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EXTENSIONS OF MINIMIZATION THEOREMS AND FIXED POINT THEOREMS ON A D^*-METRIC SPACE

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Abstract. In this paper, we introduce the new concept of w-D^*-distance on a D^*-metric space and prove a non-convex minimization theorem which improves the result of Caristi[1], Ćirić[2], Ekeland[4], Kada et al.[5] and Ume[8, 9].

1. Introduction

Caristi[1] proved a fixed point theorem on complete metric spaces which generalized the Banach contraction principle. Ekeland[4] also obtained a non-convex minimization theorem, often called the ε-variational principle for a proper lower semicontinuous function, bounded from below, on complete metric spaces. Later, Takahashi[7] proved the following minimization theorem. Let X be a complete metric space and let $f : X \to (-\infty, \infty]$ be a proper lower semicontinuous function, bounded from below. Suppose that, for each $u \in X$ with $f(u) > \inf_{x\in X} f(x)$, there exists $v \in X$ such that $v \neq u$ and $f(v) + d(u, v) \leq f(u)$. Then, there exists $x_0 \in X$ such that $f(x_0) = \inf_{x\in X} f(x)$. In 1996, Kada et al.[5] introduced the concept of w-distance on a metric space as follows. Let X be a metric space with metric d. Then a function $p : X \times X \to [0, \infty)$ is called a w-distance on X if the following are satisfied:

(1) $p(x, z) \leq p(x, y) + p(y, z)$ for any $x, y, z \in X$,
(2) for any $x \in X$, $p(x, \cdot) : X \to [0, \infty)$ is lower semicontinuous,
(3) for any $\varepsilon > 0$, there exists $\delta > 0$ such that $p(z, x) \leq \delta$ and $p(z, y) \leq \delta$ imply $d(x, y) \leq \varepsilon$.

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By using the w-distance, they improved the Caristi fixed point theorem, Ekeland variational principle, and Takahashi's minimization theorem on complete metric spaces. Recently, Shaban Sedghi et al.[6] introduced D^*-metric which is a probable modification of the definition of D-metric introduced by Dhage[3] and prove some basic properties in D^*-metric spaces.

In this paper, we introduce the new concept of w-D^*-distance on a D^*-metric space and prove a non-convex minimization theorem which improves the result of Caristi[1], Ćirić[2], Ekeland[4], Kada et al.[5] and Ume[8, 9].

2. PRELIMINARIES

Throughout this paper, we denote by $\mathbb{N}$ the set of all positive integers, by $\mathbb{R}^+$ the set of all nonnegative real numbers, and by $\mathbb{R}$ the set of all real numbers.

Definition 2.1 ([6]). Let X be a nonempty set. A *generalized metric* (or D^*-metric) on X is a function, $D^* : X \times X \times X \longrightarrow [0, \infty)$, that satisfies the following conditions for each $x, y, z, a \in X$:

(1) $D^*(x, y, z) \geq 0$,
(2) $D^*(x, y, z) = 0$ if and only if $x = y = z$,
(3) $D^*(x, y, z) = D^*(p\{x, y, z\})$, (symmetry) where p is a permutation function,
(4) $D^*(x, y, z) \leq D^*(x, y, a) + D^*(a, z, z)$.

The pair (X, D^*) is called a *generalized metric* (or *D^*-metric*) *space.*

We give some examples of D^*-metric.

Example 2.2 ([6]). Define $D^* : X \times X \times X \longrightarrow \mathbb{R}^+$ by

(a) $D^*(x, y, z) = \max\{d(x, y), d(y, z), d(z, x)\}$,
(b) $D^*(x, y, z) = d(x, y) + d(y, z) + d(z, x)$.

Here, d is the ordinary metric on X.

(c) If $X = \mathbb{R}^n$, then we define

$$D^*(x, y, z) = \left(||x - y||^p + ||y - z||^p + ||z - x||^p\right)^{\frac{1}{p}}$$

for every $p \in \mathbb{R}^+$.

(d) If $X = \mathbb{R}$, then we define

$$D^*(x, y, z) = \begin{cases} 0 & \text{if } x = y = z, \\ \max\{x, y, z\} & \text{otherwise.} \end{cases}$$

The following example shows that D^*-metric space is a proper extension of metric.

Example 2.3. Let $\psi : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}^+$ be a mapping defined as follows:

$$\psi(x,y) = 0 \text{ if } x = y, \quad \psi(x,y) = \frac{1}{2} \text{ if } x > y, \quad \psi(x,y) = \frac{1}{3} \text{ if } x < y.$$

Then clearly ψ is not metric since $\psi(1,2) \neq \psi(2,1)$.

Define $G : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}^+$ by

$$G(x,y,z) = \max\{\psi(x,y), \psi(y,z), \psi(z,x)\}.$$

Then G is a D^*-metric.

Remark 2.4 ([6]). In a D^*-metric space, we prove that $D^*(x,x,y) = D^*(x,y,y)$. For

(i) $D^*(x,x,y) \leq D^*(x,x,x) + D^*(x,y,y) = D^*(x,y,y)$ and similarly

(ii) $D^*(y,y,x) \leq D^*(y,y,y) + D^*(y,x,x) = D^*(y,x,x)$.

Hence by (i), (ii) we get $D^*(x,x,y) = D^*(x,y,y)$.

Let (X, D^*) be a D^*-metric space. For $r > 0$, define

$$B_{D^*}(x,r) = \{y \in X : D^*(x,y,y) < r\}.$$

Definition 2.5 ([6]). Let (X, D^*) be a D^*-metric space and $A \subset X$.

(1) If for every $x \in A$, there exists $r > 0$ such that $B_{D^*}(x,r) \subset A$, then subset A is called *open subset* of X.

