J. Korean Soc. Math. Educ. Ser. B: Pure Appl. Math. ISSN 1226-0657
Volume 17, Number 4 (November 2010), Pages 397–411

ON THE SUPERSTABILITY OF SOME PEXIDER TYPE FUNCTIONAL EQUATION II

GWANG HUI KIM

ABSTRACT. In this paper, we will investigate the superstability for the sine functional equation from the following Pexider type functional equation :

$$f(x+y)-g(x-y)=\lambda\cdot h(x)k(y) \quad \lambda : \text{constant},$$

which can be considered an exponential type functional equation, the mixed functional equation of the trigonometric function, the mixed functional equation of the hyperbolic function, and the Jensen type equation.

1. INTRODUCTION

In 1940, Ulam [24] proposed the stability problem of the functional equation. Next year, by Hyers [10], this problem was affirmatively solved, which is following:

Let X and Y be Banach spaces with norm $\|\cdot\|$, respectively. If $f : X \to Y$ satisfies

$$\|f(x+y)-f(x)-f(y)\| \le \varepsilon, \qquad \forall x,y \in X,$$

then there exists an unique additive mapping $A : X \to Y$ such that

$$\|f(x)-A(x)\| \le \varepsilon, \qquad \forall x \in X.$$

This result was generalized by Bourgin [7] in 1950. In 1978, the constant boundedness condition of Hyers' result was improved by Th. M. Rassias [23] to the condition bounded by 2-variables, and thereafter it again was improved by P. Găvruta [9] to the condition bounded by the function.

In 1979, Baker, Lawrence, and Zorzitto [4] showed that if f satisfies the inequality

$$|f(x+y)-f(x)f(y)| \le \varepsilon,$$

Received by the editors August 31, 2010. Revised October 29. Accepted November 22, 2010.
2000 *Mathematics Subject Classification.* 39B82, 39B52.
Key words and phrases. stability, superstability, functional equation, d'Alembert equation, (hyperbolic) cosine functional equation.

then either f is bounded or f satisfies the exponential functional equation

$$f(x+y) = f(x)f(y). \tag{1.1}$$

This method is referred to as the superstability of the functional equation (1.1).

In this paper, let $(G,+)$ be an uniquely 2-divisible Abelian group, $\mathbb{C}$ the field of complex numbers, and $\mathbb{R}$ the field of real numbers, $\mathbb{R}_+$ the set of positive reals. Whenever we only deal with (C), $(G,+)$ needs the Abelian which is not 2-divisible.

We may assume that f, g, h and k are nonzero functions, λ, ε is a nonnegative real constant, and $\varphi : G \to \mathbb{R}_+$ is a mapping.

In 1980, the superstability of the cosine functional equation (also referred the d'Alembert functional equation)

$$f(x+y) + f(x-y) = 2f(x)f(y), \tag{C}$$

was investigated by Baker [5] with the following result: let $\varepsilon > 0$. If $f : G \to C$ satisfies

$$|f(x+y) + f(x-y) - 2f(x)f(y)| \leq \varepsilon,$$

then either $|f(x)| \leq (1+\sqrt{1+2\varepsilon})/2$ for all $x \in G$ or f is a solution of Eq.(C).

His result was improved by Badora [3], Badora and Ger [4] under the condition $|f(x+y) + f(x-y) - 2f(x)f(y)| \leq \varphi(x)$ or $\varphi(y)$.

The superstability of the Wilson equation

$$f(x+y) + f(x-y) = 2f(x)g(y) \tag{C_{fg}}$$

was proved by Kannappan, and Kim ([13], [18]) under the condition $|f(x+y) + f(x-y) - 2g(x)f(y)| \leq \varepsilon$, $\varphi(x)$ or $\varphi(y)$, respectively.

In the present work, the stability question about a Pexider type functional equation motivated by some trigonometric function will be investigated. To be systematic, we first list all the functional equations that are of interest here as following:

$$f(x+y) - g(x-y) = \lambda h(x)k(y) \tag{P^{λ}_{fghk}}$$

$$f(x+y) - g(x-y) = \lambda h(x)h(y) \tag{P^{λ}_{fghh}}$$

$$f(x+y) - g(x-y) = \lambda f(x)h(y) \tag{P^{λ}_{fgfh}}$$

$$f(x+y) - g(x-y) = \lambda h(x)f(y) \tag{P^{λ}_{fghf}}$$

$$f(x+y) - g(x-y) = \lambda g(x)h(y) \tag{P^{λ}_{fggh}}$$

$$f(x+y) - g(x-y) = \lambda h(x)g(y) \tag{P^{λ}_{fghg}}$$

(P^{λ}_{fgfg}) $f(x+y)-g(x-y)=\lambda f(x)g(y)$

(P^{λ}_{fggf}) $f(x+y)-g(x-y)=\lambda g(x)f(y)$

(P^{λ}_{fgff}) $f(x+y)-g(x-y)=\lambda f(x)f(y)$

(P^{λ}_{fggg}) $f(x+y)-g(x-y)=\lambda g(x)g(y)$

(T^{λ}_{gh}) $f(x+y)-f(x-y)=\lambda g(x)h(y)$

(T^{λ}_{gg}) $f(x+y)-f(x-y)=\lambda g(x)g(y)$

(T^{λ}_{fg}) $f(x+y)-f(x-y)=\lambda f(x)g(y)$

(T^{λ}_{gf}) $f(x+y)-f(x-y)=\lambda g(x)f(y)$

(T^{λ}) $f(x+y)-f(x-y)=\lambda f(x)f(y)$

(P_{fghh}) $f(x+y)-g(x-y)=2h(x)h(y)$

(P_{fgfh}) $f(x+y)-g(x-y)=2f(x)h(y)$

(P_{fghf}) $f(x+y)-g(x-y)=2h(x)f(y)$

(P_{fggh}) $f(x+y)-g(x-y)=2g(x)h(y)$

(P_{fghg}) $f(x+y)-g(x-y)=2h(x)g(y)$

(P_{fgfg}) $f(x+y)-g(x-y)=2f(x)g(y)$

(P_{fggf}) $f(x+y)-g(x-y)=2g(x)f(y)$

(P_{fgff}) $f(x+y)-g(x-y)=2f(x)f(y)$

(P_{fggg}) $f(x+y)-g(x-y)=2g(x)g(y)$

(T_{gh}) $f(x+y)-f(x-y)=2g(x)h(y)$

(T_{gg}) $f(x+y)-f(x-y)=2g(x)g(y)$

(T_{fg}) $f(x+y)-f(x-y)=2f(x)g(y)$

(T_{gf}) $f(x+y)-f(x-y)=2g(x)f(y)$

(T) $f(x+y)-f(x-y)=2f(x)f(y)$

(J_y) $f(x+y)-f(x-y)=2f(y).$

The hyperbolic cosine function, hyperbolic sine function, hyperbolic trigonometric function, and some exponential functions also satisfy the above mentioned equations, therefore they also can be called the *hyperbolic* cosine(sine, trigonometric) functional equation, exponential, and Jensen functional equation, respectively.

For example,

$$\cosh(x+y) - \cosh(x-y) = 2\sinh(x)\sinh(y)$$
$$\sinh(x+y) - \sinh(x-y) = 2\cosh(x)\sinh(y)$$
$$\sinh^2\left(\frac{x+y}{2}\right) - \sinh^2\left(\frac{x-y}{2}\right) = \sinh(x)\sinh(y)$$
$$ca^{x+y} - ca^{x-y} = 2\frac{ca^x}{2}(a^y - a^{-y}) = 2ca^x\frac{a^y - a^{-y}}{2}$$
$$e^{x+y} - e^{x-y} = 2\frac{e^x}{2}(e^y - e^{-y}) = 2e^x\sinh(y)$$
$$(n(x+y)+c) - (n(x-y)+c) = 2(ny+c) : \text{for } f(x) = nx + c,$$

where a and c are constants.

The investigation of the T type function equations is introduced in paper [16]. In there, author investigated the superstability of the functional equations (T_{fg}, T_{gf}) related to the d'Alembert and the Wilson equation (C, C_{fg}) with the following results:

If $f, g : G \to \mathbb{C}$ satisfies

$$|f(x+y) - f(x-y) - 2g(x)f(y)| \le \varphi(x) \quad (\text{or } \varphi(y)),$$

then either f is bounded or g satisfies (C) (either g is bounded or f satisfies (C)), respectively.

Also, author and Lee [21] proved that $f : G \to \mathbb{C}$ satisfies

$$|f(x+y) - f(x-y) - 2f(x)f(y)| \le \varphi(x) \ \text{ or } \ \varphi(y),$$

then f is bounded.

Thereafter, author have treated the superstability of the generalized trigonometric type functional equations (T_{gg}, T_{gh}, P_{fgfg}, P_{fggf}, P_{fgff}, P_{fggg}) in papers ([14], [16], [17], [20]).

In 1983, Cholewa [8] investigated the superstability of the sine functional equation

$$(S) \qquad f\left(\frac{x+y}{2}\right)^2 - f\left(\frac{x-y}{2}\right)^2 = f(x)f(y),$$

with the following condition:

$$\left|f\left(\frac{x+y}{2}\right)^2 - f\left(\frac{x-y}{2}\right)^2 - f(x)f(y)\right| \le \varepsilon.$$

The superstability of the generalized sine functional equation

$$(S_{gh}) \qquad f\left(\frac{x+y}{2}\right)^2 - f\left(\frac{x-y}{2}\right)^2 = g(x)h(y)$$

was treated by author in ([15], [19]) under the condition bounded by function.

The aim of this paper is to investigate the transferred superstability for the sine functional equation from the following Pexider type functional equations :

$$(P^{\lambda}_{fghk}) \qquad f(x+y) - g(x-y) = \lambda \cdot h(x)k(y), \quad \lambda : constant.$$

on the abelian group. Furthermore, the obtained results can be extended to the Banach space.

