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SOLVABILITY FOR SOME DIRICHLET PROBLEM WITH P-LAPACIAN

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Abstract. We investigate the existence of the following Dirichlet boundary value problem

$$(|u'|^{p-2}u')' + (p-1)[\alpha|u^+|^{p-2}u^+ - \beta|u^-|^{p-2}u^-] = (p-1)h(t),$$
$$u(0) = u(T) = 0,$$

where $p > 1$, $\alpha > 0$, $\beta > 0$ and $\alpha^{-\frac{1}{p}} + \beta^{-\frac{1}{p}} = 2$, $T = \pi_p/\alpha^{\frac{1}{p}}$, $\pi_p = \frac{2\pi}{p\sin(\pi/p)}$ and $h \in L^\infty(0, T)$. The results of this paper generalize some early results obtained in [8] and [9]. Moreover, the method used in this paper is elementary and new.

1. Introduction

Consider the solvability of the following Dirichlet boundary value problem

$$(\phi_p(u'))' + (p-1)[\alpha\phi_p(u^+) - \beta\phi_p(u^-)] = (p-1)h(t), \quad t \in (0, T) \tag{1}$$

$$u(0) = u(T) = 0, \tag{2}$$

where $p > 1$, $\phi_p(u) = |u|^{p-2}u$, $u^\pm = \max\{\pm u, 0\}$, $h \in L^\infty(0,T)$ and $\alpha > 0$, $\beta > 0$ with $\alpha^{-\frac{1}{p}} + \beta^{-\frac{1}{p}} = 2$, $T = \pi_p/\alpha^{\frac{1}{p}}$ and $\pi_p = \frac{2\pi}{p\sin(\pi/p)}$.

By a solution of problem (1)-(2) we mean a real-valued function $u \in C^1[0,T]$ satisfying (1) and (2) such that $\phi_p(u')$ is absolutely continuous and (1) holds almost everywhere in $(0,T)$. Note that if $p = 2$ and $\alpha = \beta = 1$, then $T = \pi_p = \pi$ and (1)-(2) reduces to the linear problem

$$u'' + u = h(t), \quad u(0) = u(\pi) = 0 \ .$$

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The solvability of this problem is fully described, for example, by the classical linear Fredholm alternative, that is, this problem is solvable if and only if h satisfies

$$\int_0^{\pi} h(t)\sin t dt = 0.$$

In this case, the solution set is a continuum constituted by a one dimensional linear manifold. But for $p \neq 2$, the situation is quite different. Del Pino et al [8] proved that for $p \neq 2$, the condition

$$\int_0^{\pi_p} h(t)\sin_p t dt = 0, \tag{3}$$

where $u = \sin_p t$ is the unique solution of the following initial value problem

$$(\phi_p(u'))' + (p-1)\phi_p(u) = 0, \quad u(0) = 0,\, u'(0) = 1,$$

is sufficient for the solvability of the following boundary value problem

$$(\phi_p(u'))' + (p-1)\phi_p(u) = (p-1)h(t), \quad u(0) = u(\pi_p) = 0, \tag{4}$$

provided that $h \in C^1[0,\pi_p]$ and $h \not\equiv 0$. They also showed that for $p \neq 2$, the solution set of the problem (4) is bounded on $C^1[0,\pi_p]$ if (3) holds. Later, Drabek et al [9] generalized the results of [8] and replaced the condition $h \in C^1[0,\pi_p]$ by a weaker one $h \in L^\infty(0,\,\pi_p)$. For more results on this topic, see, for example, [1–7,10,11] and the references therein.

In this paper, the above existence result is generalized to (1)-(2) and the method used in this paper is elementary and different from those used in [8] and [9]. Moreover, we will give a sufficient condition for the existence of solutions for the following more general class of nonhomogeneous nonlinear equations:

$$\phi_p(u'))' + \tfrac{(p-1)q}{p}[\alpha\phi_q(u^+) - \beta\phi_q(u^-)] = (p-1)h(t), \quad u(0) = u(T) = 0,$$

where $q \geq p > 1$, $h \in L^\infty(0,\,T)$, $\alpha^{-\frac{1}{p}} + \beta^{-\frac{1}{p}} = 2 \quad \text{and} \quad T = \pi_p/\alpha^{\frac{1}{p}}$.