(2) Subset A of X is said to be *D^*-bounded* if there exists $r > 0$ such that $D^*(x,y,y) < r$ for all $x, y \in A$.

(3) A sequence $\{x_n\}$ in X converges to x if and only if

$$D^*(x_n, x_n, x) = D^*(x, x, x_n) \to 0$$

as $n \to \infty$. That is, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\forall n \geq n_0 \Longrightarrow D^*(x, x, x_n) < \varepsilon \quad (*).$$

This is equivalent; for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\forall n, m \geq n_0 \Longrightarrow D^*(x, x_n, x_m) < \varepsilon \quad (**).$$

Indeed, if $(*)$ holds, then

$$D^*(x_n, x_m, x) = D^*(x_n, x, x_m) \leq D^*(x_n, x, x) + D^*(x, x_m, x_m) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Conversely, set $m = n$ in $(**)$, then we have $D^*(x_n, x_n, x) < \varepsilon$.

(4) A sequence $\{x_n\}$ in X is called a *Cauchy sequence* if for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $D^*(x_n, x_n, x_m) < \varepsilon$ for each $n, m > n_0$. The D^*-metric space (X, D^*) is said to be *complete* if every Cauchy sequence is convergent.

Let τ be the set of all $A \subset X$ with $x \in A$ if and only if there exists $r > 0$ such that $B_{D^*}(x, r) \subset A$. Then τ is a topology on X (induced by the D^*-metric D^*).

Lemma 2.6 ([6]). *Let (X, D^*) be a D^*-metric space. If $r > 0$, then ball $B_{D^*}(x, r)$ with center $x \in X$ and radius r is open ball.*

Definition 2.7 ([6]). Let (X, D^*) be a D^*-metric space. D^* is said to be a *continuous function* on X^3 if

$$\lim_{n\to\infty} D^*(x_n, y_n, z_n) = D^*(x, y, z)$$

whenever a sequence $\{(x_n, y_n, z_n)\}$ in X^3 converges to a point $(x, y, z) \in X^3$, that is,

$$\lim_{n\to\infty} x_n = x, \quad \lim_{n\to\infty} y_n = y, \quad \lim_{n\to\infty} z_n = z.$$

Lemma 2.8 ([6]). *Let (X, D^*) be a D^*-metric space. Then D^* is a continuous function on X^3.*

Now, we introduce the new concept of w-D^*-distance on a D^*-metric space.

Definition 2.9. Let (X, D^*) be a D^*-metric space. Then a function $S : X \times X \times X \to [0, \infty)$ is called a *w-D^*-distance* on X if the following are satisfied.

(S1) $S(x, y, z) \le S(x, a, a) + S(a, y, z)$ for any $x, y, z, a \in X$,

(S2) for each $x \in X$, $S(x, y, y)$ is a lower semicontinuous at y in X,

(S3) for any $\varepsilon > 0$, there exists $\delta > 0$ such that $S(z, x, x) \le \delta$ and $S(z, y, y) \le \delta$ imply $D^*(x, y, y) \le \varepsilon$.

Let us give some examples of w-D^*-distance.

Example 2.10. Let (X, D^*) be a D^*-metric space. Then $S = D^*$ is a w-D^*-distance.

Proof. (S1) and (S2) are obvious. We show (S3). Let $\varepsilon > 0$ be given and put $\delta = \varepsilon/2$. Then if $D^*(z, x, x) \le \delta$ and $D^*(z, y, y) \le \delta$, we have

$$\begin{aligned} D^*(x, y, y) &\le D^*(x, z, z) + D^*(z, y, y) \\ &\le D^*(z, x, x) + D^*(z, y, y) \le \delta + \delta = 2\delta = \varepsilon. \end{aligned}$$

□

Example 2.11. Let (X, D^*) be a D^*-metric space. Then a function $S : X \times X \times X \to [0, \infty)$ defined by $S(x, y, z) = c$ for every $x, y, z \in X$ is a w-D^*-distance on X, where c is a positive real number.

Proof. (S1) and (S2) are obvious. To show (S3), for any $\varepsilon > 0$, put $\delta = c/2$. Then we have $S(z, x, x) \leq \delta$ and $S(z, y, y) \leq \delta$ imply $D^*(x, y, y) \leq \varepsilon$. □

Example 2.12. Let X be normed linear space with norm $\|\cdot\|$. Then a function $S : X \times X \times X \to [0, \infty)$ defined by

$$S(x, y, z) = \|x\| + \|y\| + \|z\| \quad \text{for every } x, y, z \in X$$

is a w-D^*-distance on X.

Proof. Define $D^* : X \times X \times X \to [0, \infty)$ by $D^*(x, y, z) = \|x - y\| + \|y - z\| + \|z - x\|$. Let $x, y, z \in X$. Then we have

$$\begin{aligned} S(x, y, z) &= \|x\| + \|y\| + \|z\| \\ &\leq S(x, a, a) + S(a, y, z). \end{aligned}$$

This implies (S1). (S2) is obvious. Let $\varepsilon > 0$ and put $\delta = \varepsilon/2$. Then, if $S(z, x, x) \leq \delta$ and $S(z, y, y) \leq \delta$, we have

$$\begin{aligned} D^*(x, y, y) &= \|x - y\| + \|y - y\| + \|y - x\| \\ &\leq 2(\|x\| + \|y\|) \\ &\leq S(z, x, x) + S(z, y, y) \\ &\leq 2\delta = \varepsilon. \end{aligned}$$

This implies (S3). □

Example 2.13. Let $X = \mathbb{R}^+$ be a metric space with the usual metric. Define $h : X \times X \to X$ and $D^*, S : X \times X \times X \to X$ as follows :

$$h(x, y) = x, \qquad S(x, y, z) = h(y, z)$$

$$D^*(x, y, z) = \begin{cases} 0, & \text{if } x = y = z, \\ \max\{x, y, z\}, & \text{otherwise.} \end{cases}$$

Then

(i) h is neither metric nor w-distance on X,

(ii) S is a w-D^*-distance.