As a consequence, we obtain the superstability of the functional equations $((P^{\lambda}_{fghh}), (P^{\lambda}_{fgfh}), (P^{\lambda}_{fghf}), (P^{\lambda}_{fggh}), (P^{\lambda}_{fghg}), (P^{\lambda}_{fgfg}), (P^{\lambda}_{fggf}), (P^{\lambda}_{fgff}), (P^{\lambda}_{fggg}), (T^{\lambda}_{gh}), (T^{\lambda}_{gg}), (T^{\lambda}_{fg}), (T^{\lambda}_{gf}), (T^{\lambda}))$ in order of variables $x+y, x-y, x$, and y, the number of which is $14 \times \lambda \times 4$ (i.e., $\varphi(x), \varphi(y), \min\{\varphi(x), \varphi(y)\}, \varepsilon$).

For simplicity, we will form the notations of the equations as follows :

$$(C^{\lambda}) \qquad f(x+y) + f(x-y) = \lambda f(x)f(y)$$

$$(C^{\lambda}_{fg}) \qquad f(x+y) + f(x-y) = \lambda f(x)g(y)$$

$$(C^{\lambda}_{fg}) \qquad f(x+y) + f(x-y) = \lambda g(x)f(y).$$

2. Superstability for the Sine Equation from the Pexider Type Equation (P^{λ}_{fghk})

In this section, we will investigate the superstability for the sine functional equation from the functional equation (P^{λ}_{fghk}) related to the d'Alembert equation (C), Wilson equation (C_{fg}).

Theorem 1. *Suppose that $f, g, h, k : G \to \mathbb{C}$ satisfy the inequality*

$$|f(x+y) - g(x-y) - \lambda \cdot h(x)k(y)| \leq \varphi(x) \quad \forall\, x, y \in G. \tag{2.1}$$

If k fails to be bounded, then

(i) h satisfies (S) under one of the cases $h(0) = 0$ or $f(x) = g(-x)$,

(ii) In addition, if k satisfies (C^{λ}), then h and k are solutions of Eq. (C^{λ}_{fg}):=

$$h(x+y) + h(x-y) = \lambda h(x)k(y).$$

Proof. Let k be an unbounded solution of the inequality (2.1). Then, there exists a sequence $\{y_n\}$ in G such that $0 \neq |k(y_n)| \to \infty$ as $n \to \infty$.

(i) Taking $y = y_n$ in the inequality (2.1), dividing both sides by $|\lambda k(y_n)|$, and passing to the limit as $n \to \infty$ we obtain that

$$h(x) = \lim_{n\to\infty} \frac{f(x+y_n) - g(x-y_n)}{\lambda \cdot k(y_n)}, \qquad x \in G. \tag{2.2}$$

Replacing y by $y + y_n$ and $-y + y_n$ in (2.1), we have

$$\begin{aligned}&\big|f(x+(y+y_n)) - g(x-(y+y_n)) - \lambda\cdot h(x)k(y+y_n)\\&\quad + f(x+(-y+y_n)) - g(x-(-y+y_n)) - \lambda\cdot h(x)k(-y+y_n)\big| \le 2\varphi(x)\end{aligned}$$

so that

$$\begin{aligned}&\bigg|\frac{f((x+y)+y_n) - g((x+y)-y_n)}{\lambda\cdot k(y_n)}\\&+\frac{f((x-y)+y_n) - g((x-y)-y_n)}{\lambda\cdot k(y_n)} - \lambda\cdot h(x)\cdot\frac{k(y+y_n)+k(-y+y_n)}{\lambda\cdot k(y_n)}\bigg|\\ (2.3)\qquad &\le \frac{2\varphi(x)}{\lambda\cdot|k(y_n)|}\end{aligned}$$

for all $x, y, y_n \in G$.

We conclude that, for every $y \in G$, there exists a limit function

$$l_k(y) := \lim_{n\to\infty}\frac{k(y+y_n)+k(-y+y_n)}{\lambda\cdot k(y_n)},$$

where the function $l_k : G \to \mathbb{C}$ satisfies the equation

$$h(x+y) + h(x-y) = \lambda\cdot h(x)l_k(y) \quad \forall x, y \in G. \tag{2.4}$$

Applying the case $h(0) = 0$ in (2.4), it implies that h is an odd. Keeping this in mind, by means of (2.4), we infer the equality

$$\begin{aligned}h(x+y)^2 - h(x-y)^2 &= \lambda\cdot h(x)l_k(y)[h(x+y) - h(x-y)]\\&= h(x)\big[h(x+2y) - h(x-2y)\big]\\&= h(x)\big[h(2y+x) + h(2y-x)\big]\\&= \lambda\cdot h(x)h(2y)l_k(x).\end{aligned} \tag{2.5}$$

Putting $y = x$ in (2.4) we get the equation

$$h(2x) = \lambda\cdot h(x)l_k(x), \qquad x \in G.$$

This, in return to (2.5), leads to the equation

$$h(x+y)^2 - h(x-y)^2 = h(2x)h(2y) \tag{2.6}$$

valid for all $x, y \in G$ which, in the light of the unique 2-divisibility of G, states nothing else but (S).

