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EXISTENCE OF MINIMAL SURFACES WITH PLANAR ENDS

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ABSTRACT. In this article we consider axes of a complete embedded minimal surface in $\mathbf{R}^3$ of finite total curvature, and then prove that there is no planar ends at which the Gauss map have the minimum branching order if the minimal surface has a single axis.

1. INTRODUCTION

Historically, a minimal surface was described as a surface on which each point has a neighborhood which is the surface of least area with respect to its boundary. In 1989, Osserman [6] showed that if we stand at infinity to view a complete minimal surface in $\mathbf{R}^3$ of finite total curvature, it looks like several planes and catenoids. Since the plane and the catenoid are rotational surfaces, we say that they have axes (of rotation). Like the catenoid, we can consider an axis of a catenoid type end on which the torque of the representative curve of the end vanishes, see [2]. Because most of all the known examples can be constructed by symmetry methods, it follows that each axis of a catenoid type end of such examples coincide.

In this paper, we define another type of axes for complete embedded minimal surfaces of finite total curvature and describe their properties around a planar end. Hoffman and Karcher [2] raised the question of the order of the Gauss map at a planar end of a complete embedded minimal surface of finite total curvature can be equal to two which is the minimum possible value. Using properties of axes around a planar end, we can obtain another a partial result to this problem.

Theorem 1. *If a complete embedded minimal surface in $\mathbf{R}^3$ of finite total curvature has a single axis, then it cannot have a planar end where the Gauss map has the minimum order two.*

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2. PRELIMINARIES

A minimal surface of $\mathbf{R}^3$ is a conformal harmonic immersion $X : S \hookrightarrow \mathbf{R}^3$ where S is a 2-dimensional smooth manifold, with or without boundary. We can define X by

$$X(p) = \Re \int_{p_0}^{p} \left(\frac{1}{2} f\,(1-g^2),\, \frac{i}{2} f\,(1+g^2),\, fg \right) dz + X(p_0) \tag{1}$$

where f is a holomorphic function and g a meromorphic function, which is called *the Enneper-Weierstrass representation* of X. In particular, the meromorphic function $g : S \to \mathbf{C}$ is the stereographic projection of the Gauss map of X with respect to the north pole of S^2, just say it *the Gauss map* of X.

Let X be a complete minimal surface with finite total curvature. Then, by [6], there exists a closed Riemann surface S_k of genus k and finite number of points $p_1, \cdots, p_r$ on S_k, where S is conformally equivalent to $S_k \setminus \{p_1, \cdots, p_r\}$ and the Gauss map g can be extended to S_k such that the extension

$$g : S_k \to \mathbf{C} \cup \{\infty\}$$

is a holomorphic function. Take $p \in \{p_1, \cdots, p_r\}$ and consider a conformal closed disk $D \subset S_k$ such that $p \in D$ and $p_j \notin D$ for $p_j \neq p$, respectively. Denote $D^* := D \setminus \{p\}$, then the restriction $E := X(D^*)$ is called an end of X at p. We may assume that E has the vertical limit normal, for example $g(p) = 0$, and then we can take a suitable conformal local coordinate z of the origin such that $z(0) = p$ and a domain $U \subset \mathbf{C}$ containing the origin to get

$$g(z) = z^n h(z) \tag{2}$$

where $n > 0$ and h is a holomorphic function on U with $h(0) \neq 0$. Since E is embedded, by Lemma 11.9 in [1], the conformal metric $\Lambda = \frac{1}{2}|f|(1+|g|^2)$ must have order 2 at zero. It means that in other terms

$$|f|(1+|g|^2) \sim \frac{c}{|z|^2}$$

for some constant $c > 0$. Together with the well-definedness of (1), we have

$$f(z) = a_{-2} z^{-2} + \sum_{i=0}^{\infty} a_i\, z^i \tag{3}$$

on $U^* := U \setminus \{0\}$ where $a_{-2} \neq 0$. From (1) we easily deduce, $X|_{U^*}$ is a graph $(x_1, x_2, u(x_1, x_2))$ over the x_1x_2-plane with

$$u(x_1, x_2) = \beta + \alpha \log r + r^{-2}(\gamma_1 x_1 + \gamma_2 x_2) + O(r^{-2}) \tag{4}$$

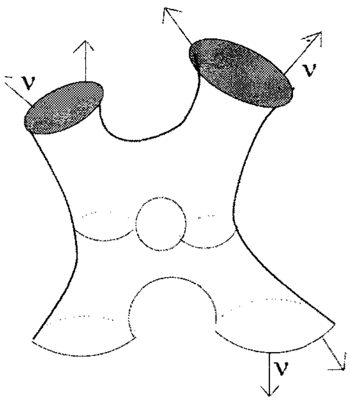

Figure 1.

for $r = (x_1^2 + x_2^2)^{\frac{1}{2}}$ sufficiently large, where β, α, γ_1 and γ_2 are real constants, see ([7]). The horizontal plane $\{(x_1, x_2, x_3) \in \mathbf{R}^3 \,|\, x_3 = \beta\}$ is called by *the limit tangent plane* of the end. We call E *a planar end* if $\alpha = 0$, or equivalently $n \geq 2$ in (2). Otherwise, then we call it *a catenoid type end.* Notice that the embedded end E is planar if and only if the corresponding puncture p is a branch point of the Gauss map g with the branching order equal to $n - 1 \geq 1$.