Proof. (i) Since $h(0, 1) \neq h(1, 0)$, h is not metric. Also, h is not w-distance on X. In fact, if h is w-distance on X, then for $\varepsilon = 1$, there exists $\delta > 0$ such that

$h(z,x) = z \le \delta$ and $h(z,y) = z \le \delta$ imply $d(x,y) = |x-y| \le 1$. Putting $z = \delta/2$, $x = 1$, and $y = 4$ in the above inequalities, we have $1 \ge |1-4| = 3 > 1$, which is a contradiction. Thus, h is not w-distance on X.

(ii) Let $x, y, z, a \in X$. We have $S(x,y,z) = y \le a + y = S(x,a,a) + S(a,y,z)$. This implies (S1). (S2) is obvious. Let $\varepsilon > 0$ and put $\delta = \varepsilon$. Then, if $S(z,x,x) \le \delta$ and $S(z,y,y) \le \delta$, we have $D^*(x,y,y) \le \max\{x,y\} \le \delta = \varepsilon$. This implies (S3). □

Example 2.14. Let X be a normed linear space with norm $\|\cdot\|$. Define $h : X \times X \longrightarrow \mathbb{R}^+$ and $D^*, S : X \times X \times X \to \mathbb{R}^+$ as follows :

$$h(x,y) = \|x\|, \qquad S(x,y,z) = \max\{h(x,y), h(y,z), h(z,x)\}$$

$$D^*(x,y,z) = \|x-y\| + \|y-z\| + \|z-x\|.$$

Then clearly h is not w-distance but S is a w-D^*-distance.

The following lemma plays important role to prove minimization theorems, fixed point theorems, and variational inequalities.

Lemma 2.15. *Let (X, D^*) be a D^*-metric space and let S be a w-D^*-distance on X. Let $\{x_n\}$ and $\{y_n\}$ be sequences in X, let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[0,\infty)$ converging to 0, and let $x, y, z \in X$. Then the following hold:*

1. *If $S(x_n, y, y) \le \alpha_n$ and $S(x_n, z, z) \le \beta_n$ for any $n \in \mathbb{N}$, then $y = z$. In particular, if $S(x,y,y) = 0$ and $S(x,z,z) = 0$, then $y = z$,*
2. *if $S(x_n, y_n, y_n) \le \alpha_n$ and $S(x_n, z, z) \le \beta_n$ for any $n \in \mathbb{N}$, then $\{y_n\}$ converges to z,*
3. *if $S(x_n, x_m, x_m) \le \alpha_n$ for any $n, m \in \mathbb{N}$ with $m > n$, then $\{x_n\}$ is a Cauchy sequence,*
4. *if $S(y, x_n, x_n) \le \alpha_n$ for any $n \in \mathbb{N}$, then $\{x_n\}$ is a Cauchy sequence.*

Proof. We first prove (2). Let $\varepsilon > 0$ be given. From the definition of w-D^*-distance, there exists $\delta > 0$ such that $S(u,v,v) \le \delta$ and $S(u,z,z) \le \delta$ imply $D^*(v,z,z) \le \varepsilon$. Choose $n_0 \in \mathbb{N}$ such that $\alpha_n \le \delta$ and $\beta_n \le \delta$ for every $n \ge n_0$. Then we have, for any $n \ge n_0$, $S(x_n, y_n, y_n) \le \alpha_n \le \delta$ and $S(x_n, z, z) \le \beta_n \le \delta$ and hence $D^*(y_n, z, z) \le \varepsilon$. This implies that $\{y_n\}$ converges to z. It follows from (2) that (1) holds. Let us prove (3). Let $\varepsilon > 0$ be given. As in the proof of (1), choose $\delta > 0$. Then for any $n, m \ge n_0 + 1$,

$$S(x_{n_0}, x_n, x_n) \le \alpha_{n_0} \le \delta \quad \text{and} \quad S(x_{n_0}, x_m, x_m) \le \alpha_{n_0} \le \delta$$

and hence $D^*(x_n, x_m, x_m) \leq \varepsilon$. This implies that $\{x_n\}$ is a Cauchy sequence. As in the proof of (3), we can prove (4). □

Lemma 2.16. *Let (X, D^*) be a D^*-metric space and let S_1, S_2 be w-D^*- distances on X. Then two functions on $X \times X \times X$ defined as follows are w-D^*-distances on X.*

(i) $S(x,y,z) = \max\{S_1(x,y,z), S_2(x,y,z)\}$ *for every $x, y, z \in X$.*

(ii) $S(x,y,z) = \alpha S_1(x,y,z) + \beta S_2(x,y,z)$ *for every $x, y, z \in X$, where α and β are nonnegative real numbers such that $\alpha \neq 0$ or $\beta \neq 0$.*

Proof. We first prove (i). Let $x, y, z, a \in X$. Then we have

$$\begin{aligned} S(x,y,z) &= \max\{S_1(x,y,z), S_2(x,y,z)\} \\ &\leq \max\{S_1(x,a,a) + S_1(a,y,z), S_2(x,a,a) + S_2(a,y,z)\} \\ &\leq \max\{S(x,a,a) + S(a,y,z), S(x,a,a) + S(a,y,z)\} \\ &= S(x,a,a) + S(a,y,z). \end{aligned}$$

It is clear that for any $x \in X$, $S(x,y,y) = \max\{S_1(x,y,y), S_2(x,y,y)\}$ is lower semicontinuous at y in X. Let $\varepsilon > 0$ be given. Then choose $\delta > 0$ such that $S_1(z,x,x) \leq \delta$ and $S_1(z,y,y) \leq \delta$ imply $D^*(x,y,y) \leq \varepsilon$. If $S(z,x,x) \leq \delta$ and $S(z,y,y) \leq \delta$, then $S_1(z,x,x) \leq \delta$ and $S_1(z,y,y) \leq \delta$. So, we have $D^*(x,y,y) \leq \varepsilon$.