2. LEMMAS

If $h \in L^\infty(0,T)$, then in a similar way as in the proof of [8], one can show that a globally defined solution of (1) satisfying the initial condition

$$u(0) = 0, \quad u'(0) = \alpha \tag{5}$$

exists for any $\alpha \in \mathbb{R}$. Therefore throughout this paper we assume the existence of a globally defined solution of (1) with the initial condition (5).

Let $u = \sin_p t$ be the unique solution of the following initial value problem:

$$\phi_p(u'))' + (p-1)\phi_p(u) = 0, \quad u(0) = 0,\, u'(0) = 1.$$

Then by [2] and [8], for $t \in [0, \pi_p/2]$, it can be described implicitly by the formula

$$t = \int_0^{\sin_p t} \frac{ds}{(1-s^p)^{\frac{1}{p}}},$$

and $\sin_p t = \sin_p(\pi_p - t)$ for $t \in [\frac{\pi_p}{2}, \pi_p]$, $\sin_p t = -\sin_p(2\pi_p - t)$ for $t \in [\pi_p, 2\pi_p]$ and $\sin_p(2k\pi_p + t) = \sin_p t\, \forall k \in Z$, $t \in [0, 2\pi_p]$, i.e., $\sin_p t \in C^2$ is $2\pi_p$-periodic. Moreover, by defining $\cos_p t = \sin_p' t$, it follows from the above formula that $\sin_p^p t + \cos_p^p t = 1$ for $t \in [0, \pi_p/2]$.

Let $S(t)$ be the solution of the following homogeneous initial value problem

$$\phi_p(u'))' + (p-1)[\alpha\phi_p(u^+) - \beta\phi_p(u^-)] = 0, \quad u(0) = 0,\, u'(0) = 1. \tag{6}$$

Then it is well-known that $S(t)$ is $2\pi_p$-periodic and can be expressed explicitly as

$$S(t) = \begin{cases} \alpha^{-\frac{1}{p}} \sin_p \alpha^{\frac{1}{p}} t, & t \in [0,\ T]; \\ -\beta^{-\frac{1}{p}} \sin_p \beta^{\frac{1}{p}} (t - T), & t \in [T,\ 2\pi_p]. \end{cases}$$

Moreover, by using (6), it is also easy to verify that $S(t)$ satisfies the following identity:

$$|S'(t)|^p + \alpha(S^+(t))^p + \beta(S^-(t))^p \equiv 1, \quad t \in \mathbb{R}. \tag{7}$$

Under the generalized polar coordinates transformation

$$T: \quad u = \rho^{\frac{1}{p-1}} S(\theta), \qquad u' = \rho^{\frac{1}{p-1}} S'(\theta), \quad \rho > 0, \quad \theta \in \mathbb{R}, \tag{8}$$

and by using (7), it is not difficult to show that equation (1) is transformed into the following first order system:

$$\begin{aligned} \frac{d\rho}{dt} &= (p-1)S'(\theta)h(t), \\ \frac{d\theta}{dt} &= 1 - \rho^{-1}S(\theta)h(t). \end{aligned} \tag{9}$$

If we consider the periodicity of $S(t)$, and by $u = \rho^{\frac{1}{p-1}} S(\theta)$, with $\rho > 0$, we can assume without loss of generality that $u(0) = 0$ implies that $\theta(0) = 0$ or $\theta(0) = T$, which, by (8), is equivalent to $u'(0) > 0$ or $u'(0) < 0$ respectively. For simplicity, we discuss the first case only, that is $\theta(0) = 0$. Now, the condition $u(T) = 0$ is equivalent to $\theta(T) = mT$ for some $m \in \mathbb{Z}$.

