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ON SOME PROPERTIES OF TOPOS $E(\Omega, \mathcal{A})$

IG SUNG KIM

ABSTRACT. Category $E(\Omega, \mathcal{A})$ forms a topos. We study on some properties of the topos $E(\Omega, \mathcal{A})$. In particular, we show that $E(\Omega, \mathcal{A})$ is well-pointed.

1. INTRODUCTION

Let $(\Omega, \mathcal{A})$ be fixed ample space. Then the set of all possibilistic set for $(\Omega, \mathcal{A})$ yield a category $E(\Omega, \mathcal{A})$. Yuan [4] showed that $E(\Omega, \mathcal{A})$ is a topos.

In this paper, we study some properties of the topos $E(\Omega, \mathcal{A})$. In particular, we show that $E(\Omega, \mathcal{A})$ is a well-pointed topos. Finally, by the use of the concepts of the topos $E(\Omega, \mathcal{A})$, we obtain the logic operators of possibilistic sets.

2. PRELIMINARIES

In this section, we state some definitions and properties which will serve as the basic tools for the arguments used to prove our results.

Definition 2.1. An *elementary topos* is a category $\mathcal{E}$ that satisfies the following;

(T1) $\mathcal{E}$ is finitely complete,

(T2) $\mathcal{E}$ has exponentiation,

(T3) $\mathcal{E}$ has a subobject classifier.

(T2) means that for every object A in $\mathcal{E}$, the endofunctor $(-) \times A$ has its right adjoint $(-)^A$. Hence for every object A in $\mathcal{E}$, there exists an object B^A, and a morphism $ev_A : B^A \times A \to B$, called the evaluation map of A, such that for any Y

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and $f : Y\times A \to B$ in $\mathcal{E}$, there exists a unique morphism g such that $ev_A\circ(g\times id) = f$;

$$\begin{array}{ccc} Y\times A & \xrightarrow{f} & B \\ {\scriptstyle g\times id}\downarrow & & \downarrow{\scriptstyle id} \\ B^A\times A & \xrightarrow[ev_A]{} & B \end{array}$$

And subobject classifier in (T3) is an $\mathcal{E}$-object Ω, together with a morphism $\top : \mathbf{1} \to \Omega$ such that for any monomorphism $h : D \to C$, there is a unique morphism $\chi_h : C \to \Omega$, called the character of $h : D \to C$ which makes the following diagram a pull-back;

$$\begin{array}{ccc} D & \xrightarrow{!} & \mathbf{1} \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle \top} \\ C & \xrightarrow[\chi_h]{} & \Omega \end{array}$$

Example 2.2. Category *Set* is a topos. $\{*\}$ is a terminal object. $\Omega = \{0,1\}$ and $\top : \{*\} \to \Omega$ with $\top(*) = 1$ is a subobject classifier. If we define

$\chi_h = 1$ if $c = h(d)$ for some $d \in D$,

$\chi_h = 0$ otherwise

then χ_h is a characteristic function of D .

Definition 2.3. A topos $\mathcal{E}$ is called *classical* if $[\top, \bot] : 1+1 \to \Omega$ is an isomorphism.

Definition 2.4. A topos $\mathcal{E}$ is called *Boolean* if for every object D in $\mathcal{E}$, $(\mathrm{Sub}(D), \in)$ is a Boolean algebra where $\mathrm{Sub}(D)$ is the class of monomorphism with common codomain D, and $g \in f$ if there exists a morphism $h : B \to A$ such that $f \circ h = g$ where $f : A \to D$ and $g : B \to D$ are monomorphisms.

Lemma 2.5 ([4]). *For any topos $\mathcal{E}$, the following statements are equivariant;*

(1) *$\mathcal{E}$ is Boolean.*

(2) *$Sub(\Omega)$ is a Boolean algebra.*

(3) *$\top : \mathbf{1} \to \Omega$ has a complement in $Sub(\Omega)$.*

(4) *$\bot : \mathbf{1} \to \Omega$ is the complement of $\top$ in $Sub(\Omega)$.*

(5) *$\top \cup \bot \simeq 1_\Omega$ in $Sub(\Omega)$.*

(6) *$\mathcal{E}$ is classical.*

(7) *$i_1 : \mathbf{1} \to \mathbf{1} + \mathbf{1}$ is a subobject classifier.*

Definition 2.6. A topos is called *well-pointed* if it satisfies the extentionality principle for morphisms, i.e., If $f, g : A \to B$ are a pair of distinct parallel morphisms, then there is an element $a : \mathbf{1} \to A$ of A such that $f \circ a \neq g \circ a$.