In case $f(x) = g(-x)$, it is enough to show that $h(0) = 0$. Suppose that this is not the case.

Putting $x = 0$ in (2.1), due to $h(0) \neq 0$ and $f(x) = g(-x)$, we obtain the inequality

$$|k(y)| \leq \frac{\varphi(0)}{\lambda \cdot |h(0)|}, \quad y \in G.$$

This inequality means that k is globally bounded – a contradiction. Thus, since the claimed $h(0) = 0$ holds, we know that h satisfies (S).

(ii) In the case k satisfies (C^{λ}), the limit l_k states nothing else but k, so, from(2.4), h and k validate Eq. (C^{λ}_{fg}). □

Theorem 2. *Suppose that $f, g, h, k : G \to \mathbb{C}$ satisfy the inequality*

$$|f(x+y) - g(x-y) - \lambda \cdot h(x)k(y)| \leq \varphi(y) \quad \forall\, x, y \in G. \tag{2.7}$$

If h fails to be bounded, then

(i) k satisfies (S) under one of the cases $k(0) = 0$ or $f(x) = g(x)$,

(ii) In addition, if h satisfies (C^{λ}), then h and k are solutions of Eq. (C^{λ}_{fg}):=

$$k(x+y) + k(x-y) = \lambda h(x)k(y).$$

Proof. (i) Taking $x = x_n$ in the inequality (2.7), dividing both sides by $|\lambda \cdot h(x_n)|$, and passing to the limit as $n \to \infty$ we obtain that

$$k(y) = \lim_{n\to\infty} \frac{f(x_n + y) - g(x_n - y)}{\lambda \cdot h(x_n)}, \quad x \in G. \tag{2.8}$$

Replacing x by $x_n + x$ and $x_n - x$ in (2.7), dividing by $\lambda \cdot h(x_n)$, then it gives us the existence of a limit function

$$l_h(x) := \lim_{n\to\infty} \frac{h(x_n + x) + h(x_n - x)}{\lambda \cdot h(x_n)}, \tag{2.9}$$

where the function $l_h : G \to \mathbb{C}$ satisfies the equation

$$k(x+y) + k(-x+y) = \lambda \cdot l_h(x)k(y) \quad \forall x, y \in G. \tag{2.10}$$

Applying the case $k(0) = 0$ in (2.10), it implies that k is an odd.

A similar procedure to that applied after (2.4) of Theorem 1 in equation (2.10) allows us to show that k satisfies (S).

The case $f(x) = g(x)$ also is same reason as Theorem 1.

(ii) In the case h satisfies (C^{λ}), the limit l_h states nothing else but h, so, from (2.10), k and h validate Eq. (C^{λ}_{fg}). □

The following corollaries follow immediately from Theorem 1 and Theorem 2.

Corollary 1. *Suppose that $f, g, h, k : G \to \mathbb{C}$ satisfy the inequality*

$$|f(x+y) - g(x-y) - \lambda \cdot h(x)k(y)| \le \min\{\phi(x), \phi(y)\} \quad \forall\, x, y \in G.$$

(a) If k fails to be bounded, then

(i) h satisfies (S) under one of the cases $h(0) = 0$ or $f(x) = g(-x)$,

(ii) In addition, if k satisfies (C^λ), then h and k are solutions of Eq. (C^λ_{fg}):=

$$h(x+y) + h(x-y) = \lambda h(x)k(y).$$

(b) If h fails to be bounded, then

(iii) k satisfies (S) under one of the cases $k(0) = 0$ or $f(x) = g(x)$,

(iv) In addition, if h satisfies (C^λ), then h and k are solutions of Eq. (C^λ_{fg}):=

$$k(x+y) + k(x-y) = \lambda h(x)k(y).$$

Corollary 2. *Suppose that $f, g, h, k : G \to \mathbb{C}$ satisfy the inequality*

$$|f(x+y) - g(x-y) - \lambda \cdot h(x)k(y)| \le \varepsilon \quad \forall\, x, y \in G.$$

(a) If k fails to be bounded, then

(i) h satisfies (S) under one of the cases $h(0) = 0$ or $f(x) = g(-x)$,

(ii) In addition, if k satisfies (C^λ), then h and k are solutions of Eq. (C^λ_{fg}):=

$$h(x+y) + h(x-y) = \lambda h(x)k(y).$$

(b) If h fails to be bounded, then

(iii) k satisfies (S) under one of the cases $k(0) = 0$ or $f(x) = g(x)$,

(iv) In addition, if h satisfies (C^λ), then h and k are solutions of Eq. (C^λ_{fg}):=

$$k(x+y) + k(x-y) = \lambda h(x)k(y).$$

3. Applications in the Reduced Equations

3.1. Applications of the Equations under Three Unknown Functions. Replacing k by one of the functions f, g, h in all the results of the Section 2, and exchanging each functions f, g, h in the above equations, we then obtain P^λ, T^λ type's 14 equations.