Lemma 1. *Let* $(\rho(t), \theta(t))$ *be the solution of (9) satisfying the initial value condition* $(\rho(0), \theta(0)) = (\rho_0, {}_0)$. *Suppose* $h \in L^\infty(0, T)$, *then*

$$\theta(T) = T + \rho_0^{-1} I_h + O(\rho_0^{-2}) \tag{10}$$

as $\rho_0 \to +\infty$, *where* $\rho_0 = \rho(0)$ *and* $O(\rho_0^{-2})$ *is uniformly with respect to all* $h \in L^\infty(0, T)$ *with* $\|h\| \le C$ *for any fixed constant* $C > 0$ *and*

$$I_h = -\int_0^T S(t)h(t)dt.$$

Proof. Since h is bounded, we obtain from the first equation of (9) that for $t \in [0, T]$,

$$\rho(t) = \rho_0 + (p-1)\int_0^t S'(\theta(\tau))h(\tau)d\tau = \rho_0 + O(1),$$

which implies that for $\rho_0 \gg 1$, $\rho(t) \gg 1$ for all $t \in [0, T]$. Introduce a new positive variable $r = \rho^{-1}$, then $\rho \gg 1$ is equivalent to $r \ll 1$ and for $r(0) = r_0 \ll 1$, one has $r(t) \ll 1$ for all $t \in [0, T]$. Under this variable transformation, system (9) is changed into the following form:

$$\begin{aligned} \frac{dr}{dt} &= -(p-1)r^2 S'(\theta)h(t), \\ \frac{d\theta}{dt} &= 1 - rS(\theta)h(t). \end{aligned} \tag{11}$$

Since $\theta(0) = 0$, for $t \in [0, T]$, we get from above equations ($r_0 \ll 1$)

$$\begin{aligned} r(t) &= r_0 + O(r_0^2), \\ \theta(t) &= t + O(r_0). \end{aligned} \tag{12}$$

Substituting (12) into (11) and integrating from 0 to t, we obtain

$$\begin{aligned} r(t) &= r_0 - (p-1)r_0^2 \textstyle\int_0^t S'(\tau)h(\tau)d\tau + O(r_0^3), \\ \theta(t) &= t - r_0 \textstyle\int_0^t S(\tau)h(\tau)d\tau + O(r_0^2). \end{aligned} \tag{13}$$

Let $t = T$, we get from the second equation of (13) that

$$\theta(T) = T + r_0 I_h + O(r_0^2),$$

which is equivalent to (10). □

Lemma 2. *If* $I_h = 0$, *then for* $\rho_0 \gg 1$, *we have the following approximation*

$$\theta(T) = T + \rho_0^{-2} J_h + O(\rho_0^{-3}), \tag{14}$$

where

$$J_h = -\frac{(p-2)}{2}\left[\int_0^{\frac{T}{2}} \frac{(\int_t^{\frac{T}{2}} S'(\tau)h(\tau))d\tau)^2}{|S'(t)|^p}dt + \int_0^{\frac{T}{2}} \frac{(\int_t^{\frac{T}{2}} S'(T-\tau)h(T-\tau)d\tau)^2}{|S'(T-t)|^p}dt\right].$$

Proof. Substituting (13) into (11) and integrating the second equation over $[0, T]$, we obtain

$$\theta(T) = T + r_0 I_h + r_0^2 J_h + O(r_0^3)$$

which is equivalent to (14), where

$$I_h = -\int_0^T S(t)h(t)dt$$

and

$$J_h = (p-1)\int_0^T S(t)h(t)\left(\int_0^t S'(\tau)h(\tau)d\tau\right)dt + \int_0^T S'(t)h(t)\left(\int_0^t S(\tau)h(\tau)d\tau\right)dt.$$

By using $I_h = 0$ and integration by parts, we obtain

$$J_h = (p-2)\int_0^T S(t)h(t)\left(\int_0^t S'(\tau)h(\tau)d\tau\right)dt.$$

Denote $a = \frac{T}{2}$ and set

$$\begin{aligned} L &= \textstyle\int_0^T S(t)h(t)\left(\int_0^t S'(\tau)h(\tau)d\tau\right)dt \\ &= \textstyle\int_0^a S(t)h(t)\left(\int_0^t S'(\tau)h(\tau)d\tau\right)dt + \int_a^T S(t)h(t)\left(\int_0^t S'(\tau)h(\tau)d\tau\right)dt \\ &=: L_1 + L_2. \end{aligned}$$

Then $J_h = (p-2)L$, where $L = L_1 + L_2$ with

$$\begin{aligned} L_1 &= \textstyle\int_0^a S(t)h(t)\left(\int_0^t S'(\tau)h(\tau)d\tau\right)dt \\ &= \textstyle\int_0^a(\int_t^a S(\tau)h(\tau)d\tau)S'(t)h(t)dt; \\ L_2 &= \textstyle\int_a^T S(t)h(t)\left(\int_0^t S'(\tau)h(\tau)d\tau\right)dt. \end{aligned}$$

