

FALLING FUZZY COMMUTATIVE IDEALS OF BCK-ALGEBRAS

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ABSTRACT. Based on the theory of falling shadows and fuzzy sets, the notion of a falling fuzzy commutative ideal of a BCK-algebra is introduced. Relations between falling fuzzy commutative ideals and falling fuzzy ideals are investigated.

1. INTRODUCTION

In the study of a unified treatment of uncertainty modelled by means of combining probability and fuzzy set theory, Goodman [1] pointed out the equivalence of a fuzzy set and a class of random sets. Wang and Sanchez [9] introduced the theory of falling shadows which directly relates probability concepts with the membership function of fuzzy sets. Falling shadow representation theory shows us the way of selection relaid on the joint degrees distributions. It is reasonable and convenient approach for the theoretical development and the practical applications of fuzzy sets and fuzzy logics. The mathematical structure of the theory of falling shadows is formulated in [8]. Tan et al. [6, 7] established a theoretical approach to define a fuzzy inference relation and fuzzy set operations based on the theory of falling shadows. Jun and Park [3] discussed the notion of a falling fuzzy subalgebra/ideal of a BCK/BCI-algebra. In this paper, we establish a theoretical approach to define a fuzzy commutative ideal in a BCK-algebra based on the theory of falling shadows. We provide relations between falling fuzzy commutative ideals and falling fuzzy ideals. We also consider relations between fuzzy commutative ideals and falling fuzzy commutative ideals.

1.1. Basic Results on BCK-algebras and Fuzzy Aspects A BCK/BCI-algebra is an important class of logical algebras introduced by K. Iséki and was extensively investigated by several researchers.

Received by the editors August 25, 2010. Accepted February 20, 2011.

2000 *Mathematics Subject Classification.* 06F35, 03G25, 08A72.

Key words and phrases. falling shadow, (commutative) ideal, fuzzy (commutative) ideal, falling fuzzy (commutative) ideal.

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An algebra $(X; *, 0)$ of type $(2, 0)$ is called a *BCI-algebra* if it satisfies the following axioms:

- (I) $(\forall x, y, z \in X) (((x * y) * (x * z)) * (z * y) = 0),$
- (II) $(\forall x, y \in X) ((x * (x * y)) * y = 0),$
- (III) $(\forall x \in X) (x * x = 0),$
- (IV) $(\forall x, y \in X) (x * y = 0, y * x = 0 \Rightarrow x = y).$

If a BCI-algebra X satisfies the following identity:

- (V) $(\forall x \in X) (0 * x = 0),$

then X is called a *BCK-algebra*. A BCK-algebra X is said to be *commutative* if $x * (x * y) = y * (y * x)$ for all $x, y \in X$.

Any BCK-algebra X satisfies the following conditions:

- (a1) $(\forall x \in X) (x * 0 = x),$
- (a2) $(\forall x, y, z \in X) (x \leq y \Rightarrow x * z \leq z, z * y \leq z * x),$
- (a3) $(\forall x, y, z \in X) ((x * y) * z = (x * z) * y),$

where $x \leq y$ if and only if $x * y = 0$.

A subset I of a BCK-algebra X is called an *ideal* of X , denoted by $I \triangleleft X$, if it satisfies:

- (b1) $0 \in I.$
- (b2) $(\forall x \in X) (\forall y \in I) (x * y \in I \Rightarrow x \in I).$

Every ideal I of a BCK-algebra X has the following assertion:

$$(1.1) \quad (\forall x \in X) (\forall y \in I) (x \leq y \Rightarrow x \in I).$$

A subset I of a BCK-algebra X is called a *commutative ideal* of X , denoted by $I \triangleleft_c X$, if it satisfies (b1) and

- (b3) $(\forall x, y, z \in X) ((x * y) * z \in I, z \in I \Rightarrow x * (y * (y * x)) \in I).$

We refer the reader to the paper [2] and book [5] for further information regarding BCK-algebras.