Let us prove (ii). Without loss of generality, we may assume $\alpha > 0$. Let $x, y, z, a \in X$. Then we have

$$\begin{aligned} S(x,y,z) &= \alpha S_1(x,y,z) + \beta S_2(x,y,z) \\ &\leq \alpha(S_1(x,a,a) + S_1(a,y,z)) + \beta(S_2(x,a,a) + S_2(a,y,z)) \\ &= S(x,a,a) + S(a,y,z). \end{aligned}$$

Since for any $x \in X$, $\alpha S_1(x,y,y)$ and $\beta S_1(x,y,y)$ are lower semicontinuous at y, $S(x,y,y) = \alpha S_1(x,y,y) + \beta S_1(x,y,y)$ is also lower semicontinuous. Let $\varepsilon > 0$ be given and then choose $\delta' > 0$ such that $S_1(z,x,x) \leq \delta'$ and $S_1(z,y,y) \leq \delta'$ imply $D^*(x,y,y) \leq \varepsilon$. Put $\delta = \alpha\delta'$. Then, if $S(z,x,x) \leq \delta$ and $S(z,y,y) \leq \delta$, we have $S_1(z,x,x) \leq \delta'$ and $S_1(z,y,y) \leq \delta'$. So, we have $D^*(x,y,y) \leq \varepsilon$. □

Lemma 2.17. *Let X be a metric space with metric d, let p be a w-distance on X and let α be a function from X into $[0,\infty)$. Define $D^*, S : X \times X \times X \to \mathbb{R}^+$ as follows :*

$$D^*(x,y,z) = \max\{d(x,y), d(y,z), d(z,x)\}, \quad S(x,y,z) = \max\{\alpha(x), p(x,y), p(x,z)\}$$

for every $x, y, z \in X$. Then S is a w-D^*-distance on X.

Proof. For every $x, y, z, a \in X$,

$$\begin{aligned} S(x,y,z) &= \max\{\alpha(x), p(x,y), p(x,z)\} \\ &\le \max\{\alpha(x)+\alpha(a), p(x,a)+p(a,y), p(x,a)+p(a,z)\} \\ &\le S(x,a,a)+S(a,y,z). \end{aligned}$$

Therefore (S1) is satisfied. (S2) is obvious. We show (S3). Let $\varepsilon > 0$ be fixed. Then since p is a w-distance on X, then exists $\delta > 0$ such that $p(z,x) \le \delta$ and $p(z,y) \le \delta$ imply $d(x,y) \le \varepsilon$. So, assume $S(z,x,x) \le \delta$ and $S(z,y,y) \le \delta$, then $p(z,x) \le \delta$ and $p(z,y) \le \delta$. Therefore, $D^*(x,y,y) = d(x,y) \le \varepsilon$. This implies (S3). □

3. Minimization Theorems and Its Applications

Theorem 3.1. *Let (X, D^*) be a complete D^*-metric space, and let $f : X \to (-\infty, \infty]$ be a proper lower semicontinuous function, bounded from below. Assume that there exists a w-D^*-distance S on X such that for any $u \in X$ with $\inf_{x\in X} f(x) < f(u)$, there exists $v \in X$ with $v \ne u$ and*

$$f(v) + S(u,v,v) \le f(u).$$

Then there exists $x_0 \in X$ such that $\inf_{x\in X} f(x) = f(x_0)$.

Proof. Suppose $\inf_{x\in X} f(x) < f(y)$ for every $y \in X$ and choose $u \in X$ with $f(u) < \infty$. Then we define inductively a sequence $\{u_n\}$ in X, starting with $u_1 = u$. Suppose $u_n \in X$ is known. Then choose $u_{n+1} \in S(u_n)$ such that

$$S(u_n) = \{x \in X : f(x) + S(u_n, x, x) \le f(u_n)\},$$

$$k(u_n) = \inf_{x \in S(u_n)} f(x)$$

and

$$f(u_{n+1}) \le k(u_n) + \frac{1}{n}.$$

Since

$$f(u_{n+1}) + S(u_n, u_{n+1}, u_{n+1}) \le f(u_n),$$

$\{f(u_n)\}$ is nonincreasing. So, $\lim_{n\to\infty} f(u_n)$ exists. Put $k = \lim_{n\to\infty} f(u_n)$. We claim that $\{u_n\}$ is a Cauchy sequence. In fact, if $n < m$, then

$$\begin{aligned} S(u_n, u_m, u_m) &\leq \sum_{j=n}^{m-1} S(u_j, u_{j+1}, u_{j+1}) \\ &\leq \sum_{j=n}^{m-1} \{f(u_j) - f(u_{j+1})\} \qquad (3.1) \\ &= f(u_n) - f(u_m) \leq f(u_n) - k. \end{aligned}$$