We will only illustrate the results for the cases of Eqs.(P^λ_{fghh}, P^λ_{fgfh}) in the obtained equations. The other cases are similar to these, thus their illustrations will be omitted.

Corollary 3. *Suppose that $f, g, h : G \to \mathbb{C}$ satisfy the inequality*

$$|f(x+y) - g(x-y) - \lambda \cdot h(x)h(y)| \le \begin{cases} \varphi(x) \\ \varphi(y) \\ \min\{\varphi(x), \varphi(y)\} \\ \varepsilon \end{cases} \quad \forall\, x, y \in G. \tag{3.1}$$

If h fails to be bounded, then, under one of the cases $h(0) = 0$, $f(x) = g(x)$, and $f(x) = g(-x)$, h satisfies (S).

Corollary 4. *Suppose that $f, g, h : G \to \mathbb{C}$ satisfy the inequality*

$$|f(x+y) - g(x-y) - \lambda \cdot f(x)h(y)| \le \varphi(x) \quad \forall\, x, y \in G. \tag{3.2}$$

If h fails to be bounded, then

(i) f satisfies (S) under one of the cases $f(0) = 0$ or $f(x) = g(-x)$,

(ii) In addition, if h satisfies (C^λ), then f and h are solutions of Eq. (C_{fg}):=

$$f(x+y) + f(x-y) = \lambda \cdot f(x)h(y).$$

Corollary 5. *Suppose that $f, g, h : G \to \mathbb{C}$ satisfy the inequality*

$$|f(x+y) - g(x-y) - \lambda \cdot f(x)h(y)| \le \varphi(y) \quad \forall\, x, y \in G. \tag{3.3}$$

If f fails to be bounded, then

(i) h satisfies (S) under one of the cases $h(0) = 0$ or $f(x) = g(x)$,

(ii) In addition, if f satisfies (C^λ), then h and f are solutions of Eq. (C^λ_{fg}):=

$$h(x+y) + h(x-y) = \lambda \cdot f(x)h(y).$$

Corollary 6. *Suppose that $f, g, h : G \to \mathbb{C}$ satisfy the inequality*

$$|f(x+y) - g(x-y) - \lambda \cdot f(x)h(y)| \le \begin{cases} \min\{\varphi(x), \varphi(y)\} \\ \varepsilon \end{cases} \quad \forall\, x, y \in G. \tag{3.4}$$

(a) If h fails to be bounded, then

(i) f satisfies (S) under one of the cases $f(0) = 0$ or $f(x) = g(-x)$,

(ii) In addition, if h satisfies (C^λ), then f and h are solutions of Eq. (C_{fg}):=

$$f(x+y) + f(x-y) = \lambda \cdot f(x)h(y).$$

(b) If f fails to be bounded, then

(i) h satisfies (S) under one of the cases $h(0) = 0$ or $f(x) = g(x)$,

(ii) In addition, if f satisfies (C^λ), then h and f are solutions of Eq. (C^λ_{fg}):= $h(x+y) + h(x-y) = \lambda \cdot f(x)h(y)$.

Remark 1. As the above corollaries, we obtain the stability results of 12×4 $(\varphi(x), \varphi(y), \min\{\varphi(x), \varphi(y)\}, \varepsilon)$ numbers for the other 12 equations, which are the followings : ((P^{λ}_{fghf}), (P^{λ}_{fggh}), (P^{λ}_{fghg}), (P^{λ}_{fgfg}), (P^{λ}_{fggf}), (P^{λ}_{fgff}), (P^{λ}_{fggg}), (T^{λ}_{gh}), (T^{λ}_{gg}), (T^{λ}_{fg}), (T^{λ}_{gf}), (T^{λ})).

3.2. Applications of the Case $\lambda = 2$ in P^{λ} Type Equations. Let us apply the case $\lambda = 2$ in all P^{λ} type equations considered in the Section 2 and Subsection 3.1.

Then, we obtain the P type equations

$$(P_{fghk}) \qquad f(x+y) - g(x-y) = 2h(x)k(y),$$

and $((P_{fghh})$, (P_{fgfh}), (P_{fghf}), (P_{fggh}), (P_{fghg}), (P_{fgfg}), (P_{fggf}), (P_{fgff}), $(P_{fggg}))$, and T and J types ((T_{fg}), (T_{gf}), (T_{gg}), (T_{gh}), (T), (J_y)), which are concerned with the (hyperbolic) cosine, sine, exponential functions, and Jensen equation.

In papers (Aczel [1], Aczel and Dhombres [2], Kannappan ([11], [12]), Kannappan and author [13]), we can find that the Wilson equation and the sine equations can be represented by the composition of a homomorphism. By applying these results, we also obtain, additionally, the explicit solutions of the considered functional equations.