A fuzzy set μ in a BCK-algebra X is called a *fuzzy ideal* of X if it satisfies:

- (c1) $(\forall x \in X) (\mu(0) \geq \mu(x)).$
- (c2) $(\forall x, y \in X) (\mu(x) \geq \min\{\mu(x * y), \mu(y)\}).$

A fuzzy set μ in a BCK-algebra X is called a *fuzzy commutative ideal* of X (see [4]) if it satisfies (c1) and

- (c3) $(\forall x, y, z \in X) (\mu(x * (y * (y * x))) \geq \min\{\mu((x * y) * z), \mu(z)\}).$

Note that every fuzzy commutative ideal is a fuzzy ideal, but the converse is not true.

Proposition 1.1. *Let μ be a fuzzy set in a BCK-algebra X . Then μ is a fuzzy commutative ideal of X if and only if*

$$(\forall t \in [0, 1]) (\mu_t \neq \emptyset \implies \mu_t \triangleleft_c X)$$

where $\mu_t := \{x \in X \mid \mu(x) \geq t\}$ (see [4]).

1.2. The Theory of Falling Shadows We first display the basic theory on falling shadows. We refer the reader to the papers [1, 6, 7, 8, 9] for further information regarding falling shadows.

Given a universe of discourse U , let $\mathcal{P}(U)$ denote the power set of U . For each $u \in U$, let

$$(1.2) \quad \dot{u} := \{E \mid u \in E \text{ and } E \subseteq U\},$$

and for each $E \in \mathcal{P}(U)$, let

$$(1.3) \quad \dot{E} := \{\dot{u} \mid u \in E\}.$$

An ordered pair $(\mathcal{P}(U), \mathcal{B})$ is said to be a *hyper-measurable structure* on U if \mathcal{B} is a σ -field in $\mathcal{P}(U)$ and $\dot{U} \subseteq \mathcal{B}$. Given a probability space (Ω, \mathcal{A}, P) and a hyper-measurable structure $(\mathcal{P}(U), \mathcal{B})$ on U , a *random set* on U is defined to be a mapping $\xi : \Omega \rightarrow \mathcal{P}(U)$ which is \mathcal{A} - \mathcal{B} measurable, that is,

$$(1.4) \quad (\forall C \in \mathcal{B}) (\xi^{-1}(C) = \{\omega \mid \omega \in \Omega \text{ and } \xi(\omega) \in C\} \in \mathcal{A}).$$

Suppose that ξ is a random set on U . Let

$$\tilde{H}(u) := P(\omega \mid u \in \xi(\omega)) \text{ for each } u \in U.$$

Then \tilde{H} is a kind of fuzzy set in U . We call \tilde{H} a *falling shadow* of the random set ξ , and ξ is called a *cloud* of \tilde{H} .

For example, $(\Omega, \mathcal{A}, P) = ([0, 1], \mathcal{A}, m)$, where \mathcal{A} is a Borel field on $[0, 1]$ and m is the usual Lebesgue measure. Let \tilde{H} be a fuzzy set in U and $\tilde{H}_t := \{u \in U \mid \tilde{H}(u) \geq t\}$ be a t -cut of \tilde{H} . Then

$$\xi : [0, 1] \rightarrow \mathcal{P}(U), \quad t \mapsto \tilde{H}_t$$

is a random set and ξ is a cloud of \tilde{H} . We shall call ξ defined above as the *cut-cloud* of \tilde{H} (see [1]).

2. FALLING FUZZY COMMUTATIVE IDEALS

In what follows let X denote a BCK-algebra unless otherwise specified.

Definition 2.1. Let (Ω, \mathcal{A}, P) be a probability space, and let

$$\xi : \Omega \rightarrow \mathcal{P}(X)$$

be a random set. If $\xi(\omega)$ is an ideal of X for any $\omega \in \Omega$, then the falling shadow \tilde{H} of the random set ξ , i.e.,

$$(2.1) \quad \tilde{H}(x) = P(\omega \mid x \in \xi(\omega))$$

is called a *falling fuzzy ideal* of X (see [3]).