Set $U(t) = \int_t^a S'(\tau)h(\tau)d\tau,\ V(t) = \int_t^a S(\tau)h(\tau)d\tau$, then

$$\begin{aligned} L_1 &= -\textstyle\int_0^a U'(t)V(t)dt \\ &= U(0)V(0) + \textstyle\int_0^a U(t)V'(t)dt \\ &= U(0)V(0) + \textstyle\int_0^a \frac{U(t)U'(t)S(t)dt}{S'(t)} \\ &= U(0)V(0) + \frac{1}{2}\frac{U^2(t)S(t)}{S'(t)}|_0^a - \frac{1}{2}\textstyle\int_0^a U^2(t)\left(1 - \frac{S(t)S''(t)}{(S'(t))^2}\right)dt. \end{aligned}$$

Claim 1. $\lim_{t\to a} \frac{U^2(t)S(t)}{S'(t)} = 0.$

In fact, by the definition of $U(t)$, $S(t)$ and by using L' Hospital's rule, we get

$$\lim_{t\to a}\frac{U^2(t)S(t)}{S'(t)}$$
$$=\lim_{t\to a}\frac{U^2(t)}{S'(t)}\lim_{t\to a}S(t)=\alpha^{-\frac{1}{p}}\lim_{t\to a}\frac{U^2(t)}{S'(t)}$$
$$=\alpha^{-\frac{1}{p}}\lim_{t\to a}\frac{2U(t)U'(t)}{S''(t)}=\alpha^{-\frac{1}{p}}\lim_{t\to a}\frac{-2U(t)|S'(t)|^{p-2}U'(t)}{\alpha|S(t)|^{p-2}S(t)}=0.$$

Claim 2. $1-\frac{S(t)S''(t)}{(S'(t))^2}=\frac{1}{|S'(t)|^p}$, $t\in(0,T)$.

In fact, since $S(t)>0$ on $(0,T)$, we get from (6) and (7),

$$|S'(t)|^{p-2}S''(t)=-\alpha|S(t)|^{p-2}S(t),\quad \text{and}\quad |S'(t)|^p+\alpha(S(t))^p\equiv 1.$$

From above equations, we obtain

$$1-\frac{SS''}{(S')^2}=\frac{(S')^2+\alpha|S|^p/|S'|^{p-2}}{(S')^2}=\frac{|S'|^p+\alpha|S|^p}{|S'|^p}=\frac{1}{|S'|^p}.$$

By using Claim 1 and Claim 2, we get

$$L_1=U(0)V(0)-\frac{1}{2}\int_0^a\frac{U^2(t)}{|S(t)|^p}dt=U(0)V(0)-\frac{1}{2}\int_0^a\frac{(\int_t^a S'(\tau)h(\tau)d\tau)^2}{|S'(t)|^p}dt$$

Now we calculate L_2:

Let $F(t)=\int_0^t S'(\tau)h(\tau)d\tau,\ G(t)=\int_0^t S(\tau)h(\tau)d\tau$, then $G(T)=I_h=0$ and

$$L_2 =\int_a^T G'(t)F(t)dt=-G(a)F(a)-\int_a^T F'(t)G(t)dt$$
$$=-G(a)F(a)-\int_a^T S'(t)h(t)(\int_0^t S(\tau)h(\tau)d\tau)dt.$$

Since

$$\int_a^T S'(t)h(t)(\int_0^t S(\tau)h(\tau)d\tau)dt$$
$$\overset{t=T-x}{=}\int_0^a S'(T-x)h(T-x)\left(\int_0^{T-x}S(\tau)h(\tau)d\tau\right)dx$$
$$\overset{\tau=T-y}{=}\int_0^a S'(T-x)h(T-x)\left(\int_x^T S(T-y)h(T-y)dy\right)dx$$

and $\int_0^T S(T-y)h(T-y)dy=\int_0^T S(t)h(t)dt=0$, we get

$$\int_x^T S(T-y)h(T-y)dy=-\int_0^x S(T-y)h(T-y)dy.$$

This implies that

$$\int_a^T S'(t)h(t)\left(\int_0^t S(\tau)h(\tau)d\tau\right)dt$$
$$=\int_0^a S'(T-x)h(T-x)\left(\int_0^x S(T-y)h(T-y)dy\right)dx.$$