From Lemma 2.15, $\{u_n\}$ is a Cauchy sequence. Let $u_n \to v_0$. Then, if $m \to \infty$ in (3.1), we have

$$S(u_n, v_0, v_0) \leq f(u_n) - k \leq f(u_n) - f(v_0).$$

On the other hand, by hypothesis, there exists $v_1 \in X$ such that $v_1 \neq v_0$ and $f(v_1) + S(v_0, v_1, v_1) \leq f(v_0)$. Hence, we obtain

$$\begin{aligned} f(v_1) + S(u_n, v_1, v_1) &\leq f(v_1) + S(u_n, v_0, v_0) + S(v_0, v_1, v_1) \\ &\leq f(v_0) + S(u_n, v_0, v_0) \qquad (3.2) \\ &\leq f(u_n) \end{aligned}$$

and hence $v_1 \in S(u_n)$. Since

$$f(v_0) \leq f(u_{n+1}) \leq k(u_n) + \frac{1}{n} \leq f(v_1) + \frac{1}{n}$$

for every $n \in \mathbb{N}$, we have $f(v_0) \leq f(v_1)$. Then, $f(v_0) = f(v_1)$. So, we have $S(v_0, v_1, v_1) = 0$. By hypothesis, there exists $v_2 \in X$ such that $v_2 \neq v_1$ and $f(v_2) + S(v_1, v_2, v_2) \leq f(v_1)$. As in (3.2), we have $f(v_2) + S(u_n, v_2, v_2) \leq f(u_n)$ and hence $v_2 \in S(u_n)$. So, we have $f(v_1) = f(v_0) \leq f(v_2)$. This implies $S(v_1, v_2, v_2) = 0$. From $S(v_0, v_2, v_2) \leq S(v_0, v_1, v_1) + S(v_1, v_2, v_2) = 0$, we have $S(v_0, v_2, v_2) = 0$. Hence, from $S(v_0, v_1, v_1) = 0, S(v_0, v_2, v_2) = 0$ and Lemma 2.15, we have $v_1 = v_2$. This is a contradiction. □

Corollary 3.2 ([5]). *Let (X, d) be a complete metric space, and let $f : X \to (-\infty, \infty]$ be a proper lower semicontinuous function, bounded from below. Assume that there exists a w-distance p on X such that for any $u \in X$ with $\inf_{x \in X} f(x) < f(u)$, there exists $v \in X$ with $v \neq u$ and $f(v) + p(u, v) \leq f(u)$. Then there exists $x_0 \in X$ such that $\inf_{x \in X} f(x) = f(x_0)$.*

Proof. Define $D^* : X \times X \times X \to \mathbb{R}^+$ by $D^*(x, y, z) = \max\{d(x, y), d(y, z), d(z, x)\}$ for all $x, y, z \in X$ and define $S : X \times X \times X \to \mathbb{R}^+$ by $S(x, y, z) = \max\{p(x, y), p(x, z)\}$ for all $x, y, z \in X$. Then, X, D^*, S and f satisfy the suppositions in Theorem 3.1. Therefore, Corollary 3.2 follows from Theorem 3.1. □

The following example shows that Theorem 3.1 is more general that Corollary 3.2.

Example 3.3. Let X, h, D^*, and S be as in Example 2.13. Define $f : X \to (-\infty, \infty]$ as follows:

$$f(x) = \begin{cases} 2x+1, & \text{if } 0 \le x < 2, \\ -1, & \text{if } x = 2, \\ 3x+1, & \text{if } 2 < x. \end{cases}$$

It is clear that all of the conditions except inequality in Theorem 3.1 are satisfied. To show that inequality in Theorem 3.1 is satisfied, we need to consider several possible cases as follows.

(1) For $u = 0$ in X, there exists $v = 2$ in X such that

$$f(v) + S(u,v,v) = f(v) + v = f(2) + 2 = -1 + 2 = 1 = f(0) = f(u).$$

(2) For $u \in X$ with $0 < u < 2$, there exists $v \in (0, (1/2)u)$ such that

$$f(v) + S(u,v,v) = f(v) + v = 2v + 1 + v < u + 1 + u = 2u + 1 = f(u).$$

(3) For $u \in X$ with $2 < u$, there exists $v \in (0, 2)$, such that

$$f(v) + S(u,v,v) = f(v) + v = 2v + 1 + v = 3v + 1 < 3u + 1 = f(u).$$

Hence, for $u \in X$ with $\inf_{x \in X} f(x) < f(u)$, there exists $v \in X$ with $v \neq u$ such that

$$f(v) + S(u,v,v) \le f(u),$$

that is, inequality in Theorem 3.1 is satisfied. Thus, all of the conditions in Theorem 3.1 are satisfied and therefore, there exists $2 \in X$ such that $\inf_{x \in X} f(x) = f(2)$.

Remark 3.4. Since $S(x,y,z) = h(y,z)$ is neither metric nor w-distance, Corollary 3.2 cannot be applicable. This means that Theorem 3.1 is a proper extension of Corollary 3.2.

The following theorem extends, improves, and unifies many known results due to Caristi[1], Kada et al.[5], Takahashi[7], and Ume[8].

Theorem 3.5. *Let (X, D^*) be a complete D^*-metric space and let $f : X \to (-\infty, \infty]$ be a proper lower semicontinuous function, bounded from below. Let T be a mapping from X into itself. Suppose that there exists a w-D^*-distance S on X such that*

$$f(Tx) + S(x, Tx, Tx) \le f(x)$$

for every $x \in X$. Then, there exists $x_0 \in X$ such that $Tx_0 = x_0$ and $S(x_0, x_0, x_0) = 0$.