Let (Ω, \mathcal{A}, P) be a probability space and let

$$F(X) := \{f \mid f : \Omega \rightarrow X \text{ is a mapping}\},$$

where X is a BCK-algebra. Define an operation \otimes on $F(X)$ by

$$(\forall \omega \in \Omega) ((f \otimes g)(\omega) = f(\omega) * g(\omega))$$

for all $f, g \in F(X)$. Let $\theta \in F(X)$ be defined by $\theta(\omega) = 0$ for all $\omega \in \Omega$. Then $(F(X); \otimes, \theta)$ is a BCK-algebra (see [3]).

Definition 2.2. Let (Ω, \mathcal{A}, P) be a probability space and let

$$\xi : \Omega \rightarrow \mathcal{P}(X)$$

be a random set. If $\xi(\omega)$ is a commutative ideal of X for any $\omega \in \Omega$, then the falling shadow \tilde{H} of the random set ξ , i.e.,

$$(2.2) \quad \tilde{H}(x) = P(\omega \mid x \in \xi(\omega))$$

is called a *falling fuzzy commutative ideal* of X .

For any subset A of X and $f \in F(X)$, let

$$A_f := \{\omega \in \Omega \mid f(\omega) \in A\}$$

and

$$\xi : \Omega \rightarrow \mathcal{P}(F(X)), \omega \mapsto \{f \in F(X) \mid f(\omega) \in A\}.$$

Then $A_f \in \mathcal{A}$.

Theorem 2.3. *If A is a commutative ideal of X , then*

$$\xi(\omega) = \{f \in F(X) \mid f(\omega) \in A\}$$

is a commutative ideal of $F(X)$.

Table 1. Cayley table

*	0	a	b	c	d
0	0	0	0	0	0
a	a	0	0	a	a
b	b	a	0	b	b
c	c	c	c	0	c
d	d	d	d	d	0

Proof. Assume that A is a commutative ideal of X and let $\omega \in \Omega$. Since $\theta(\omega) = 0 \in A$, we see that $\theta \in \xi(\omega)$. Let $f, g, h \in F(X)$ be such that $(f \oplus g) \oplus h \in \xi(\omega)$ and $h \in \xi(\omega)$. Then

$$(f(\omega) * g(\omega)) * h(\omega) = ((f \oplus g) \oplus h)(\omega) \in A \text{ and } h(\omega) \in A.$$

Since $A \triangleleft_c X$, it follows from (b3) that $(f \oplus (g \oplus (g \oplus f)))(\omega) = f(\omega) * (g(\omega) * (g(\omega) * f(\omega))) \in A$ so that $f \oplus (g \oplus (g \oplus f)) \in \xi(\omega)$. Hence $\xi(\omega)$ is a commutative ideal of $F(X)$. \square

Since

$$\xi^{-1}(\dot{f}) = \{\omega \in \Omega \mid f \in \xi(\omega)\} = \{\omega \in \Omega \mid f(\omega) \in A\} = A_f \in \mathcal{A},$$

we see that ξ is a random set on $F(X)$. Let

$$\tilde{H}(f) = P(\omega \mid f(\omega) \in A).$$

Then \tilde{H} is a falling fuzzy commutative ideal of $F(X)$.

