a_2 q}, \frac{b_2 \overline{u}}{a_2 c_1}(a_1 - b_1 \overline{u}))$ if the positive constants a_i, b_i $(i = 1, 2)$, s, and q satisfy that

$$s < \frac{b_2}{a_2}, \quad (b_2 - a_2 s)^2 - 4a_2^2 q = 0, \quad \text{and} \quad \frac{b_2}{a_2} < s + \frac{2a_1}{b_1} q.$$

□

3. Linearized Stabilities of the General Predator-prey system

In this section we investigate the linearized stabilities of the non-negative constant steady-states $(0,0)$, $(K,0)$, and $(\overline{u},\overline{v})$ for the general predator-prey system (1.1). We assume the following conditions for the reaction functions in the general predator-prey system (1.1):

(i) $f(0)>0,\ f'(0)\le 0,\ f'(u)<0$ for $u>0,\ f(K)=0$ for some $K>0$,

(ii) $p(u)\ge 0$ for $u\ge 0$,

(iii) $g(0)>0,\ g'(u)\le 0$ for $u\ge 0,\ \lim_{u\to\infty} g(u)=g_\infty>0$,

(iv) $r(0)=0,\ r'(0)\le 0,\ r'(u)>0$ for $u>0$.

Let us denote that

$$F(u,v)=uf(u)-v\,p(u) \qquad \text{and} \qquad G(u,v)=v(-g(u)+c\,p(u)-r(v)).$$

Then system (1.1) is rewritten as

$$\begin{cases} u_t=F(u,v) & \text{for } t\in(0,\infty),\\ v_t=G(u,v) & \text{for } t\in(0,\infty),\\ u(0)=u_0\ge 0,\quad v(0)=v_0\ge 0, & \end{cases}$$

To linearize system (1.1) about the steady-state (η,ξ) we write

$$x(t)=u(t)-\eta,\quad y(t)=v(t)-\xi$$

which on substituting into system (1.1), linearizing with small $|x|$ and $|y|$ gives

$$\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = A_{(\eta,\xi)} \begin{pmatrix} x \\ y \end{pmatrix},$$

where

$$\begin{aligned} A_{(\eta,\xi)} &= \begin{pmatrix} \frac{dF}{du} & \frac{dF}{dv} \\ \frac{dG}{du} & \frac{dG}{dv} \end{pmatrix}_{(\eta,\xi)} \\ &= \begin{pmatrix} f(\eta)+\eta f'(\eta)-\xi\,p'(\eta) & -p(\eta) \\ \xi(-g'(\eta)+c\,p'(\eta)) & -g(\eta)+c\,p(\eta)-r(\xi)-\xi\,r'(\xi) \end{pmatrix}. \end{aligned}$$

When (η,ξ) is one of the nonnegative constant steady-states, the eigenvalues of the the characteristic equation $\lambda^2-(\mathrm{tr}\,A)\lambda+\det A=0$ determine the linearized stability of the steady-states (η,ξ). If $\det A<0$ at a steady-state (η,ξ), then (η,ξ) is a saddle point. When $\det A > 0$ at a steady-state (η, ξ), then (η, ξ) is linearly asymptotically stable if $\operatorname{tr} A < 0$, and unstable if $\operatorname{tr} A > 0$.

Lemma 4 below shows the stability properties of the nonnegative constants steady-state $(0, 0)$ by simple observations.

Lemma 4. *The steady-state* $(0, 0)$ *is a saddle point of the general predator-prey system* (1.1).

Proof. The community matrix $A_{(0,0)}$ of the linearization about the steady-state $(0, 0)$ is

$$A_{(0,0)} = \begin{pmatrix} f(0) & -p(0) \\ 0 & -g(0) + c\,p(0) - r(0) \end{pmatrix} = \begin{pmatrix} f(0) & 0 \\ 0 & -g(0) \end{pmatrix}.$$

By observing that $\det A = -f(0) \cdot g(0) < 0$, we see that

$$\begin{aligned} \det A &< 0, \\ (\operatorname{tr} A)^2 - 4 \det A &> 0. \end{aligned}$$

Thus $A_{(0,0)}$ has two real eigenvalues with opposite signs, and so the steady-state $(0, 0)$ is a saddle point. □

4. Linearized Stabilities of the Predator-prey System with Monotone Response Functions

For the predator-prey system (1.1) with monotone response functions the linearized stabilities of the non-negative constant steady-states $(K, 0)$ and $(\overline{u}, \overline{v})$ are investigated in Lemma 5 and Theorem 7 in this section. We assume conditions (i) to (iv) of Section 3 for the reaction functions in the general predator-prey system (1.1) as well as the following condition on the monotonicity of the function $p(u)$:

(v) $p(0) = 0$, $p'(0) \geq 0$, $p'(u) > 0$ for $u > 0$.