Similar to the calculation of L_1, we obtain

$$L_2=-G(a)F(a)-\frac{1}{2}\int_0^a\frac{(\int_t^a S'(T-\tau)h(T-\tau)d\tau)^2}{|S'(T-t)|^p}dt.$$

It is evident that

$$F(a) = U(0), G(a) = V(0).$$

Now it follows from the expressions of L_1 and L_2 that

$$L = L_1 + L_2 = -\frac{1}{2}\left[\int_0^a \frac{\left(\int_t^a S'(\tau)h(\tau)d\tau\right)^2}{|S'(t)|^p}dt + \int_0^a \frac{(\int_t^a S'(T-\tau)h(T-\tau)d\tau)^2}{|S'(T-t)|^p}dt\right].$$

□

Remark 1. Let $\alpha = \beta = 1$, then $S(t) = \sin_p t$, $S'(t) = \sin_p' t = \cos_p t$, $T = \pi_p$, $a = \frac{\pi_p}{2}$, Lemma 2 reduces the

$$J_h = -\frac{(p-2)}{2}\left[\int_0^{\frac{\pi_p}{2}} \frac{(\int_t^{\frac{\pi_p}{2}} h(\tau)\cos_p \tau d\tau)^2 + (\int_t^{\frac{\pi_p}{2}} h(\pi_p - \tau)\cos_p \tau d\tau)^2}{\cos_p^p t}dt\right],$$

which differs only by a constant from the one defined in [8]. Besides, it should point out that the expression of J_h in [8] contains a typing error: π_p should be $\pi_p/2$ in the upper limit of the second integral.

3. Main Results

In this section, by using a similar method used in [8] and [9], we give and prove an existence result of (1)-(2).

Let $X = C_0^1[0,T] = \{u \in C^1[0,T] : u(0) = u(T) = 0\}$ and $R^+ = [0,+\infty)$. For $u \in X$, $h \in L^\infty(0,T)$ and $\lambda \in R^+$, define an operator $G_{\lambda,h} : X \to X$ by $G_{\lambda,h}(v) = u$ if and only

$$(15)\qquad \begin{aligned} &(\phi_p(u'))' = \lambda[h(\lambda^{\frac{1}{p}}t) - \alpha\phi_p(v^+) + \beta\phi_p(v^-)],\\ &u(0) = u(T) = 0. \end{aligned}$$

Standard arguments based on the Arzela-Ascoli theorem imply that $G_{\lambda,h}$ is a well-defined operator which is compact from X into X^*. Moreover, $G_{\lambda,h}$ depends continuously on the perturbations of h and λ.

Lemma 3. *Let* $\deg[I - G_{\lambda,h}; B_R(0), 0]$ *be the Leray-Schauder degree of* $I - G_{\lambda,h}$ *with respect to* $B_R(0)$ *and 0, where* $R > 0$ *and* $B_R(0) = \{u \in X; \|u\| < R\}$*, I is the identity operator. Then for small* $\varepsilon > 0$ *and any* $R > 0$,

$$(16)\qquad \begin{aligned} &\deg[I - G_{1-\varepsilon,0}; B_R(0), 0] = 1,\\ &\deg[I - G_{1+\varepsilon,0}; B_R(0), 0] = -1. \end{aligned}$$

Proof. The result of Lemma 3 is a direct consequence of the results of [8] and the invariance of the Leray-Schauder degree under homotopy since α, β lie in the curve $\alpha^{-\frac{1}{p}} + \beta^{-\frac{1}{p}} = 2$ which passes the point $\lambda_1 = (p-1)$. □

Theorem 1. *Assume $h \in L^\infty(0,\, T)$ and $h \not\equiv 0$, $I_h = \int_0^T S(t)h(t)dt = 0$. Then the boundary value problem $(1)-(2)$ has at least one solution. Moreover, if $p \neq 2$, then the set of all possible solutions is bounded in $C^1[0,T]$.*

Proof. By using the homogeneity and the boundary condition in (15), we see that for fixed $h \in L^\infty(0,\, T)$, we can take $R > 0$ so large that (16) extend to

$$\begin{aligned} &\deg[I - G_{1-\varepsilon,h}; B_R(0), 0] = 1, \\ &\deg[I - G_{1+\varepsilon,h}; B_R(0), 0] = -1. \end{aligned} \tag{17}$$

First, we consider the case $1 < p < 2$, then by Lemma 2, $J_h > 0$ and for $t \geq T$ we extend h to $[0, 2T]$ as a L^∞ function.