Proof. Since f is proper, there exists $u \in X$ such that $f(u) < \infty$. Put

$$Y = \{x \in X : f(x) \leq f(u)\}.$$

Then, since f is lower semicontinuous, Y is closed. Hence Y is complete D^*-metric space. Let $x \in Y$. Then, since $f(Tx) + S(x, Tx, Tx) \leq f(x) \leq f(u)$, we have $Tx \in Y$. So Y is invariant under T. Assume that $Tx \neq x$ for every $x \in Y$. Then, by Theorem 3.1, there exists $v_0 \in Y$ such that $f(v_0) = \inf_{x \in Y} f(x)$. Since

$$f(Tv_0) + S(v_0, Tv_0, Tv_0) \leq f(v_0)$$

and

$$f(v_0) = \inf_{x \in Y} f(x),$$

we have $f(Tv_0) = f(v_0) = \inf_{x \in Y} f(x)$ and $S(v_0, Tv_0, Tv_0) = 0$. Similarly, we obtain

$$f(T^2 v_0) = f(Tv_0) = \inf_{x \in Y} f(x)$$

and

$$S(Tv_0, T^2 v_0, T^2 v_0) = 0.$$

Since

$$S(v_0, T^2 v_0, T^2 v_0) \leq S(v_0, Tv_0, Tv_0) + S(Tv_0, T^2 v_0, T^2 v_0) = 0,$$

we have $S(v_0, T^2 v_0, T^2 v_0) = 0$. Hence, from

$$S(v_0, Tv_0, Tv_0) = 0, S(v_0, T^2 v_0, T^2 v_0) = 0$$

and Lemma 2.15, we have $Tv_0 = T^2 v_0$. This is a contradiction. Therefore, T has a fixed point x_0 in Y. Since $f(x_0) < \infty$ and

$$f(x_0) + S(x_0, x_0, x_0) = f(Tx_0) + S(x_0, Tx_0, Tx_0) \leq f(x_0),$$

we have $S(x_0, x_0, x_0) = 0$. □

We give an example to support Theorem 3.5.

Example 3.6. Let X, h, D^* and S be as in Example 3.3, Define $T : X \to X$ and $f : X \to (-\infty, \infty]$ as follows:

$$Tx = \frac{1}{2}x \quad \forall x \in X, \qquad f(x) = \begin{cases} 2x + 1, & \text{if } 0 \leq x \leq 4, \\ 3x + 1, & \text{if } 4 < x. \end{cases}$$

Clearly, f is a proper lower semicontinuous function, bounded from below. Now, we show that inequality in Theorem 3.5 is satisfied. There are several possible cases which we need to consider.

(1) For $x \in X$ with $0 \leq Tx = (1/2)x \leq 2$, we have

$$\begin{aligned} f(Tx) + S(x, Tx, Tx) &= f(Tx) + Tx = f\left(\frac{1}{2}x\right) + \frac{1}{2}x \\ &= 2 \times \frac{1}{2}x + 1 + \frac{1}{2}x = \frac{3}{2}x + 1 \\ &\leq 2x + 1 = f(x). \end{aligned}$$

(2) For $x \in X$ with $2 < Tx = (1/2)x \leq 4$, we have

$$\begin{aligned} f(Tx) + S(x, Tx, Tx) &= f(Tx) + Tx = f\left(\frac{1}{2}x\right) + \frac{1}{2}x \\ &= 2 \times \frac{1}{2}x + 1 + \frac{1}{2}x = \frac{3}{2}x + 1 \\ &\leq 3x + 1 = f(x). \end{aligned}$$

(3) For $x \in X$ with $4 < Tx = (1/2)x$, we have

$$\begin{aligned} f(Tx) + S(x, Tx, Tx) &= f(Tx) + Tx = f\left(\frac{1}{2}x\right) + \frac{1}{2}x \\ &= 3 \times \frac{1}{2}x + 1 + \frac{1}{2}x = 2x + 1 \\ &\leq 3x + 1 = f(x). \end{aligned}$$

Hence, $f(Tx) + S(x, Tx, Tx) \leq f(x)$ for all $x \in X$. Thus, all of the conditions in Theorem 3.5 are satisfied and, therefore, there exists $0 \in X$ such that $T0 = 0$ and $S(0, 0, 0) = 0$.

Remark 3.7. Since $S(x, y, z) = h(y, z)$ is neither metric nor w-distance, fixed point theorems of Caristi[1], Kada et al.[5], Takahashi[7], and Ume[8] cannot be applicable. Therefore, Theorem 3.5 is a proper extension of results due to Caristi[1], Kada et al.[5], Takahashi[7], and Ume[8].

The following theorem is a generalization of the corresponding results in [4,5,7,8].

Theorem 3.8. *Let (X, D^*) be a complete D^*-metric space, let S be a w-D^*-distance on X and let $f : X \to (-\infty, \infty]$ be a proper lower semicontinuous function, bounded from below. Then the following (i) and (ii) hold :*

(i) *For any $u \in X$ with $f(u) < \infty$, there exists $v \in X$ such that $f(v) \leq f(u)$ and $f(z) > f(v) - S(v, z, z)$ for every $z \in X$ with $z \neq v$,*

(ii) *for any $\varepsilon > 0$ and $u \in X$ with $S(u, u, u) = 0$ and $f(u) \leq \inf_{x \in X} f(x) + \varepsilon$, there exists $v \in X$ such that $f(v) \leq f(u)$, $S(u, v, v) \leq 1$, and $f(z) > f(v) - \varepsilon \cdot S(v, z, z)$ for every $z \in X$ with $z \neq v$.*