On the other hand let $\gamma \subset S$ be a closed curve and let ν be the outward unit conormal, see Figure 1, then we can define *the flux* and *the torque* of X along γ that;

$$\begin{aligned} Flux(\gamma) &= \int_\gamma \nu \, ds \\ Torque_O(\gamma) &= \int_\gamma X \wedge \nu \, ds \end{aligned}$$

where $O = (0, 0, 0)$ is the base point of the position vector X. If we move the base point from O to $W \in \mathbf{R}^3$ then the position vector based on W is changed to $X - W$, and the torque is

$$\begin{aligned} Torque_W(\gamma) &= \int_\gamma (X - W) \wedge \nu \, ds \\ &= Torque_O(\gamma) - W \wedge Flux(\gamma). \end{aligned}$$

It follows that the torque of γ does not depend upon the base point of X if the flux of γ vanishes.

Balancing Formula ([5]). *Let S be a compact minimal surface. Then*

$$Flux(\partial S) = (0, 0, 0)$$

$$Torque_W(\partial S) \;=\; (0,0,0)$$

for all base points $W \in \mathbf{R}^3$.

Note that we can compute the flux and the torque of an embedded end as those of a representative curve, for example $E \cap \partial B$ for a sufficiently large ball B.

Proposition 1 ([4]). *If E is an embedded end which is defined by the graph of a function u in (4) over a horizontal plane, where the order of Gauss map n is given in (2). In the case of $n = 1$, E is a catenoid type end and that*

$$Flux(E) \;=\; 2\pi(0,0,\alpha) \tag{5}$$

$$Torque_O(E) \;=\; \pi(\gamma_2, -\gamma_1, 0). \tag{6}$$

If $n \geq 2$ in (2), then the end E is a planar end and

$$Flux(E) = (0,0,0) \tag{7}$$

which follows that the torque of E is well-defined independent upon the base points. Precisely, we get;

$$Torque(E) \;=\; \begin{cases} \pi(\gamma_2, -\gamma_1, 0) \neq (0,0,0) & \text{if } n = 2 \\ (0,0,0) & \text{if } n > 2. \end{cases} \tag{8}$$

3. Proof of Main Result

In this section we assume that $M \subset \mathbf{R}^3$ is a complete embedded minimal surface of finite total curvature with horizontal limit tangent planes at the ends. Denote some notations by;

- $\Pi_t := \{(x_1, x_2, x_3) | \, x_3 = t\}$ is a horizontal plane at the height $t \in \mathbf{R}$
- $\gamma_t := M \cap \Pi_t$ is the intermediate curve of M with Π_t
- $S(a, b) := \{(x_1, x_2, x_3) | \, a < x_3 < b\}$ is a slab for $a < b$.

Let M have parallel planar ends $E_{P_1}, \cdots, E_{P_k}$ of heights $h_1, \cdots, h_k$, respectively, with

$$h_1 > \cdots > h_k.$$

Let $t \in \mathbf{R} \setminus \{h_1, \cdots, h_k\}$, then γ_t is compact and divides the surface M into two components $M_t^{\pm}$ where

$$M_t^+ : \;=\; M \cap \{(x_1, x_2, x_3) \in \mathbf{R}^3 \,|\, x_3 > t\}$$

$$M_t^- : \;=\; M \cap \{(x_1, x_2, x_3) \in \mathbf{R}^3 \,|\, x_3 < t\}.$$

Consider a sufficiently large closed region $K \subset \mathbf{R}^3$ such that the boundary of the compact minimal sub-surface $M_t^+ \cap K$ satisfies

$$\partial(M_t^+ \cap K) = \gamma_t \cup \Gamma_{t_1} \cup \cdots \cup \Gamma_{t_m}$$

where $\Gamma_{t_1}, \cdots, \Gamma_{t_m}$ are respectively in the same homology class as the representative curves of each end of M_t^+, i.e., the representative curves of ends. Then from the Balancing Formula;

$$Flux(\gamma_t) + \sum_{i=1}^{m} Flux(\Gamma_{t_i}) = (0,0,0).$$

Recall that the flux of each end of M must be either vertical or zero, see (5) and (7); hence the flux of γ_t must be vertical.

Proposition 2. *For all $t \in \mathbf{R} \setminus \{h_1, \cdots, h_k\}$, there is a unique horizontal vector $W_t = (w_t^1, w_t^2, 0) \in \mathbf{R}^3$ such that;*

$$Torque_{W_t}(\gamma_t) = (0,0,0).$$

We call W_t the base point of M at the height t.