We claim that there exists a constant $R > 0$ such that for any $\lambda \in [1, 1+\varepsilon]$ the boundary value problem

$$\begin{aligned} &(\phi_p(u'))' + (p-1)\lambda[\alpha\phi_p(u^+) - \beta\phi_p(u^-)] = (p-1)\lambda h(\lambda^{1/p}t)], \\ &u(0) = u(T) = 0, \end{aligned} \tag{18}$$

has no solution with $\|u\|_{C^1[0,T]} \geq R$.

Suppose on the contrary that there exist sequence $\{u_n\}_{n=1}^\infty \subset C_0^1[0,T]$, $\{\lambda_n\}_{n=1}^\infty \subset [1, 1+\varepsilon]$, such that $\lambda_n \to \bar{\lambda} \in [1, 1+\varepsilon]$, and $\|u_n\|_{C^1[0,T]} \to \infty$ and u_n, λ_n satisfy (18). From (8), we know that $\rho_n(0) \to +\infty$. In this case, $v_n(t) := u_n(\lambda^{\frac{-1}{p}} t)$ solves the equation

$$(\phi_p(v_n'))' + (p-1)[\alpha\phi_p(v_n^+) - \beta\varphi_p(v_n^-)] = (p-1)h(t), v_n(0) = 0,$$

with $\rho_n(0) \to +\infty$ and $u_n(T) = v_n(T\lambda_n^{\frac{1}{p}}) = 0$. But Lemma 2 and $I_h = 0$ imply $\theta_n(T) > T$ for n large enough. This contradicts the fact $u_n(T) = v_n(T\lambda_n^{\frac{1}{p}}) = 0$ because $1 \leq \lambda_n \leq 1 + \varepsilon$ for any $n \in \mathbb{N}$. Thus the claim is verified.

For this claim we see that for $\varepsilon > 0$ small the homotopy $\overline{H} : X \times [1,\, 1+\varepsilon] \to X$ defined by $\overline{H}(u,\, \lambda) = u - G_{\lambda, h_\lambda}(u)$, where $h_\lambda = h(\lambda^{\frac{1}{p}} t)$, satisfies $\overline{H}(u,\, \lambda) \neq 0$ for all $\lambda \in [1,\, 1+\varepsilon]$ and $\|u\|_{C^1[0,\,T]} \geq R$. Thus, from the homotopy invariance property of the Leray-Schauder degree, we obtain by (17)

$$\deg[I - G_{1,h}; B_R(0), 0] = \deg[I - G_{1+\varepsilon, h_{1+\varepsilon}}; B_R(0), 0] = -1.$$

This proves that for given h satisfying $I_h = 0$, the boundary value problem (1)-(2) has at least one solution. Moreover, it follows from our discussions that all possible solutions of (1)-(2) are bounded in the $C^1[0,T]$ norm. The case $p > 2$ can be proved similarly. □

Theorem 2. *Define a functional* $E : W_0^{1,p}(0,T) \to R$ *given by*

$$E(u) = \frac{\int_0^T |u'|^p}{p} - \frac{[\alpha \int_0^T |u^+|^p + \beta \int_0^T |u^-|^p]}{p} + \int_0^T hu$$

where u *is a solution of* $(1) - (2)$, $\alpha > 0, \beta > 0$, $\alpha^{-\frac{1}{p}} + \beta^{-\frac{1}{p}} = 2$, *and* $T = \pi_p/\alpha^{\frac{1}{p}}$.

Assume that $h \in L^\infty(0, T)$, *and* $I_h = 0$, $h(t) \not\equiv 0$.

(i) *for* $1 < p < 2$, *the functional* E *is unbounded from below. The set of its critical points is nonempty and bounded.*

(ii) *for* $p > 2$, *the functional* E *is bounded from below and has a global minimizer. The set of its critical points is bounded.*

The proof of Theorem 2 is similar to the proof of Theorem 1.2 in [8], so we omit it.