Lemma 5. *Assume condition* (2.1) *on the general predator-prey system* (1.1). *Then the steady-state* $(K, 0)$ *is a saddle point of system* (1.1).

Proof. The community matrix $A_{(K,0)}$ of the linearization about the steady-state $(K, 0)$ is

$$A = A_{(K,0)} = \begin{pmatrix} f(K) + K f'(K) & -p(K) \\ 0 & -g(K) + c\,p(K) - r(0) \end{pmatrix}$$

$$= \begin{pmatrix} Kf'(K) & -p(K) \\ 0 & -g(K) + c\,p(K) \end{pmatrix}.$$

By observing

$$\begin{aligned} \operatorname{tr} A &= Kf'(K) - g(K) + c\,p(K), \\ \det A &= Kf'(K)\,[-g(K) + c\,p(K)]\,, \end{aligned}$$

we see that

(4.1)
$$\begin{aligned} (\operatorname{tr} A)^2 - 4\det A &= [Kf'(K) - g(K) + c\,p(K)]^2 - 4\,Kf'(K)\,[-g(K) + c\,p(K)] \\ &= K^2(f'(K))^2 + (g(K))^2 + c^2\,(p(K))^2 \\ &\quad -2Kf'(K)g(K) - 2c\,g(K)p(K) + 2cKf'(K)p(K) \\ &\quad +4\,Kf'(K)g(K) - 4cKf'(K)p(K) \\ &= K^2(f'(K))^2 + (g(K))^2 + c^2\,(p(K))^2 \\ &\quad +2Kf'(K)g(K) - 2c\,g(K)p(K) - 2cKf'(K)p(K) \\ &= [Kf'(K) + g(K) - c\,p(K)]^2\,. \end{aligned}$$

From the assumptions (i) to (v) and condition (2.1) on the general predator-prey system (1.1) we have the following inequalities.

$p'(u) > 0$ for $u > 0$, $g'(u) \le 0$ for $u \ge 0$, and $g(\overline{u}) - c\,p(\overline{u}) = 0$ for some $\overline{u} \in (0, K)$.

Thus it is reduced that $g(K) - c\,p(K) < 0$. Since $f'(u) < 0$ for for $u > 0$, we conclude that

$$\begin{aligned} \det A &< 0, \\ (\operatorname{tr} A)^2 - 4\det A &> 0. \end{aligned}$$

Therefore $A = A_{(K,0)}$ has two distinct real eigenvalues with opposite signs, and so the steady-state $(K, 0)$ is a saddle point. □

Now we are interested in the linearized stability of the positive steady-state $(\overline{u}, \overline{v})$ for systems (1.3), (1.4), and (1.5). The following result is from [16] on the stability and Hopf bifurcation property of system (1.3), the predator-prey system with Holling type II reaction.

Theorem 6 ([16, Theorem 2]). *Assume that* $a_2 < 0$ *and* $0 \le q < \frac{b_2}{a_2} - \frac{b_1}{a_1}$ *for system* (1.3). *Also suppose that*

$$\left(\frac{b_2}{a_2} - \frac{b_1}{a_1}\right)^2 - 4\frac{b_1 b_2}{a_1 a_2} > 0. \tag{4.2}$$

Then for system (1.3) *Hopf bifurcation near* $(\overline{u}, \overline{v})$ *occurs at the parameter values* $q = q^*_{\pm}$*, where*

$$q^*_{\pm} = \frac{1}{2}\left(\left(\frac{b_2}{a_2} - \frac{b_1}{a_1}\right) \pm \sqrt{\left(\frac{b_2}{a_2} - \frac{b_1}{a_1}\right)^2 - 4\frac{b_1 b_2}{a_1 a_2}}\right).$$

That is, $(\overline{u}, \overline{v})$ *is asymptotically stable for system* (1.3) *if* $0 < q < q^*_-$ *or* $q > q^*_+$*, and unstable if* $q^*_- < q < q^*_+$*.*

In the following theorem we investigate the linearized stability of the positive constant steady-state $(\overline{u}, \overline{v})$ for system (1.4), the predator-prey system with Holling type III reaction.