Proof. We first prove (i). Let $u \in X$ be such that $f(u) < \infty$ and let $Y = \{x \in X : f(x) \leq f(u)\}$. Then, Y is nonempty closed and complete D^*-metric space. Hence, we may prove that there exists an element $v \in Y$ such that $f(z) > f(v) - S(v,z,z)$ for every $z \in X$ with $z \neq v$. Suppose not. Then, for every $x \in Y$, there exists $z \in X$ such that $z \neq x$ and $f(z) + S(x,z,z) \leq f(x)$. Since $f(z) \leq f(x) \leq f(u)$, $z \in X$ is an element of Y. So, by Theorem 3.1, there exists $x_0 \in Y$ such that $f(x_0) = \inf_{x\in Y} f(x)$. We also have that for each x_0, there exists $x_1 \in Y$ such that $x_1 \neq x_0$ and $f(x_1) + S(x_0,x_1,x_1) \leq f(x_0)$. Hence, we have $f(x_1) = f(x_0) = \inf_{x\in Y} f(x)$ and $S(x_0,x_1,x_1) = 0$. Similarly, there exists $x_2 \in Y$ such that $x_2 \neq x_1$ and $S(x_1,x_2,x_2) = 0$. From $S(x_0,x_2,x_2) \leq S(x_0,x_1,x_1) + S(x_1,x_2,x_2) = 0$, we have $S(x_0,x_2,x_2) = 0$. Hence, from $S(x_0,x_1,x_1) = 0, S(x_0,x_2,x_2) = 0$ and Lemma 2.15, we have $x_1 = x_2$. This is a contraction. Therefore, there exists $v \in Y$ such that $f(z) > f(v) - S(v,z,z)$ for every $z \in X$ with $z \neq v$.

Let us prove (ii). Let $Z = \{x \in X : f(x) \leq f(u) - \varepsilon \cdot S(u,x,x)\}$. Then Z is nonempty closed and complete D^*-metric space. Note that $\varepsilon \cdot S$ is a w-D^*-distance by Lemma 2.16. Then, as in the proof of (i), we have that there exists $v \in Z$ such that $f(z) > f(v) - \varepsilon \cdot S(v,z,z)$ for every $z \in X$ with $z \neq v$. On the other hand, since $v \in Z$, we have $f(v) \leq f(u) - \varepsilon \cdot S(u,v,v) \leq f(u)$ and

$$S(u,v,v) \leq \frac{1}{\varepsilon}[f(u) - f(v)] \leq \frac{1}{\varepsilon}\Big[f(u) - \inf_{x\in X} f(x)\Big] \leq \frac{1}{\varepsilon} \cdot \varepsilon = 1.$$

The proof is complete. □

The following is an example to support Theorem 3.8.

Example 3.9. Let X, h, D^*, S, and f be as in Example 3.6. Taking $v = 0$ in X, (i) and (ii) of Theorem 3.8 hold.

Remark 3.10. Since $S(x,y,z) = h(y,z)$ is neither metric nor w-distance, theorems in [4, 5, 7, 8] cannot be applicable. Therefore, Theorem 3.8 is a generalization of the corresponding results in [4, 5, 7, 8].

4. Fixed Point Theorems

The following theorem is a generalization of the corresponding results in [2,5,9]

Theorem 4.1. *Let (X, D^*) be a complete D^*-metric space, let S be a w-D^*-distance on X and let T be a mapping from X into itself. Suppose that there exists $r \in [0,1)$ such that*

$$S(Tx, T^2x, T^2x) \leq r \cdot S(x, Tx, Tx)$$

for every $x \in X$ and that

$$\inf\{S(x,y,y) + S(x,Tx,Tx) : x \in X\} > 0$$

for every $y \in X$ with $y \neq Ty$. Then, there exists $z \in X$ such that $z = Tz$. Moreover, if $v = Tv$, then $S(v,v,v) = 0$.

Proof. Let $u \in X$ and define the sequence $\{u_n\}_{n=0}^{\infty}$ satisfying the following: $u_0 = u$ and $u_n = T^n u$ for any $n \in \mathbb{N}$. Then, we have, for any $n \in \mathbb{N}$,

$$\begin{aligned} S(u_n, u_{n+1}, u_{n+1}) &\leq r \cdot S(u_{n-1}, u_n, u_n) \\ &\leq r^2 \cdot S(u_{n-2}, u_{n-1}, u_{n-1}) \\ &\vdots \\ &\leq r^n \cdot S(u, u_1, u_1). \end{aligned}$$

So, if $m > n$,

$$\begin{aligned} S(u_n, u_m, u_m) &\leq S(u_n, u_{n+1}, u_{n+1}) + \cdots + S(u_{m-1}, u_m, u_m) \\ &\leq r^n \cdot S(u, u_1, u_1) + \cdots + r^{m-1} \cdot S(u, u_1, u_1) \\ &\leq \frac{r^n}{1-r} S(u, u_1, u_1). \end{aligned}$$

By Lemma 2.15, $\{u_n\}$ is a Cauchy sequence. Since X is complete, $\{u_n\}$ converges to some point $z \in X$. Let $n \in \mathbb{N}$ be fixed. Then, since $\{u_n\}$ converges to z and $S(u_n, z, z)$ is lower semicontinuous at z in X, we have

$$S(u_n, z, z) \leq \liminf_{m\to\infty} S(u_n, u_m, u_m) \leq \frac{r^n}{1-r} S(u, u_1, u_1).$$

Assume that $z \neq Tz$. Then, by hypothesis, we have

$$\begin{aligned} 0 &< \inf\{S(x,z,z) + S(x,Tx,Tx) : x \in X\} \\ &\leq \inf\{S(u_n,z,z) + S(u_n,u_{n+1},u_{n+1}) : n \in \mathbb{N}\} \\ &\leq \inf\Big\{\frac{r^n}{1-r} S(u,u_1,u_1) + r^n S(u,u_1,u_1) : n \in \mathbb{N}\Big\} \\ &= 0. \end{aligned}$$

This is a contradiction. Therefore we have $z = Tz$. If $v = Tv$, we have

$$S(v,v,v) = S(Tv, T^2v, T^2v) \leq r \cdot S(v, Tv, Tv) = r \cdot S(v,v,v)$$

and hence $S(v,v,v) = 0$. □

Corollary 4.2 ([9]). *Let (X, d) be a complete metric space with a w-distance p and let T be a self-mapping of X. Suppose that there exists $r \in [0, 1)$ such that*

$$p(Tx, Ty) \leq r \cdot \max\{p(x, y), p(x, Tx), p(y, Ty), p(x, Ty), p(y, Tx)\} \tag{4.1}$$

for every $x, y \in X$ and that

$$\inf\{p(x, y) + p(x, Tx) : x \in X\} > 0 \tag{4.2}$$

for every $y \in X$ with $y \neq Ty$. Then, there exists $z \in X$ such that $z = Tz$. Moreover, if $v = Tv$, then $p(v, v) = 0$.