4. More General Nonhomogeneous Problems

In this section, we deal with the existence of solutions to the following nonhomogeneous boundary value problem:

$$(19)\qquad \begin{aligned} &(\phi_p(u'))' + \tfrac{(p-1)q}{p}[\alpha\phi_q(u^+) - \beta\phi_q(u^-)] = (p-1)h(t), \\ &u(0) = u(T) = 0, \end{aligned}$$

where $q \geq p > 1$, $\pi_{pq} = \int_0^1 \frac{ds}{(1-s^q)^{\frac{1}{p}}} = \frac{2}{q}B(\frac{1}{q}, \frac{1}{p^*})$, $p^* = \frac{p}{p-1}$, $\alpha^{-\frac{1}{q}} + \beta^{-\frac{1}{q}} = 2$, $T = \pi_{pq}/\alpha^{\frac{1}{q}}$ and $B(r,s) = \int_0^1 t^{r-1}(1-t)^{s-1}dt$ is the β function for $r > 0$, $s > 0$ and $h \in L^\infty(0, T)$.

If $q = p$, then (19) reduces to (1). Therefore we consider the case $q > p$ only. Similar to the results of [7], we can define (with minor modification) the following $2\pi_{pq}$-periodic function $u = \sin_{pq} t$ which is the solution of the following initial value problem:

$$(20)\qquad \begin{aligned} &(\phi_p(u'))' + \tfrac{(p-1)q}{p}\phi_q(u) = 0, \\ &u(0) = 0, \quad u'(0) = 1 \end{aligned}$$

which for $t \in [0, \frac{\pi_{pq}}{2}]$ can be given implicitly by the formula

$$t = \int_0^{\sin_{pq} t} \frac{ds}{(1-s^q)^{\frac{1}{p}}}$$

and $\sin_{pq} t = \sin_{pq}(\pi_{pq} - t)$ for $t \in [\frac{\pi_{pq}}{2}, \pi_{pq}]$; $\sin_{pq} t = -\sin_{pq}(2\pi_{pq} - t)$ for $t \in [\pi_{pq}, 2\pi_{pq}]$. Define also $\cos_{pq} t = \frac{d}{dt}(\sin_{pq} t)$. Then

$$|\sin_{pq} t|^q + |\cos_{pq} t|^p \equiv 1, \qquad \forall t \in R.$$

Let $\overline{S}(t)$ be the solution of the following initial value problem:

$$\begin{aligned} &(\phi_p(u'))' + \tfrac{(p-1)q}{p}[\alpha\phi_q(u^+) - \beta\phi_q(u^-)] = 0, \\ &u(0) = 0, \quad u'(0) = 1. \end{aligned} \tag{21}$$

Then it is easy to see that $\overline{S}$ is $2\pi_{pq}$-periodic and can be expressed explicitly as

$$\overline{S}(t) = \begin{cases} \alpha^{-\frac{1}{q}} \sin_{pq} \alpha^{\frac{1}{q}} t, & t \in [0,\ T]; \\ -\beta^{-\frac{1}{q}} \sin_{pq} \beta^{\frac{1}{q}} (t - T), & t \in [T,\ 2\pi_{pq}]. \end{cases}$$

Moreover, it is also easy to verify by using (21) that $\overline{S}(t)$ satisfies the following identity:

$$|S'(t)|^p + \alpha(S^+(t))^q + \beta(S^-(t))^q \equiv 1. \tag{22}$$

For $\rho > 0$, $\theta \in \mathbb{R}$, define the following generalized polar coordinates transformation $\overline{T}$ as:

$$u = \rho^\sigma \overline{S}(\theta), \qquad u' = \rho^{\frac{1}{p-1}} \overline{S}'(\theta), \tag{23}$$

where

$$\sigma = \frac{q-p}{(p-1)q} > 0.$$

Then by using (22), we can show that (19) is changed into the following system:

$$\begin{aligned} \frac{d\rho}{dt} &= (p-1)\overline{S}'(\theta)h(t), \\ \frac{d\theta}{dt} &= \rho^\sigma - \frac{p}{q}\rho^{-1}\overline{S}(\theta)h(t), \end{aligned} \tag{24}$$