Theorem 7. *Assume that* $q < \frac{b_2}{a_2}$, $(\frac{b_1}{a_1})^2 < \frac{b_2}{a_2} - q$*, and* $s < (\frac{b_2}{a_2} - q)\frac{a_1}{b_1} - \frac{b_1}{a_1}$*. Also assume that*

$$\frac{a_1}{b_1} < \frac{q}{s}.$$

Then for system (1.4) *the positive constant steady-state* $(\overline{u}, \overline{v})$ *is asymptotically stable.*

Proof. For system (1.4) we have

$$\begin{aligned} F(u,v) &= u(a_1 - b_1 u - \tfrac{c_1 uv}{1+su+qu^2}), \\ G(u,v) &= v(-a_2 + \tfrac{b_2 u^2}{1+su+qu^2}), \end{aligned}$$

and thus

$$A = \begin{pmatrix} \frac{dF}{du} & \frac{dF}{dv} \\ \frac{dG}{du} & \frac{dG}{dv} \end{pmatrix} = \begin{pmatrix} a_1 - 2b_1 u - \frac{c_1 uv(2+su)}{(1+su+qu^2)^2} & -\frac{c_1 u^2}{1+su+qu^2} \\ \frac{b_2 uv(2+su)}{(1+su+qu^2)^2} & -a_2 + \frac{b_2 u^2}{1+su+qu^2} \end{pmatrix}.$$

Using the equations $a_1 - b_1\overline{u} - \frac{c_1\overline{u}\overline{v}}{1+s\overline{u}+q\overline{u}^2} = 0$ and $-a_2 + \frac{b_2\overline{u}^2}{1+s\overline{u}+q\overline{u}^2} = 0$ we may simplify the components of the community matrix A at $(\overline{u}, \overline{v})$ as

$$\begin{aligned} A = A_{(\overline{u},\overline{v})} &= \begin{pmatrix} a_1 - 2b_1\overline{u} - \frac{c_1\overline{u}\overline{v}(2+s\overline{u})}{(1+s\overline{u}+q\overline{u}^2)^2} & -\frac{c_1\overline{u}^2}{1+s\overline{u}+q\overline{u}^2} \\ \frac{b_2\overline{u}\overline{v}(2+s\overline{u})}{(1+s\overline{u}+q\overline{u}^2)^2} & -a_2 + \frac{b_2\overline{u}^2}{1+s\overline{u}+q\overline{u}^2} \end{pmatrix} \\ &= \begin{pmatrix} -b_1\overline{u} + \frac{c_1\overline{u}\overline{v}}{1+s\overline{u}+q\overline{u}^2} - \frac{c_1\overline{u}\overline{v}(2+s\overline{u})}{(1+s\overline{u}+q\overline{u}^2)^2} & -\frac{c_1\overline{u}^2}{1+s\overline{u}+q\overline{u}^2} \\ \frac{b_2\overline{u}\overline{v}(2+s\overline{u})}{(1+s\overline{u}+q\overline{u}^2)^2} & 0 \end{pmatrix} \end{aligned}$$

$$
\begin{aligned}
&= \begin{pmatrix} -b_1\overline{u} + \frac{c_1\overline{u}\overline{v}(q\overline{u}^2-1)}{(1+s\overline{u}+q\overline{u}^2)^2} & -\frac{c_1\overline{u}^2}{1+s\overline{u}+q\overline{u}^2} \\ \frac{b_2\overline{u}\overline{v}(2+s\overline{u})}{(1+s\overline{u}+q\overline{u}^2)^2} & 0 \end{pmatrix} \\
&= \begin{pmatrix} -b_1\overline{u} + \frac{(q\overline{u}^2-1)(a_1-b_1\overline{u})}{1+s\overline{u}+q\overline{u}^2} & -\frac{c_1\overline{u}^2}{1+s\overline{u}+q\overline{u}^2} \\ \frac{b_2\overline{u}\overline{v}(2+s\overline{u})}{(1+s\overline{u}+q\overline{u}^2)^2} & 0 \end{pmatrix}.
\end{aligned}
$$

