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SOME FIXED POINT THEOREMS AND EXAMPLE IN $\mathcal{M}$-FUZZY METRIC SPACE

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Abstract. We introduce the concept of semi-compatible and weak-compatible in $\mathcal{M}$-fuzzy metric space, and prove some fixed point theorem for four self maps satisfying some conditions in $\mathcal{M}$-fuzzy metric space.

1. Introduction

The concept of fuzzy metric space was introduced by Kramosil and Michalek[4]. George and Veeramani[3] studied the properties on fuzzy metric space. Grabiec[2] obtained the Banach contraction principle for this spaces.

Recently, Sedghi et.al.[7] introduced the concept of $\mathcal{M}$-fuzzy metric space which is a generalization of fuzzy metric space due to George and Veeramani[3] and proved common fixed point theorems for two mappings in complete $\mathcal{M}$-fuzzy metric space, and Park et.al.[5] introduced the concept of compatible mapping of type(*) and gave common fixed point theorems satisfying some conditions in $\mathcal{M}$-fuzzy metric space.

In this paper, we introduce the concept of semi-compatible and weak-compatible in $\mathcal{M}$-fuzzy metric space, and prove some fixed point theorem for four self maps satisfying some conditions in this space. Also, we recall the example satisfying all conditions of Theorem 3.4.

2. Preliminaries

Definition 2.1 ([1]). Let X be a nonempty set. A *generalized metric* (or *D-metric*) on X is a function $D : X^3 \to \mathbf{R}^+$ satisfying the following conditions;

(a) $D(x, y, z) \geq 0$,

(b) $D(x, y, z) = 0$ if and only if $x = y = z$,

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(c) $D(x,y,z) = D(p\{x,y,z\})$ (symmetry), where p is a permutation function,

(d) $D(x,y,z) \leq D(x,y,a) + D(a,z,z)$ for all $x,y,z,a \in X$.

The pair (X,D) is called a *generalized metric*(or *D-metric*) space.

Immediate example of D-metric space is $D(x,y,z) = \max\{d(x,y), d(y,z), d(z,x)\}$ where d is the ordinary metric on X.

Let us recall (see [8]) that a continuous $t-$norm is a binary operation $* : [0,1] \times [0,1] \to [0,1]$ which satisfies the following conditions; (a)$*$ is commutative and associative, (b)$*$ is continuous, (c)$a * 1 = a$ for all $a \in [0,1]$, (d)$a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ $(a,b,c,d \in [0,1])$.

Definition 2.2 ([7])**.** The $3-$tuple $(X, \mathcal{M}, *)$ is said to be a $\mathcal{M}$-*fuzzy metric space* if X is an arbitrary set, $*$ is a continuous $t-$norm, and $\mathcal{M}$ is a fuzzy set on $X^3 \times (0,\infty)$ satisfying the following conditions; for all $x,y,z,a \in X$ and $t,s > 0$,

(a)$\mathcal{M}(x,y,z,t) > 0$,

(b)$\mathcal{M}(x,y,z,t) = 1$ if and only if $x = y = z$,

(c)$\mathcal{M}(x,y,z,t) = \mathcal{M}(p\{x,y,z\},t)$(symmetry), where p is a permutation function,

(d)$\mathcal{M}(x,y,a,t) * \mathcal{M}(a,z,z,s) \leq \mathcal{M}(x,y,z,t+s)$,

(e)$\mathcal{M}(x,y,z,\cdot) : (0,\infty) \to [0,1]$ is continuous.

We know that both a D-metric and a fuzzy metric induce a $\mathcal{M}$-fuzzy metric as following.

Example 2.3. Let (X,D) be a $D-$metric space. Denote $a*b = ab$ for all $a,b \in [0,1]$ and for all $x,y,z \in X$ and $t > 0$,

$$\mathcal{M}(x,y,z,t) = \frac{t}{t + D(x,y,z)}.$$

Then $(X,\mathcal{M},*)$ is a $(X,\mathcal{M},*)$ is a $\mathcal{M}$-fuzzy metric space.