Definition 2.7. Let X be a set and $\mathcal{A}$ be a subset of power set $P(X)$ of X. If (1) $X \in \mathcal{A}$

(2) $A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A}$

(3) For any index set I, $A_i \in \mathcal{A} \Rightarrow \cup A_i \in \mathcal{A}$

Then A is called an ample field over X and $(X, \mathcal{A})$ is called an ample space.

Definition 2.8. Let $(X, \mathcal{A})$ be an ample space, then

$[x] = \bigcap \{A | x \in A \in \mathcal{A}\}$

is called an atom of $\mathcal{A}$ containing the element $x \in X$.

Definition 2.9. Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be two ample spaces. Let

$y' = \{B | y \in B \in \mathcal{B}\}$, $Y' = \{y' | y \in Y\}$

Let $\bar{\mathcal{B}}$ be an ample field generated by Y'. If set-valued mapping $\xi : X \to P(Y)$ satisfies $B \in \bar{\mathcal{B}} \Rightarrow \xi^{-1}(B) \in \mathcal{A}$

then ξ is called a possibilistic set from $(X, \mathcal{A})$ to $(Y, \mathcal{B})$ and $E(X, \mathcal{A}; Y, \mathcal{B})$ denote a set of all possibilistic sets from $(X, \mathcal{A})$ to $(Y, \mathcal{B})$.

Let $(\Omega, \mathcal{A})$ be a fixed ample field and $\xi \in E(\Omega, \mathcal{A}; X, \mathcal{B})$ be denoted as $(X, \mathcal{B}, \xi)$. Let $E(\Omega, \mathcal{A})$ be a category, its objects be possibilistic sets $(X, \mathcal{B}, \xi)$ satisfying $X = \cup \xi(\omega)$; a morphism from $(X_1, \mathcal{B}_1, \xi_1)$ to $(X_2, \mathcal{B}_2, \xi_2)$ be a mapping $f : X_1 \to X_2$ satisfying $f^{-1}(\xi_2(\omega)) = \xi_1(\omega)$ for all $\omega \in \Omega$. Then we have that the category $E(\Omega, \mathcal{A})$ is a topos [ ].

3. Some Properties of Topos $E(\Omega, \mathcal{A})$

Theorem 3.1. *Topos $E(\Omega, \mathcal{A})$ is bivalent.*

Proof. Let $\Delta = \mathcal{A}^* \times \mathbf{2}$, where $\mathcal{A}^* = \mathcal{A}/\phi$ and $\mathbf{2} = \{0, 1\}$. Then $(\Delta, \mathcal{B}_\Delta, \delta)$, where $\delta : \Omega \to P(\Delta)$ is a mapping defined by $\delta(\omega) = \omega' \times \mathbf{2}$, is a subobject classifier. And $(\mathcal{A}^*, \mathcal{B}^*, \varepsilon^*)$, where $\mathcal{B}^* = [\Omega']$ with $\Omega' = \{\omega' \mid \omega \in \Omega\}$ and $\varepsilon^*(\omega) = \omega' = \{A \mid \omega \in A \in \mathcal{A}\}$, is a terminal object. For any $\varepsilon^* : \Omega \to P(\mathcal{A}^*)$ and $\delta : \Omega \to P(\Delta)$, there exist $\top : \mathcal{A}^* \to \Delta$ with $\top(A) = (A, 1)$ and $\bot : \mathcal{A}^* \to \Delta$ with $\bot(A) = (A, 0)$.

$$\begin{array}{ccc} \Omega & \xrightarrow{\varepsilon^*} & P(\mathcal{A}^*) \\ \| & & \downarrow \\ \Omega & \xrightarrow[\delta]{} & P(\Delta) \end{array}$$