Hence for the corresponding characteristic equation $\lambda^2 - (\operatorname{tr} A)\lambda + \det A = 0$ we find that

$$
\begin{aligned}
\det A &= \tfrac{b_2c_1\overline{u}^3\overline{v}(2+s\overline{u})}{(1+s\overline{u}+q\overline{u}^2)^3} > 0 \\
\operatorname{tr} A &= -b_1\overline{u} + \tfrac{(q\overline{u}^2-1)(a_1-b_1\overline{u})}{1+s\overline{u}+q\overline{u}^2}
\end{aligned}
$$

Hence $(\overline{u}, \overline{v})$ is linearly asymptotically stable if $\operatorname{tr} A < 0$, and unstable if $\operatorname{tr} A > 0$. Now, through simple computations we note that

$$
\begin{aligned}
\operatorname{tr} A &= -b_1\overline{u} + \tfrac{(q\overline{u}^2-1)(a_1-b_1\overline{u})}{1+s\overline{u}+q\overline{u}^2} \\
&= \tfrac{-b_1\overline{u}(1+s\overline{u}+q\overline{u}^2)+(q\overline{u}^2-1)(a_1-b_1\overline{u})}{1+s\overline{u}+q\overline{u}^2} \\
&= \tfrac{-2b_1q\overline{u}^3-(b_1s-a_1q)\overline{u}^2-a_1}{1+s\overline{u}+q\overline{u}^2} \\
&< 0
\end{aligned}
\tag{4.3}
$$

from the condition $\frac{a_1}{b_1} < \frac{q}{s}$, and so we conclude that $(\overline{u}, \overline{v})$ is linearly asymptotically stable. □

Note. As an example set of the coefficients for system (1.4), the predator-prey system with Holling type III reaction, let $a_1 = 1$, $a_2 = 1$, $b_1 = 1$, $s = 1$, $q = 2$, b_2 be any value that $b_2 > 4$, and c_1 be any positive real number for system (1.4).

Then we have that $\frac{b_2}{a_2} - q = b_2 - 2 > 2$. Hence

$$
\overline{u} = \frac{s + \sqrt{s^2 + 4(\frac{b_2}{a_2} - q)}}{2(\frac{b_2}{a_2} - q)} = \tfrac{1+\sqrt{1+4(b_2-2)}}{2(b_2-2)} \qquad \text{and thus } \ 0 < \overline{u} < 1 \ \text{ for } \ b_2 > 4.
$$

Therefore

$$
\overline{v} = \frac{b_2\overline{u}}{a_2c_1}(a_1 - b_1\overline{u}) = \frac{b_2\overline{u}}{c_1}(1 - \overline{u}) > 0 \quad \text{for } \ b_2 > 4.
$$

The following inequalities are also satisfied ;

$$
\tfrac{b_2}{a_2} - q - (\tfrac{b_1}{a_1})^2 = b_2 - 3 > 0, \ (\tfrac{b_2}{a_2} - q)\tfrac{a_1}{b_1} - \tfrac{b_1}{a_1} - s = b_2 - 4 > 0, \ \tfrac{q}{s} - \tfrac{a_1}{b_1} = 2 - 1 > 0.
$$

Thus by Theorem 7 the positive constant steady-state $(\overline{u}, \overline{v})$ with is asymptotically stable for system (1.4) when $a_1 = 1$, $a_2 = 1$, $b_1 = 1$, $s = 1$, $q = 2$, b_2 is any value that $b_2 > 4$, and c_1 is any positive real number.

Note. As another example set of the coefficients for system (1.4), the predator-prey system with Holling type III reaction, let $a_1 = 1$, $a_2 = 1$, $b_1 = 1$, $b_2 = 5$, q and s be any values that $0 < q < 4$, $0 < s < \min\{q, 4-q\}$, and c_1 be any positive real number for system (1.4).