For simplicity, we will only show the result for the case of Eqs.$(P^{\lambda}_{fghk}, P^{\lambda}_{fghh})$ in the above equations. The other cases are similar to this, thus their illustrations will be omitted.

Corollary 7. *Suppose that $f, g, h, k : G \to \mathbb{C}$ satisfy the inequality*

$$(3.5) \qquad |f(x+y) - g(x-y) - 2h(x)k(y)| \le \varphi(x) \quad \forall\, x, y \in G.$$

If k fails to be bounded, then

(i) h satisfies (S) under one of the cases $h(0) = 0$ or $f(x) = g(-x)$, and h is of the form

$$h(x) = A(x) \quad or \quad h(x) = c(E(x) - E^*(x)),$$

where $A : G \to \mathbb{C}$ is an additive function, $c \in \mathbb{C}$, $E : G \to \mathbb{C}^*$ is a homomorphism and $E^* = 1/E(x)$.

(ii) In addition, if k satisfies (C), then h and k are solutions of the of Eq. (C_{fg}) and h, k are given by

$$k(x) = \frac{E(x) + E^*(x)}{2}, \quad h(x) = c(E(x) - E^*(x)) + \frac{d(E(x) + E^*(x))}{2},$$

where $c, d \in \mathbb{C}$, E and E^* are as in (i).

Proof. The proof is enough from Theorem 1 except for the explicit solutions, then they are immediate from the following:

(i) Appealing to the solutions of (S) in ([12], p.153) (see also [13,14]).

(ii) The given explicit solutions are taken from [11] and [12](pp. 148) (see also [1][2]). □

Corollary 8. *Suppose that $f, g, h, k : G \to \mathbb{C}$ satisfy the inequality*

$$|f(x+y) - g(x-y) - 2h(x)k(y)| \leq \varphi(y) \quad \forall\, x, y \in G. \tag{3.6}$$

If h fails to be bounded, then

(i) k satisfies (S) under one of the cases $h(0) = 0$ or $f(x) = g(-x)$, and k is of the form

$$k(x) = A(x) \quad or \quad k(x) = c(E(x) - E^*(x)),$$

where $A : G \to \mathbb{C}$ is an additive function, $c \in \mathbb{C}$, $E : G \to \mathbb{C}^*$ is a homomorphism and $E^* = 1/E(x)$.

(ii) In addition, if h satisfies (C), then k and h are solutions of the of Eq. (C_{fg}) and k, h are given by

$$h(x) = \frac{E(x) + E^*(x)}{2}, \quad k(x) = c(E(x) - E^*(x)) + \frac{d(E(x) + E^*(x))}{2},$$

where $c, d \in \mathbb{C}$, E and E^* are as in (i).

Corollary 9. *Suppose that $f, g, h, k : G \to \mathbb{C}$ satisfy the inequality*

$$|f(x+y) - g(x-y) - 2h(x)k(y)| \leq \begin{cases} \min\{\varphi(x), \varphi(y)\} \\ \varepsilon \end{cases} \quad \forall\, x, y \in G. \tag{3.7}$$

(a) If k fails to be bounded, then

(i) h satisfies (S) under one of the cases $h(0) = 0$ or $f(x) = g(-x)$, and h is of the form

$$h(x) = A(x) \quad or \quad h(x) = c(E(x) - E^*(x)),$$

where $A : G \to \mathbb{C}$ is an additive function, $c \in \mathbb{C}$, $E : G \to \mathbb{C}^*$ is a homomorphism and $E^* = 1/E(x)$.

(ii) In addition, if k satisfies (C), then h and k are solutions of the of Eq. (C_{fg}) and h, k are given by

$$k(x) = \frac{E(x) + E^*(x)}{2}, \quad h(x) = c(E(x) - E^*(x)) + \frac{d(E(x) + E^*(x))}{2},$$

where $c, d \in \mathbb{C}$, E and E^* are as in (i).

(b) If h fails to be bounded, then

(i) k satisfies (S) under one of the cases $h(0)=0$ or $f(x)=g(-x)$, and k is of the form

$$k(x)=A(x) \quad or \quad k(x)=c(E(x)-E^*(x)),$$

where $A:G\to\mathbb{C}$ is an additive function, $c\in\mathbb{C}$, $E:G\to\mathbb{C}^*$ is a homomorphism and $E^*=1/E(x)$.

(ii) In addition, if h satisfies (C), then k and h are solutions of the of Eq. (C_{fg}) and k,h are given by

$$h(x)=\frac{E(x)+E^*(x)}{2}, \quad k(x)=c(E(x)-E^*(x))+\frac{d(E(x)+E^*(x))}{2},$$

where $c,d\in\mathbb{C}$, E and E^* are as in (i).