Proof. By Lemma 2.4 in [9], for every $x \in X$

$$\sup\{p(T^i x, T^j x) : i, j \in \mathbb{N} \cup \{0\}\} < \infty.$$

Define $D^* : X \times X \times X \to [0, \infty)$ by

$$D^*(x, y, z) = \max\{d(x, y), d(y, z), d(z, x)\}$$

for all $x, y, z \in X$ and define $S : X \times X \times X \to [0, \infty)$ by

$$S(x, y, z) = \max\{\sup\{p(T^i x, T^j x) : i, j \in \mathbb{N} \cup \{0\}\}, p(x, y), p(x, z)\}$$

for all $x, y, z \in X$. By Lemma 2.17, S is a w-D^*-distance on X. Let $x \in X$. Then we have, using Lemma 2.4, in [9],

$$\begin{aligned} S(Tx, T^2x, T^2x) &= \max\{\sup\{p(T^i x, T^j x) : i, j \in \mathbb{N}\}, p(Tx, T^2x)\} \\ &= \sup\{p(T^i x, T^j x) : i, j \in \mathbb{N}\} \\ &\leq r \cdot \sup\{p(T^i x, T^j x) : i, j \in \mathbb{N} \cup \{0\}\} \\ &= r \cdot \max\{\sup\{p(T^i x, T^j x) : i, j \in \mathbb{N} \cup \{0\}\}, p(x, Tx)\} \\ &= r \cdot S(x, Tx, Tx). \end{aligned}$$

Since $\inf\{S(x, y, y) + S(x, Tx, Tx) : x \in X\} \geq \inf\{p(x, y) + p(x, Tx) : x \in X\}$ for every $y \in X$ with $y \neq Ty$, $\inf\{S(x, y, y) + S(x, Tx, Tx) : x \in X\} > 0$. So, all conditions of Theorem 4.1 are satisfied. Therefore, there exists $z \in X$ such that $z = Tz$. Moreover, if $v = Tv$, then $S(v, v, v) = p(v, v) = 0$. □

Corollary 4.3 ([2]). *Let (X, d) be a complete metric space and let $T : X \to X$ be a mapping such that*

$$d(Tx, Ty) \leq r \cdot \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\} \tag{4.3}$$

for all $x, y \in X$ and some $r \in [0, 1)$. Then, T has a unique fixed point.

Proof. Since a metric d is a w-distance, (4.1) implies (4.3). By Lemma 2.5 in [9], (4.2) is satisfied. Therefore, by Corollary 4.2, the result follows. □

Corollary 4.4. *Let (X,d) be a complete metric space, let T be a continuous mapping from X into itself, and let $f : X \to (-\infty,\infty]$ be a proper lower semicontinuous function, bounded from below. Assume that there exists a w-distance p on X such that for any $u \in X$ with $\inf_{x\in X} f(x) < f(u)$, there exists $v \in X$ with $v \neq u$ and*

$$f(v) + \max\{p(Tu,v), p(Tu,Tv)\} \leq f(u).$$

Then, there exists $x_0 \in X$ such that $\inf_{x\in X} f(x) = f(x_0)$.

Proof. Define $D^* : X \times X \times X \to \mathbb{R}^+$ by $D^*(x,y,z) = \max\{d(x,y), d(x,z), d(y,z)\}$ for all $x,y,z \in X$ and define $S : X \times X \times X \to \mathbb{R}^+$ by $S(x,y,z) = \max\{p(Tx,Ty),$ $p(Tx,Tz), p(Tx,y),\ p(Tx,z)\}$ for all $x,y,z \in X$. Then, X, D^*, S and f satisfy the suppositions in Theorem 3.1. Therefore, Corollary 4.4 follows from Theorem 3.1. □

As a consequence of Corollary 4.4, we have the following Corollary.

Corollary 4.5 ([5]). *Let (X,d) be a complete metric space, let T be a continuous mapping from X into itself, and let $f : X \to (-\infty,\infty]$ be a proper lower semicontinuous function, bounded from below. Assume that for any $u \in X$ with $\inf_{x\in X} f(x) = f(u)$, there exists $v \in X$ with $v \neq u$ and*

$$f(v) + \max\{d(Tu,v), d(Tu,Tv)\} \leq f(u).$$

Then, there exists $x_0 \in X$ such that $\inf_{x\in X} f(x) = f(x_0)$.

The following is an example to support Theorem 4.1.

Example 4.6. Let X, h, S, D^* and T be as in Example 3.6. Taking $r = 2/3$, all of conditions in Theorem 4.1 are satisfied. Therefore, there exists $0 \in X$ such that $0 = T0$, If $v = Tv = (1/2)v$, then $S(v,v,v) = h(v,v) = v = 0$.

Remark 4.7. Since $S(x,y,z) = h(y,z)$ is neither metric nor w-distance, theorems in [2, 5, 9] cannot be applicable. Therefore, Theorem 4.1 is a generalization of the corresponding results in [2, 5, 9].

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