Proof. By the halfspace theorem [3], M cannot be contained in a halfspace. Let $t > h_1$, then M_t^+ can have only catenoid type ends $E_{C_1}, \cdots, E_{C_\ell}$ and we can choose a closed region K such that;

$$\partial(M_t^+ \cap K) = \gamma_t \cup \Gamma_{C_1} \cup \cdots \cup \Gamma_{C_\ell}$$

where Γ_{C_i}, $i = 1, \cdots, \ell$, is a representative curve of E_{C_i}, respectively. From the balancing formula, we have

$$Torque_O(\gamma_t) = -\sum_{i=1}^{\ell} Torque_O(E_{C_i})$$

which is horizontal by (6). Recall that for $W \in \mathbf{R}^3$,

$$Torque_W(\gamma_t) = Torque_O(\gamma_t) - W \wedge Flux(\gamma_t).$$

Since the flux of γ_t is vertical, we can take a (unique) horizontal vector $W_t = (w_t^1, w_t^2, 0)$ for which

$$Torque_{W_t}(\gamma_t) = (0,0,0).$$

Now let $s < h_1$ and M_s^+ have planar ends $E_{P_1}, \cdots, E_{P_{k_\ell}}$ for $k_\ell \le k$. Take a closed region K such that

$$\partial(M \cap K) = \gamma_s \cup \Gamma_{P_1} \cup \cdots \cup \Gamma_{P_{k_\ell}} \cup \Gamma_{C_1} \cup \cdots \cup \Gamma_{C_\ell}$$

where Γ_{P_i}, $i = 1, \cdots, k_\ell$, is the representative curve of E_{P_i}, respectively. Then the torque of γ_s is horizontal by (6) and (8), and

$$Torque_O(\gamma_s) = -\sum_{i=1}^{\ell} Torque_O(E_{C_i}) - \sum_{j=1}^{k_\ell} Torque(E_{P_j}).$$

As before the torque vanished in this case, too. □

If we consider a slab $S(a,b)$ such that $M \cap S(a,b)$ is compact, then for the base points W_a and W_b at the heights a, b, respectively, we have;

$$\begin{aligned} (0,0,0) &= Torque_{W_a}(\gamma_a) + Torque_{W_a}(\gamma_b) \\ &= Torque_{W_a}(\gamma_a) + Torque_{W_b}(\gamma_b) - (W_a - W_b) \wedge Flux(\gamma_b) \\ &= -(W_a - W_b) \wedge Flux(\gamma_b). \end{aligned} \tag{9}$$

Since all the base points are horizontal and the flux of γ_b is vertical, it implies that $W_a = W_b$. Therefore, there is a unique vertical line on which the torque of γ_t vanishes for all $a \le t \le b$.

Definition 1. For a slab $S(h_i, h_{i-1})$, $i = 1, \cdots, k+1$, where $h_0 = \infty$ and $h_{k+1} = -\infty$, the (unique) vertical line ℓ_i, on which the torque of γ_t vanishes for all $h_i < t < h_{i-1}$, is called the i-th axis of M.

To compare two consecutive axes around a planar end, consider a slab $S(c,d)$ which contains only one planar end $E \in \{E_{P_1}, \cdots, E_{P_k}\}$ with the limit tangent plane Π. Then for a closed region K we have the boundary;

$$\partial(M \cap S(c,d) \cap K) = \gamma_c \cup \gamma_d \cup \Gamma$$

where Γ is the representative curve of E. Let us denote the upper axis ℓ_+ and the lower axis ℓ_- around E, then for the (horizontal) base points $W_\pm \in \ell_\pm$, respectively, we have;

$$\begin{aligned} Torque_{W_-}(\gamma_c) &= (0,0,0) \\ Torque_{W_+}(\gamma_d) &= (0,0,0). \end{aligned}$$

Apply the balancing formula to the compact surface $M \cap S(c,d) \cap K$, then we get

$$
\begin{aligned}
&Torque_{W_-}(\gamma_c) + Torque_{W_-}(\gamma_d) + Torque(E)\\
&\quad = Torque_{W_+}(\gamma_d) - (W_+ - W_-) \wedge Flux(\gamma_d) + Torque(E)\\
&\quad = -(W_+ - W_-) \wedge Flux(\gamma_d) + Torque(E) \qquad (10)\\
&\quad = (0,0,0).
\end{aligned}
$$

If E has the the minimum branching order then the torque does not vanish by (8), so together with (10), we have

$$Torque(E) = (W_+ - W_-) \wedge Flux(\gamma_\epsilon) \neq (0,0,0) \qquad (11)$$

which follows that $W_+ \neq W_-$. Therefore, since $\ell_+ \neq \ell_-$, the minimal surface M cannot have a single axis. This completes the proof of the main theorem.

Remark. Let E be a planar end of a minimal surface M with the minimum branching order which is defined by (4), then for the limit tangent plane Π_β the intersection $M \cap \Pi_\beta$ is asymptotically parallel to a line

$$\gamma_1 x_1 + \gamma_2 x_2 = 0$$

in Π_β. Therefore, by (8), the torque of E is the direction of $M \cap \Pi$ at infinity. Observe that, together with (11), we can say that the difference of base points $W_+ - W_-$ is perpendicular to the direction of $M \cap \Pi$ asymptotically.

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