Corollary 10. *Suppose that $f,g,h:G\to\mathbb{C}$ satisfy the inequality*

$$(3.8) \quad |f(x+y)-g(x-y)-2\cdot h(x)h(y)|\leq\begin{cases}\varphi(x)\\ \varphi(y)\\ \min\{\varphi(x),\varphi(y)\}\\ \varepsilon\end{cases} \qquad \forall\, x,y\in G.$$

If h fails to be bounded, then, under one of the cases $h(0)=0$, $f(x)=g(x)$, and $f(x)=g(-x)$, h satisfies (S).

In here, h is of the form

$$h(x)=A(x) \quad or \quad h(x)=c(E(x)-E^*(x)),$$

where $A:G\to\mathbb{C}$ is an additive function, $c\in\mathbb{C}$, $E:G\to\mathbb{C}^*$ is a homomorphism and $E^*=1/E(x)$.

Remark 2. (i)) Except for Eqs.(P_{fghk}, P_{fghh}) in P type equations, we obtain the results in the same types for the Eqs.((P_{fgfh}), (P_{fghf}), (P_{fggh}), (P_{fghg}), (P_{fgfg}), (P_{fggf}), (P_{fgff}), (P_{fggg}), (T_{fg}), (T_{gf}), (T_{gg}), (T_{gh})).

Some of the obtained results are found in papers ([4], [14], [16], [17], [20]).

(ii) But, for the trigonometric functional equation (T), we can not assume an unboundedness, namely, it has to be bounded. (see, [21]).

4. Extension to the Banach Space

All the results presented in the Section 2 and Section 3 can be apply to a semisimple commutative Banach space. Hence, interesting results about operators can be obtained. We will represent just for the main equation (P^{λ}_{fghk}).

Theorem 3. *Let $(B, \|\cdot\|)$ be a semisimple commutative Banach space. Assume that $f, g, h, k : G \to B$ satisfy one of each inequalities*

$$\|f(x+y) - g(x-y) - \lambda \cdot h(x)k(y)\| \le \varphi(x) \tag{4.1}$$

$$\|f(x+y) - g(x-y) - \lambda \cdot h(x)k(y)\| \le \varphi(y) \tag{4.2}$$

for all $x, y \in G$. Let an arbitrary linear multiplicative functional $x^* \in B^*$.

(a) case (4.1).

Suppose that $x^* \circ k$ fails to be bounded, then

(i) h satisfies (S) under one of the cases $(x^* \circ h)(0) = 0$ or $(x^* \circ f)(x) = (x^* \circ g)(-x)$,

(ii) In addition, if k satisfies (C^λ), then h and k are solutions of Eq. (C^λ_{fg}).

(b) case (4.2).

Suppose that $x^* \circ h$ fails to be bounded, then

(iii) k satisfies (S) under one of the cases $(x^* \circ k)(0) = 0$ or $(x^* \circ f)(x) = (x^* \circ g)(x)$,

(iv) In addition, if $x^* \circ h$ satisfies (C^λ), then h and k are solutions of Eq. (C^λ_{fg}).

Proof. (i) of (a). Assume that (4.1) holds, and arbitrarily fix a linear multiplicative functional $x^* \in B^*$. As is well known, we have $\|x^*\| = 1$, hence, for every $x, y \in G$, we have

$$\begin{aligned}\varphi(x) &\ge \|f(x+y) - g(x-y) - \lambda \cdot h(x)k(y)\| \\ &= \sup_{\|y^*\|=1} \left|y^*\big(f(x+y) - g(x-y) - \lambda \cdot h(x)k(y)\big)\right| \\ &\ge \left|x^*\big(f(x+y)\big) - x^*\big(g(x-y)\big) - \lambda \cdot x^*\big(h(x)\big)x^*\big(k(y)\big)\right|,\end{aligned}$$

which states that the superpositions $x^* \circ f$, $x^* \circ g$, $x^* \circ h$ and $x^* \circ k$ yield a solution of inequality (2.1) in Theorem 1. Since, by assumption, the superposition $x^* \circ k$ with $(x^* \circ h)(0) = 0$ is unbounded, an appeal to Theorem 1 shows that the two results hold.

First, the function $x^* \circ h$ solves (S). Namely

$$(x^* \circ h)\Big(\frac{x+y}{2}\Big)^2 - (x^* \circ h)\Big(\frac{x-y}{2}\Big)^2 = (x^* \circ h)(x)(x^* \circ h)(y)$$

In other words, bearing the linear multiplicativity of x^* in mind, for all $x, y \in G$, the difference $\mathcal{D}S(x, y) : G \times G \to \mathbb{C}$ defined by

$$\mathcal{D}S(x,y) := h(\frac{x+y}{2})^2 - h(\frac{x-y}{2})^2 - h(x)h(y),$$

falls into the kernel of x^*. Therefore, in view of the unrestricted choice of x^*, that implies that

$$\mathcal{D}S(x,y) \in \bigcap\{\ker x^* : x^* \text{ is a multiplicative member of } E^*\}$$

for all $x, y \in G$. Since the space B is semisimple, that is

$$\mathcal{D}S(x,y) = 0 \quad \text{for all} \quad x, y \in G,$$

as claimed.