Then we have that $\frac{b_2}{a_2} - q = 5 - q > 0$. Hence

$$\overline{u} = \frac{s + \sqrt{s^2 + 4(\frac{b_2}{a_2} - q)}}{2(\frac{b_2}{a_2} - q)} = \frac{s+\sqrt{s^2+4(5-q)}}{2(5-q)} > 0$$

For $0 < q < 4$ and $0 < s < \min\{q, 4-q\}$, the following inequalities are satisfied ;

$$\tfrac{b_2}{a_2} - q - (\tfrac{b_1}{a_1})^2 = 4 - q > 0, \quad (\tfrac{b_2}{a_2} - q)\tfrac{a_1}{b_1} - \tfrac{b_1}{a_1} - s = 4 - q - s > 0,$$

$$\frac{q}{s} - \frac{a_1}{b_1} = \frac{q}{s} - 1 > 0,$$

and it also holds that $0 < \overline{u} < 1$. Thus

$$\overline{v} = \frac{b_2\overline{u}}{a_2c_1}(a_1 - b_1\overline{u}) = \frac{5\overline{u}}{c_1}(1 - \overline{u}) > 0.$$

Therefore by Theorem 7 the positive constant steady-state $(\overline{u}, \overline{v})$ with is asymptotically stable for system (1.4) when $a_1 = 1$, $a_2 = 1$, $b_1 = 1$, $b_2 = 5$, q and s be any values that $0 < q < 4$, $0 < s < \min\{q, 4-q\}$, and c_1 is any positive real number.

5. Linearized Stabilities of Holling type IV Predator-prey System

In the following theorem we investigate the linearized stability of the positive constant steady-state $(\overline{u}, \overline{v})$ for system (1.5), the predator-prey system with Holling type IV reaction.

Theorem 8. *Assume that* $s < \frac{b_2}{a_2}$, $(b_2 - a_2s)^2 - 4a_2^2q > 0$, $\frac{b_1}{a_1} < \frac{b_2}{a_2} - s$, *and* $q < (\frac{b_2}{a_2} - s)\frac{b_1}{a_1} - (\frac{b_1}{a_1})^2$. *Then the positive constant steady-state* $(\overline{u}, \overline{v})$ *with*

$$\overline{u} = \frac{(b_2 - a_2s) - \sqrt{(b_2 - a_2s)^2 - 4a_2^2q}}{2a_2q}, \qquad \overline{v} = \frac{b_2\overline{u}}{a_2c_1}(a_1 - b_1\overline{u})$$

is asymptotically stable for system (1.5) *provided that*

$$\overline{u}^2 < \frac{1}{q}, \quad \text{and} \quad \frac{q}{s} < \frac{a_1}{b_1} < \frac{1}{s}.$$

Proof. For system (1.5) we have

$$\begin{aligned} F(u,v) &= u(a_1 - b_1 u - \tfrac{c_1 v}{1+su+qu^2}), \\ G(u,v) &= v(-a_2 + \tfrac{b_2 u}{1+su+qu^2}), \end{aligned}$$

and thus

$$A = \begin{pmatrix} \frac{dF}{du} & \frac{dF}{dv} \\ \frac{dG}{du} & \frac{dG}{dv} \end{pmatrix} = \begin{pmatrix} a_1 - 2b_1 u - \frac{c_1 v(1-qu^2)}{(1+su+qu^2)^2} & -\frac{c_1 u}{1+su+qu^2} \\ \frac{b_2 v(1-qu^2)}{(1+su+qu^2)^2} & -a_2 + \frac{b_2 u}{1+su+qu^2} \end{pmatrix}.$$

Using the equations $a_1 - b_1\overline{u} - \frac{c_1\overline{v}}{1+s\overline{u}+q\overline{u}^2} = 0$ and $-a_2 + \frac{b_2\overline{u}}{1+s\overline{u}+q\overline{u}^2} = 0$ we may simplify the components of the community matrix A at $(\overline{u},\overline{v})$ as