Lemma 2.4 ([7])**.** *Let $(X,\mathcal{M},*)$ be a $\mathcal{M}$-fuzzy metric space. For any $x,y,z \in X$ and $t > 0$, we have ,*

(a) $\mathcal{M}(x,x,y,t) = \mathcal{M}(x,y,y,t)$. (b)$\mathcal{M}(x,y,z,\cdot)$ is nondecreasing.

Definition 2.5 ([5])**.** Let X be a $\mathcal{M}$-fuzzy metric space and a sequence $\{x_n\} \subset X$.

(a)$\{x_n\}$ is *convergent* to a $x \in X$ if $\lim_{n\to\infty} \mathcal{M}(x,x,x_n,t) = 1$ for all $t > 0$.

(b)$\{x_n\}$ is called a *Cauchy sequence* if $\lim_{n\to\infty} \mathcal{M}(x_{n+p},x_{n+p},x_n,t) = 1$ for all $t > 0$ and $p > 0$.

(c)A $\mathcal{M}$-fuzzy metric in which every Cauchy sequence is convergent is said to be *complete.*

Remark 2.6. Since $*$ is continuous, it follows from (d) of Definition 2.2 that the limit of a sequence is uniquely determined.

Let $(X, \mathcal{M}, *)$ be a $\mathcal{M}$-fuzzy metric space with the following condition;

$$\lim_{t\to\infty} \mathcal{M}(x,y,z,t) = 1 \text{ for all } x,y,z \in X, \quad t > 0. \tag{2.1}$$

Lemma 2.7 ([5])**.** *Let $\{x_n\}$ be a sequence in a $(X, \mathcal{M}, *)$ with condition (2.1). If there exists $k \in (0,1)$ such that $\mathcal{M}(x_{n+2}, x_{n+1}, x_{n+1}, kt) \geq \mathcal{M}(x_{n+1}, x_n, x_n, t)$ for all $t > 0$ and $n = 1, 2, \cdots$, then $\{x_n\}$ is a Cauchy sequence in X.*

Lemma 2.8 ([5])**.** *Let $\{x_n\}$ be a sequence in a $\mathcal{M}$-fuzzy metric space $(X, \mathcal{M}, *)$ with condition (2.1). If for all $x, y \in X$ and for a number $k \in (0,1)$, $\mathcal{M}(x,y,z,kt) \geq \mathcal{M}(x,y,z,t)$, then $x = y = z$.*

3. Main Result and Example

Definition 3.1. ([7])Let A, B be mappings from $(X, \mathcal{M}, *)$ into itself. The mappings are said to be *compatible* if $\lim_{n\to\infty} \mathcal{M}(ABx_n, BAx_n, BAx_n, t) = 1$ for all $t > 0$, whenever $\{x_n\} \subset X$ such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = z$ for some $z \in X$.

Definition 3.2. Let A, B be mappings from $\mathcal{M}$-fuzzy metric space $(X, \mathcal{M}, *)$ into itself. The mappings are called *semi-compatible* if $\lim_{n\to\infty} \mathcal{M}(ABx_n, Bz, Bz, t) = 1$, $\lim_{n\to\infty} \mathcal{M}(BAx_n, Az, Az, t) = 1$ for all $t > 0$, whenever $\{x_n\} \subset X$ such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = z$ for some $z \in X$.

Definition 3.3. Let A, B be mappings from $(X, \mathcal{M}, *)$ into itself. The mappings are said to be *weak-compatible* if they commute at their coincidence points. that is, $Az = Bz$ implies $ABz = BAz$.