Theorem 3. *Suppose $q > p$, $h \in L^\infty(0, T)$. Then the boundary value problem* (19) *has infinitely many solutions $u_n(t)$ and the number of zeros of u_n in $(0, T)$ increases to ∞ as $n \to \infty$, moreover, $\|u_n\| \to \infty$ as $n \to \infty$.*

Proof. From above discussion, (19) is changed into (24). Now suppose $\theta(0) = 0$, then the second equation $u(T) = 0$ in (2) is equivalent to $\theta(T) = kT$ for some $k \in \mathbb{Z}$. The assumption $q > p$ implies $\sigma > 0$. Let $\rho(0) = \rho_0 \gg 1$, then it follows from the first equation of (24) that

$$\begin{aligned} &\rho(t) = \rho_0 + O(1), \\ &\rho^{-1}(t) = \rho_0^{-1} + o(\rho_0^{-1}), \quad t \in (0,\ T) \end{aligned} \tag{25}$$

Substituting (25) into the second equation of (24) and integrating from 0 to T, we get

$$\theta(T) = \rho_0^\sigma T + o(\rho_0^\sigma) = \rho_0^\sigma T(1 + o(1)). \tag{26}$$

It follows from (26) and the fact that $\theta(T)$ depends continuously on ρ_0 that there exist infinitely many $n \in \mathbb{N}$ such

$$\theta(T) = n\pi_{pq}$$

and $\rho_n(0) \to \infty$ as $n \to \infty$. □

Let $\alpha = \beta = 1$, then $T = \pi_{pq}$ and $\overline{S}(t) = \sin_{pq}(t)$. In this case, Theorem 3 reduces to

Corollary 4. *Let $q > p > 1$, $h \in L^\infty(0, \pi_{pq})$. Then the following boundary value problem*

$$(\phi_p(u'))' + \tfrac{(p-1)q}{p}\phi_q(u) = (p-1)h(t),$$
$$u(0) = u(\pi_{pq}) = 0,$$

has infinitely many solutions $u_n(t)$ and the number of zeros of u_n in $(0, \pi_{pq})$ increases to ∞ as $n \to \infty$, moreover, $\|u_n\| \to \infty$ as $n \to \infty$.

Remark 2. Let us compare the key approximation formulas in [8] and in this paper. Let $\alpha = \beta = 1$, $S(t) = \sin_p t$. In [8], for $|\alpha| \gg 1$, the initial condition is

$$u(0) = 0, \quad u'(0) = \alpha = \pm|\alpha|, \tag{27}$$

In our paper, the initial condition is

$$u(0) = 0,\ u'(0) = \rho_0^{\frac{1}{p-1}} S'(\theta(0)) = \pm\rho_0^{\frac{1}{p-1}}. \tag{28}$$

Comparing (27) with (28), we get $\rho_0 = |\alpha|^{p-1} \gg 1$ and $\alpha > 0$ is equivalent to $\theta(0) = 0$, and $\alpha < 0$ is equivalent to $\theta(0) = \pi_p = T$. Moreover, as $\rho = |\alpha| \gg 1$, we have $\frac{d\theta}{dt} = 1 + O(\rho_0^{-1}) \approx 1$, hence it is easy to verify that the following two key approximations are equivalent:

Assume $\int_0^{\pi_p} \sin_p t h(t)dt = 0$, then in [8]

$$t_1^\alpha = \pi_p + (p-2)J_h|\alpha|^{2(1-p)} + o(|\alpha|^{2(1-p)}), \tag{29}$$

where t_1^α is the first positive zero of $u(t)$, and $J_h > 0$. While in our paper,

$$\theta(\pi_p) = \pi + \frac{(2-p)}{2}L\rho_0^{-2} + o(\rho_0^{-2}), \tag{30}$$

where $L > 0$. It is now easy for us to see that (29) and (30) are equivalent. Besides, the results of our paper remains valid when we replace the function $h(t)$ in the right side of (1) by a continuous and bounded function $f(t,u)$, provided that the limits $\lim_{u\to\pm\infty} f(t,u) = f(t,\pm\infty) \in L^\infty(0,\,T)$ exists and

$$\lim_{u\to\pm\infty} |u|^{p-1}[f(t,u) - f(t,\pm\infty)] = 0.$$

Finally, we end up this paper with a remark that the existence of solution of (19) when $1 < q < p$ is left as an open question.

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