□

Lemma 3.2. *Finite coproducts exist in $E(\Omega, \mathcal{A})$.*

Proof. Let $(X_1, \mathcal{B}_1, \varepsilon_1)$ and $(X_2, \mathcal{B}_2, \varepsilon_2)$ be two objects in $E(\Omega, \mathcal{A})$. Let $\mathcal{B} = [\mathcal{B}_1 \bigsqcup \mathcal{B}_2] \bigcap (X_1 \bigsqcup X_2)$ where $X_1 \bigsqcup X_2$ is the disjoint union of X_1 and X_2, and $\varepsilon : \Omega \to P(X_1 \bigsqcup X_2)$ is a set valued mapping defined by $\varepsilon(\omega) = \varepsilon_1(\omega) \bigsqcup \varepsilon_2(\omega)$. Then $((X_1 \bigsqcup X_2), \mathcal{B}, \varepsilon)$ is an object in $E(\Omega, \mathcal{A})$. Since $i_1^{-1} \circ \varepsilon = \varepsilon_1$ and $i_2^{-1} \circ \varepsilon = \varepsilon_2$ where $i_1 : X_1 \to X_1 \bigsqcup X_2$ and $i_2 : X_2 \to X_1 \bigsqcup X_2$, $((X_1 \bigsqcup X_2, \mathcal{B}, \varepsilon), i_1, i_2)$ is a finite coproduct of $(X_1, \mathcal{B}_1, \varepsilon_1)$ and $(X_2, \mathcal{B}_2, \varepsilon_2)$. □

Theorem 3.3. *Topos $E(\Omega, \mathcal{A})$ is classical.*

Proof. Since $(\mathcal{A}^*, \mathcal{B}^*, \varepsilon^*)$, where $\mathcal{B}^* = [\Omega']$ with $\Omega' = \{\omega' \mid \omega \in \Omega\}$ and $\varepsilon^*(\omega) = \omega' = \{A \mid \omega \in A \in \mathcal{A}\}$, is a terminal object, by lemma 3.2 $((\mathcal{A}^* \bigsqcup \mathcal{A}^*, \mathcal{B}, \varepsilon), i_1, i_2)$ is a coproduct of $(\mathcal{A}^*, \mathcal{B}^*, \varepsilon^*)$ and $(\mathcal{A}^*, \mathcal{B}^*, \varepsilon^*)$.

$$
\begin{array}{ccc}
\Omega & \xrightarrow{\varepsilon^*} & P(\mathcal{A}^*) \\
\| & & \downarrow \\
\Omega & \xrightarrow{\varepsilon} & P(\mathcal{A}^* \bigsqcup \mathcal{A}^*) \\
\| & & \uparrow \\
\Omega & \xrightarrow{\varepsilon^*} & P(\mathcal{A}^*)
\end{array}
$$

For any $\delta : \Omega \to P(\Delta)$ and $\top : \mathcal{A}^* \to \Delta$, $\bot : \mathcal{A}^* \to \Delta$, there exists a $f : \mathcal{A}^* \bigsqcup \mathcal{A}^* \to \Delta$ defined by $f(A, 0) = (A, 0)$ and $f(0, A) = (A, 1)$ such that $f \circ i_1 = \top$ and $f \circ i_2 = \bot$. Since f is bijective, $E(\Omega, \mathcal{A})$ is classical. □

Theorem 3.4. *Topos $E(\Omega, \mathcal{A})$ is Boolean.*

Proof. Lemma 2.5 and Theorem 3.3. □

Theorem 3.5. *Topos $E(\Omega, \mathcal{A})$ is well-pointed.*

Proof. Let $(X_1, \mathcal{B}_1, \varepsilon_1)$ and $(X_2, \mathcal{B}_2, \varepsilon_2)$ be two objects in $E(\Omega, \mathcal{A})$. Let $f, g : X_1 \to X_2$ such that $f \neq g$. Then there exists $x \in X_1$ such that $f(x) \neq g(x)$. Since $X_1 = \bigcup \varepsilon_1(\omega)$, there exists $\omega \in \Omega$ such that $\varepsilon_1(\omega) = x$. Construct $k : \mathcal{A}^* \to X_1$ defined by $k(A) = x$, then we get that $k(A) = \varepsilon_1(\omega)$. So we have that $f \circ k \neq g \circ k$.

$$
\begin{array}{ccccc}
\Omega & = & \Omega & = & \Omega \\
\downarrow{\scriptstyle \varepsilon^*} & & \downarrow{\scriptstyle \varepsilon_1} & & \downarrow{\scriptstyle \varepsilon_2} \\
P(\mathcal{A}^*) & \longrightarrow & P(X_1) & \longrightarrow & P(X_2)
\end{array}
$$

□

4. Logic Operations of Topos $E(\Omega, \mathcal{A})$

(1) Negation:

$\neg : \Delta \to \Delta$ is the characteristic function of the $\perp : \mathcal{A}^* \to \Delta$ where $\perp(A) = (A, 0)$, that is, the following diagram

$$\begin{array}{ccc} P(\mathcal{A}^*) & \longrightarrow & P(\mathcal{A}^*) \\ \downarrow & & \downarrow \\ P(\Delta) & \longrightarrow & P(\Delta) \end{array}$$

is a pull-back. Since $\delta : \Omega \to P(\Delta)$ defined by $\delta(\omega) = \omega' \times \mathbf{2}$ and $\varepsilon^* : \Omega \to P(\mathcal{A}^*)$ defined by $\varepsilon^*(\omega) = \omega'$, we have that $\perp^{-1} \circ \delta(\omega) = \perp^{-1}(\omega' \times \mathbf{2}) = \{A|(A, 0) \in \omega' \times \mathbf{2}\} = \{A|A \in \omega'\} = \omega' = \varepsilon^*(\omega)$.