Second, in particular, if $x^* \circ k$ satisfies (C^λ), then $x^* \circ h$ and $x^* \circ k$ are solutions of the Wilson type equation $h(x+y) + h(x-y) = \lambda h(x)k(y)$. This means that

$$\mathcal{D}C^\lambda_{hk}(x,y) := h(x+y) + h(x-y) - \lambda h(x)k(y),$$

falls into the kernel of x^*. Through the above process, we obtain

$$\mathcal{D}C^\lambda_{hk}(x,y) = 0 \quad \text{for all} \quad x, y \in G,$$

as claimed.

(ii) Case (4.2) also runs along the proof of case (4.1). □

Remark 3. (i) In the same processing of Theorem 3, we obtain similar results for the case $||\mathcal{D}P^\lambda_{fghk}(x,y)|| := \|f(x+y) - g(x-y) - \lambda \cdot h(x)k(y)\| \leq \min\{\varphi(x), \varphi(y)\}$ or ε.

(ii) All results of the Section 2 and Section 3 containing the Remarks 1, 2 can be applied to the semisimple commutative Banach space in the same method. Then we can obtain same results for the all equations of the P^λ, P, T types except for (T) and (T^λ). Some of them are found in papers ([4], [14], [17], [20]).

REFERENCES

1. J. Aczel: *Lectures on Functional Equations and their Applications.* Academic Press, New York, 1966.
2. J. Aczel & J. Dhombres: *Functional Equations in Several Variables.* Cambridge University Press, Cambridge, 1988.
3. R. Badora: On the stability of cosine functional equation. *Rocznik Naukowo-Dydak. Prace Mat.* **15** (1998), 1–14.
4. R. Badora & R. Ger: On some trigonometric functional inequalities. *Functional Equations- Results and Advances* (2002), 3–15.
5. J.A. Baker: The stability of the cosine equation. *Proc. Amer. Math. Soc.* **80** (1980), 411–416.

6. J.A. Baker, J. Lawrence & F. Zorzitto: The stability of the equation $f(x+y) = f(x)f(y)$. *Proc. Amer. Math. Soc.* **74** (1979), 242–246.
7. D.G. Bourgin: Approximately isometric and multiplicative transformations on continuous function rings. *Duke Math. J.* **16** (1949), 385–397.
8. P.W. Cholewa: The stability of the sine equation. *Proc. Amer. Math. Soc.* **88** (1983), 631–634.
9. P. Găvruta: *On the stability of some functional equations.* in : Th.M. Rassias and J. Tabor, Stability of mappings of Hyers-Ulam type, Hadronic Press, 1994, pp. 93–98.
10. D.H. Hyers: On the stability of the linear functional equation. *Proc. Natl. Acad. Sci.* **27** (1941), 222-224.
11. Pl. Kannappan: The functional equation $f(xy)+f(xy^{-1}) = 2f(x)f(y)$ for groups. *Proc. Amer. Math. Soc.* **19** (1968), 69–74.
12. Pl. Kannappan: *Functional equations and inequailitis with applications.* Springer, 2009.
13. ______& G.H. Kim: On the stability of the generalized cosine functional equations. *Ann. Acad. Pedagog. Crac. Stud. Math.* **1** (2001), 49–58.
14. G.H. Kim: On the Stability of the Pexiderized trigonometric functional equation. *Appl. Math. Comput.* **203** (2008), 99–105
15. ______: On the Stability of the generalized sine functional equations. *Acta Math. Sin., Engl. Ser.* **25** (2009), 29-38.
16. ______: On the Stability of trigonometric functional equations, Ad. Diff. Eq., Vol 2007, Article ID 90405, (2007).
17. ______: On the stability of Mixed Trigonometric Functional Equations. *Banach J. Math. Anal.* **1** (2007), no. 2, 227-236.
18. ______: The stability of the d'Alembert and Jensen type functional equations. *J. Math. Anal. Appl.* **325** (2007), 237–248.
19. ______: A stability of the generalized sine functional equations. *J. Math. Anal. Appl.* **331** (2007), 886–894.
20. ______& Y.H. Lee: The superstability of the Pexider type trigonometric functional equation. *Aust. J. Math. Anal.*, preprint
21. ______& Y.H. Lee: Boundedness of approximate trigonometric functional equations. *Appl. Math. Lett.* **331** (2009), 439–443.
22. ______& Sever S. Dragomir: On the stability of generalized d'Alembert and Jensen functional equation. *Intern. J. Math. & Math. Sci.* 2006 Article ID 43185(2006), 1-12.
23. Th.M. Rassias: On the stability of the linear mapping in Banach spaces. *Proc. Amer. Math. Soc.* **72** (1978), 297-300.
24. S.M. Ulam: "*Problems in Modern Mathematics*" Chap. VI, Science editions, Wiley, New York, 1964

Department of Mathematics, Kangnam University, Yongin, Gyounggi, 446-702, Korea
Email address: ghkim@kangnam.ac.kr