Theorem 3.4. *Let A, B, S and T be self mappings of a complete $\mathcal{M}$-fuzzy metric space $(X, \mathcal{M}, *)$ with condition (2.1) satisfying*

(a) *$A(X) \subset T(X)$, $B(X) \subset S(X)$,*

(b) *A or S is continuous,*

(c) *(A, S) is semi-compatible and (B, T) is weak-compatible,*

(d) *there exists $k \in (0,1)$ such that for all $x, y \in X$ and $t > 0$,*

$$\begin{aligned}&\mathcal{M}(Ax, By, By, kt) \\ &\geq \min\{\mathcal{M}(By, Ty, Ty, t), \mathcal{M}(Sx, Ty, Ty, t), \mathcal{M}(Ax, Sx, Sx, t)\}.\end{aligned}$$

Then A, B, S and T have a common fixed point in X.

Proof. Since $A(X) \subset T(X)$ and $B(X) \subset S(X)$, for any $x_0 \in X$, we can choose points $x_1, x_2 \in X$ such that $Ax_0 = Tx_1$, $Bx_1 = Sx_2$. Thus by induction, we can define sequences $\{x_n\}, \{y_n\} \subset X$ such that $y_{2n+1} = Ax_{2n} = Tx_{2n+1}$, $y_{2n+2} = Bx_{2n+1} = Sx_{2n+2}$ for $n = 0, 1, 2, \cdots$. Now using (d) with $x = x_{2n}, y = x_{2n+1}$,

$$\begin{aligned}
&\mathcal{M}(Ax_{2n}, Bx_{2n+1}, Bx_{2n+1}, kt)\\
&= \mathcal{M}(y_{2n+1}, y_{2n+2}, y_{2n+2}, kt)\\
&\geq \min\{\mathcal{M}(Bx_{2n+1}, Tx_{2n+1}, Tx_{2n+1}, t), \mathcal{M}(Sx_{2n}, Tx_{2n+1}, Tx_{2n+1}, t),\\
&\qquad \mathcal{M}(Ax_{2n}, Sx_{2n}, Sx_{2n}, t)\}\\
&= \min\{\mathcal{M}(y_{2n+2}, y_{2n+1}, y_{2n+1}, t), \mathcal{M}(y_{2n}, y_{2n+1}, y_{2n+1}, t), \mathcal{M}(y_{2n+1}, y_{2n}, y_{2n}, t)\}.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\mathcal{M}(y_{2n+1}, y_{2n+2}, y_{2n+2}, kt)\\
&\geq \cdots \geq \min\{\mathcal{M}(y_{2n+2}, y_{2n+1}, y_{2n+1}, \frac{t}{k^m}),\\
&\qquad \mathcal{M}(y_{2n}, y_{2n+1}, y_{2n+1}, t), \mathcal{M}(y_{2n+1}, y_{2n}, y_{2n}, \frac{t}{k^m})\}.
\end{aligned}$$

Taking limit as $m \to \infty$, we get for all $t > 0$,

$$\mathcal{M}(y_{2n+1}, y_{2n+2}, y_{2n+2}, kt) \geq \mathcal{M}(y_{2n}, y_{2n+1}, y_{2n+1}, t).$$

Similarly, we get $\mathcal{M}(y_{2n+2}, y_{2n+3}, y_{2n+3}, kt) \geq \mathcal{M}(y_{2n+1}, y_{2n+2}, y_{2n+2}, t)$.

Thus for all n and $t > 0$, $\mathcal{M}(y_n, y_{n+1}, y_{n+1}, kt) \geq \mathcal{M}(y_{n-1}, y_n, y_n, t)$. Therefore

$$\mathcal{M}(y_n, y_{n+1}, y_{n+1}, t) \geq \mathcal{M}(y_{n-1}, y_n, y_n, \frac{t}{k}) \geq \cdots \geq \mathcal{M}(y_0, y_1, y_1, \frac{t}{k^n}).$$

Hence $\lim_{n\to\infty} \mathcal{M}(y_n, y_{n+1}, y_{n+1}, t) = 1$ for all $t > 0$.