(2) Conjunction:

$\bigcap : \Delta \times \Delta \to \Delta$ is the characteristic function of the $(\top, \top) : \mathcal{A}^* \to \Delta \times \Delta$ where $(\top, \top)(A) = ((A, 1), (A, 1))$, that is, the following diagram

$$\begin{array}{ccc} P(\mathcal{A}^*) & \longrightarrow & P(\mathcal{A}^*) \\ \downarrow & & \downarrow \\ P(\Delta \times \Delta) & \longrightarrow & P(\Delta) \end{array}$$

is a pull-back. Since $\theta : \Omega \to P(\Delta \times \Delta)$ defined by $\theta(\omega) = (\omega' \times \mathbf{2}, \omega' \times \mathbf{2})$ and $\varepsilon^* : \Omega \to P(\mathcal{A}^*)$ defined by $\varepsilon^*(\omega) = \omega'$, we have that $(\top, \top)^{-1} \circ \delta(\omega) = (\top, \top)^{-1}(\omega' \times \mathbf{2}) = \{A|((A, 1), (A, 1)) \in (\omega' \times \mathbf{2}) \times (\omega' \times \mathbf{2})\} = \{A|A \in \omega'\} = \omega' = \varepsilon^*(\omega)$.

(3) Implication:

$\Rightarrow : \Delta \times \Delta \to \Delta$ is the characteristic function of the inclusion $h : Im \to \Delta \times \Delta$ where $Im = \{((\omega', s), (\omega', t)) \in (\omega' \times \mathbf{2}) \times (\omega' \times \mathbf{2})|s, t = 0, 1, s \le t,\}$ and $h(\omega', q) = (\omega', q)$, that is, the following diagram

$$\begin{array}{ccc} P(Im) & \longrightarrow & P(\mathcal{A}^*) \\ \downarrow & & \downarrow \\ P(\Delta \times \Delta) & \longrightarrow & P(\Delta) \end{array}$$

is a pull-back. Since $\theta : \Omega \to P(\Delta \times \Delta)$ defined by $\theta(\omega) = ((\omega' \times \mathbf{2}), (\omega' \times \mathbf{2}))$ and $\eta : \Omega \to Im$ defined by $\eta(\omega) = \{((\omega', s), (\omega', t)) \in (\omega' \times \mathbf{2}) \times (\omega' \times \mathbf{2})|s, t = 0, 1, s \le t\}$, we have that $h^{-1} \circ \theta(\omega) = \eta(\omega)$.

(4) Disjunction:

$\bigcup : \Delta \times \Delta \to \Delta$ is the characteristic function of the inclusion $k : D \to \Delta \times \Delta$ where $D = \{((\omega', s), (\omega', t)) \in (\omega' \times \mathbf{2}) \times (\omega' \times \mathbf{2})|s, t = 0, 1, s + t \ge 1\}$ and $k(\omega', q) = (\omega', q)$, that is, the following diagram

$$\begin{array}{ccc} P(D) & \longrightarrow & P(\mathcal{A}^*) \\ \downarrow & & \downarrow \\ P(\Delta \times \Delta) & \longrightarrow & P(\Delta) \end{array}$$

is a pull-back. Since $\theta : \Omega \to P(\Delta \times \Delta)$ defined by $\theta(\omega) = ((\omega' \times \mathbf{2}), (\omega' \times \mathbf{2}))$ and $\psi : \Omega \to D$ defined by $\psi(\omega) = \{((\omega', s), (\omega', t)) \in (\omega' \times \mathbf{2}) \times (\omega' \times \mathbf{2}) | s, t = 0, 1, s \leq t\}$, we have that $k^{-1} \circ \theta(\omega) = \psi(\omega)$.

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DEPARTMENT OF DATA INFORMATION, SANGJI UNIVERSITY, WONJU, KANGWON 220-702, KOREA
Email address: iskim@sangji.ac.kr