$$\begin{aligned} A = A_{(\overline{u},\overline{v})} &= \begin{pmatrix} a_1 - 2b_1\overline{u} - \frac{c_1\overline{v}(1-q\overline{u}^2)}{(1+s\overline{u}+q\overline{u}^2)^2} & -\frac{c_1\overline{u}}{1+s\overline{u}+q\overline{u}^2} \\ \frac{b_2\overline{v}(1-q\overline{u}^2)}{(1+s\overline{u}+q\overline{u}^2)^2} & -a_2 + \frac{b_2\overline{u}}{1+s\overline{u}+q\overline{u}^2} \end{pmatrix} \\ &= \begin{pmatrix} -b_1\overline{u} + \frac{c_1\overline{v}}{1+s\overline{u}+q\overline{u}^2} - \frac{c_1\overline{v}(1-q\overline{u}^2)}{(1+s\overline{u}+q\overline{u}^2)^2} & -\frac{c_1\overline{u}}{1+s\overline{u}+q\overline{u}^2} \\ \frac{b_2\overline{v}(1-q\overline{u}^2)}{(1+s\overline{u}+q\overline{u}^2)^2} & 0 \end{pmatrix} \\ &= \begin{pmatrix} -b_1\overline{u} + \frac{c_1\overline{u}\overline{v}(s+2q\overline{u})}{(1+s\overline{u}+q\overline{u}^2)^2} & -\frac{c_1\overline{u}}{1+s\overline{u}+q\overline{u}^2} \\ \frac{b_2\overline{v}(1-q\overline{u}^2)}{(1+s\overline{u}+q\overline{u}^2)^2} & 0 \end{pmatrix} \\ &= \begin{pmatrix} -b_1\overline{u} + \frac{\overline{u}(s+2q\overline{u})(a_1-b_1\overline{u})}{1+s\overline{u}+q\overline{u}^2} & -\frac{c_1\overline{u}}{1+s\overline{u}+q\overline{u}^2} \\ \frac{b_2\overline{v}(1-q\overline{u}^2)}{(1+s\overline{u}+q\overline{u}^2)^2} & 0 \end{pmatrix}. \end{aligned}$$

Hence for the corresponding characteristic equation $\lambda^2 - (\mathrm{tr}\,A)\lambda + \det A = 0$ we find that

$$\begin{aligned} \det A &= \frac{b_2 c_1\overline{u}\overline{v}(1-q\overline{u}^2)}{(1+s\overline{u}+q\overline{u}^2)^3} \\ \mathrm{tr}\,A &= -b_1\overline{u} + \frac{\overline{u}(s+2q\overline{u})(a_1-b_1\overline{u})}{1+s\overline{u}+q\overline{u}^2} \end{aligned}$$

From the condition $\overline{u}^2 < \frac{1}{q}$ it holds that $\det A > 0$. Through simple computations we also note that

$$\begin{aligned} \mathrm{tr}\,A &= -b_1\overline{u} + \frac{\overline{u}(s+2q\overline{u})(a_1-b_1\overline{u})}{1+s\overline{u}+q\overline{u}^2} \\ &= \frac{-b_1\overline{u}(1+s\overline{u}+q\overline{u}^2)+\overline{u}(s+2q\overline{u})(a_1-b_1\overline{u})}{1+s\overline{u}+q\overline{u}^2} \\ &= \frac{-3b_1q\overline{u}^2-2(a_1q-b_1s)\overline{u}-(b_1-a_1s)}{1+s\overline{u}+q\overline{u}^2} \\ &< 0 \end{aligned} \tag{5.1}$$

from the condition $\frac{q}{s} < \frac{a_1}{b_1} < \frac{1}{s}$.

Hence for system (1.5) we have that $\det A > 0$ and $\mathrm{tr}\,A < 0$ at $(\overline{u},\overline{v})$. Thus we conclude that $(\overline{u},\overline{v})$ is linearly asymptotically stable for system (1.5). □

Note. As an example set of the coefficients for system (1.5), the predator-prey system with Holling type IV reaction, let $a_1 = 2$, $a_2 = 1$, $b_1 = 1$, $s = 1/3$, $q = 1/2$, b_2 be any value that $b_2 > \frac{11}{6}$, and c_1 be any positive real number for system (1.5). Note that $\frac{11}{6} > \frac{1}{3} + \sqrt{2}$. Then we have that

$$(b_2 - a_2 s)^2 - 4a_2^2 q = (b_2 - \tfrac{1}{3})^2 - 2 > 0.$$

Thus

$$\overline{u} = \frac{(b_2 - a_2 s) - \sqrt{(b_2 - a_2 s)^2 - 4a_2^2 q}}{2a_2 q} = b_2 - \tfrac{1}{3} - \sqrt{(b_2 - \tfrac{1}{3})^2 - 2} \; > 0.$$