Now, for any integer $p \in \mathbf{N}$,

$$\mathcal{M}(y_n, y_{n+p}, y_{n+p}, t) \geq \mathcal{M}(y_1, y_2, y_2, \frac{t}{pk^{n-1}}) * \cdots * \mathcal{M}(y_1, y_2, y_2, \frac{t}{pk^{n+p-2}}).$$

Therefore $\lim_{n\to\infty} \mathcal{M}(y_n, y_{n+p}, y_{n+p}, t) \geq 1$. Hence $\{y_n\}$ is a Cauchy sequence in X which is complete. Therefore $\{y_n\}$ converges to $z \in X$. Since $\{Ax_{2n}\}$, $\{Bx_{2n+1}\}$, $\{Sx_{2n}\}$, $\{Tx_{2n+1}\}$ are subsequences of $\{y_n\}$, they also converges to the point z. Let A be continuous. Then $\lim_{n\to\infty} AAx_{2n} = Az$, $\lim_{n\to\infty} ASx_{2n} = Az$. Since (A, S) is semi-compatibility, $\lim_{n\to\infty} ASx_{2n} = Sz$. Because of unique of limit, we have $Az = Sz$. From the reference [6], we can see that A, B, S and T have the unique common fixed point on a $\mathcal{M}$-fuzzy metric space. $\square$

Example 3.5. Let (X, d) be the metric space with $X = [0, 1]$. Denote $a * b = ab$ and let $\mathcal{M}$ be fuzzy set on $X^3 \times (0, \infty)$ defined as follows;

$$\mathcal{M}(x, y, z, t) = \frac{t}{t + D(x, y, z)}.$$

Then $(X, \mathcal{M}, *)$ is a $\mathcal{M}$-fuzzy metric space. Define self mappings A, B, S and T by

$$A(X) = \begin{cases} 0 & \text{if } 0 \le x \le \frac{1}{2} \\ \frac{1}{3} & \text{otherwise} \end{cases}, \quad B(X) = \begin{cases} 0 & \text{if } 0 \le x \le \frac{1}{4} \\ \frac{1}{3} & \text{otherwise} \end{cases},$$

$$S(X) = \begin{cases} 2x & \text{if } 0 \le x \le \frac{1}{2} \\ 1 & \text{otherwise} \end{cases}, \quad T(X) = \begin{cases} 0 & \text{if } x = 0 \\ \frac{1}{3} & \text{if } 0 < x \le \frac{1}{4} \\ 1 & \text{if } \frac{1}{4} < x \le 1. \end{cases}$$

Then S is continuous, (A, S) is semi-compatible and (B, T) is weak-compatible. Also, $A(X) = B(X) = \{0, \frac{1}{3}\}$, $S(X) = [0, 1]$ and $T(X) = \{0, \frac{1}{3}, 1\}$ satisfy the conditions of Theorem 3.4. Therefore we know that 0 is the unique common fixed point of A, B, S and T.

References

1. B.C. Dhage: Generalized metric spaces and mappings with fixed point. *Bulletin of Calcutta Mathematical Society* **84** (1992), 329-336.
2. M. Grabiec: Fixed point in fuzzy metric spaces. *Fuzzy Sets and Systems* **27** (1988), 385-389.
3. A. George & P. Veeramani: On some results in fuzzy metric spaces. *Fuzzy Sets and Systems* **64** (1994), 395-399.
4. J. Kramosil & J. Michalek: Fuzzy metric and statistical metric spaces. *Kybernetica* **11** (1975), 326-334.
5. J.H. Park, J.S. Park & Y.C. Kwun: Fixed points in $\mathcal{M}$-fuzzy metric space. *Fuzzy Optimization and Decision Making(Springer)* **7** (2008), no. 4, 305-315.
6. J.S. Park: Some results for four self mappings in intuitionistic fuzzy 2,3-metric space using compatibility. *JP J. fixed point Theory & Appl.* **3** (2008), no. 1, 63-83.
7. S. Sedghi, N. Shobe & J.H. Park: A common fixed point theorem in $\mathcal{M}$-fuzzy metric spaces. (Submitted).
8. B. Schweizer & A. Sklar: Statistical metric spaces. *Pacific J. Math.* **10** (1960), 314-334.

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