Since $\overline{u} = 2$ if $b_2 = \frac{1}{3} + \sqrt{2}$, and $\frac{d\overline{u}}{db_2} = 1 - \frac{b_2 - \frac{1}{3}}{\sqrt{(b_2 - \frac{1}{3})^2 - 2}} < 0$ for $b_2 > \frac{1}{3} + \sqrt{2}$, it holds that

$$0 < \overline{u} < 2 \quad \text{for} \quad b_2 > \tfrac{1}{3} + \sqrt{2}.$$

Hence

$$\overline{v} = \frac{b_2 \overline{u}}{a_2 c_1}(a_1 - b_1 \overline{u}) = \frac{b_2 \overline{u}}{c_1}(2 - \overline{u}) > 0 \quad \text{for} \quad b_2 > \tfrac{1}{3} + \sqrt{2}.$$

The following inequalities are also satisfied ;

$$\tfrac{b_2}{a_2} - s = b_2 - \tfrac{1}{3} > 0, \qquad \tfrac{b_2}{a_2} - s - \tfrac{b_1}{a_1} = b_2 - \tfrac{5}{6} > 0,$$
$$(\tfrac{b_2}{a_2} - s)\tfrac{b_1}{a_1} - (\tfrac{b_1}{a_1})^2 - q = \tfrac{1}{2}b_2 - \tfrac{11}{12} > 0, \qquad \tfrac{1}{q} - \overline{u}^2 = 2 - \overline{u}^2 > 0,$$
$$\tfrac{a_1}{b_1} - \tfrac{q}{s} = 2 - \tfrac{3}{2} > 0, \qquad \tfrac{1}{s} - \tfrac{a_1}{b_1} = 3 - 2 = 1 > 0.$$

Thus by Theorem 8 the positive constant steady-state $(\overline{u}, \overline{v})$ with is asymptotically stable for system (1.5) when $a_1 = 2$, $a_2 = 1$, $b_1 = 1$, $s = 1/3$, $q = 1/2$, b_2 is any value that $b_2 > \frac{11}{6}$, and c_1 is any positive real number.

Note. As another example set of the coefficients for system (1.5), the predator-prey system with Holling type IV reaction, let $a_1 = 2$, $a_2 = 1$, $b_1 = 1$, $b_2 = 2$, q and s be any values that $0 < s < \frac{1}{2}$, $0 < q < \min\{2s, \; \frac{3}{4} - \frac{s}{2}\}$, and c_1 be any positive real number for system (1.5).

Then we have that

$$\tfrac{b_2}{a_2} - s = 2 - s > 0, \qquad (b_2 - a_2 s)^2 - 4a_2^2 q = (2 - s)^2 - 4q > 0.$$

Thus

$$\overline{u} = \frac{(b_2 - a_2 s) - \sqrt{(b_2 - a_2 s)^2 - 4a_2^2 q}}{2a_2 q} = \frac{(2 - s) - \sqrt{(2 - s)^2 - 4q}}{2q} \; > 0.$$

For $0 < s < \frac{1}{2}$ and $0 < q < \min\{2s, \; \frac{3}{4} - \frac{s}{2}\}$, the following inequalities are satisfied ;

$$\tfrac{b_2}{a_2} - s - \tfrac{b_1}{a_1} = \tfrac{3}{2} - s > 0, \qquad (\tfrac{b_2}{a_2} - s)\tfrac{b_1}{a_1} - (\tfrac{b_1}{a_1})^2 - q = \tfrac{3}{4} - \tfrac{s}{2} - q > 0.$$

Hence we have that $0 < \overline{u} < 2 = \frac{a_1}{b_1}$, and thus $\overline{v} = \frac{b_2\overline{u}}{a_2c_1}(a_1 - b_1\overline{u}) = \frac{2\overline{u}}{c_1}(2 - \overline{u}) > 0$. And it also holds that

$$0 < \overline{u} < \frac{1}{\sqrt{q}}, \qquad \frac{a_1}{b_1} - \frac{q}{s} = 2 - \frac{q}{s} > 0, \qquad \frac{1}{s} - \frac{a_1}{b_1} = \frac{1}{s} - 2 > 0.$$

Therefore by Theorem 8 the positive constant steady-state $(\overline{u}, \overline{v})$ with is asymptotically stable for system (1.5) when $a_1 = 2$, $a_2 = 1$, $b_1 = 1$, $b_2 = 2$, q and s be any values that $0 < s < \frac{1}{2}$, $0 < q < \min\{2s,\ \frac{3}{4} - \frac{s}{2}\}$, and c_1 is any positive real number.

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Department of Mathematics, Sungshin Women's University, Seoul, 136-742, Korea
Email address: shims@sungshin.ac.kr