Example 2.4. (1) Consider a BCK-algebra $X = \{0, a, b, c, d\}$ with a Cayley table which is given by Table 1 (see [5, p.263]). Let $(\Omega, \mathcal{A}, P) = ([0, 1], \mathcal{A}, m)$ and let $\xi m : [0, 1] \rightarrow \mathcal{P}(X)$ be defined by

$$\xi(t) := \begin{cases} \{0, c\} & \text{if } t \in [0, 0.15), \\ \{0, d\} & \text{if } t \in [0.15, 0.45), \\ \{0, a, b\} & \text{if } t \in [0.45, 0.75), \\ \{0, c, d\} & \text{if } t \in [0.75, 1]. \end{cases}$$

Then $\xi(t) \triangleleft_c X$ for all $t \in [0, 1]$. Hence \tilde{H} , which is given by $\tilde{H}(x) = P(t \mid x \in \xi(t))$, is a falling fuzzy commutative ideal of X , and it is represented as follows:

$$\tilde{H}(x) = \begin{cases} 1 & \text{if } x = 0, \\ 0.3 & \text{if } x \in \{a, b\} \\ 0.4 & \text{if } x = c, \\ 0.55 & \text{if } x = d. \end{cases}$$

Table 2. Cayley table

*	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	0	a
b	b	b	0	0	b
c	c	b	a	0	c
d	d	d	d	d	0

Then

$$\tilde{H}_t = \begin{cases} \{0\} & \text{if } t \in (0.55, 1], \\ \{0, d\} & \text{if } t \in (0.4, 0.55], \\ \{0, c, d\} & \text{if } t \in (0.3, 0.4], \\ X & \text{if } t \in [0, 0.3], \end{cases}$$

and so $\tilde{H}_t \triangleleft_c X$ for all $t \in [0, 1]$. Hence \tilde{H} is a fuzzy commutative ideal of X by Proposition 1.1.

(2) Consider a BCK-algebra $X = \{0, a, b, c, d\}$ with a Cayley table which is given by Table 2 (see [5, p.332]). Let $(\Omega, \mathcal{A}, P) = ([0, 1], \mathcal{A}, m)$ and let $\xi : [0, 1] \rightarrow \mathcal{P}(X)$ be defined by

$$\xi(t) := \begin{cases} \{0, a\} & \text{if } t \in [0, 0.2), \\ \{0, b\} & \text{if } t \in [0.2, 0.55), \\ \{0, b, d\} & \text{if } t \in [0.55, 0.75), \\ \{0, a, b, c\} & \text{if } t \in [0.75, 1]. \end{cases}$$

Then $\xi(t)$ is a commutative ideal of X for all $t \in [0, 1]$. Hence \tilde{H} , which is given by $\tilde{H}(x) = P(t \mid x \in \xi(t))$, is a falling fuzzy commutative ideal of X , and it is represented as follows:

$$\tilde{H}(x) = \begin{cases} 1 & \text{if } x = 0, \\ 0.45 & \text{if } x = a, \\ 0.8 & \text{if } x = b, \\ 0.25 & \text{if } x = c, \\ 0.2 & \text{if } x = d. \end{cases}$$

But \tilde{H} is not a fuzzy commutative ideal of X since

$$\tilde{H}(c * (d * (d * c))) = \tilde{H}(c) = 0.25 < 0.45 = \min\{\tilde{H}((c * d) * b), \tilde{H}(b)\}.$$

Theorem 2.5. *Every fuzzy commutative ideal of X is a falling fuzzy commutative ideal of X .*

Table 3. Cayley table

*	0	a	b	c	d
0	0	0	0	0	0
a	a	0	0	a	0
b	b	a	0	b	0
c	c	c	c	0	c
d	d	d	d	d	0

Proof. Consider the probability space $(\Omega, \mathcal{A}, P) = ([0, 1], \mathcal{A}, m)$, where \mathcal{A} is a Borel field on $[0, 1]$ and m is the usual Lebesgue measure. Let μ be a fuzzy commutative ideal of X . Then μ_t is a commutative ideal of X for all $t \in [0, 1]$ by Proposition 1.1. Let

$$\xi : [0, 1] \rightarrow \mathcal{P}(X)$$

be a random set and $\xi(t) = \mu_t$ for every $t \in [0, 1]$. Then μ is a falling fuzzy commutative ideal of X . \square

Example 2.4(2) shows that the converse of Theorem 2.5 is not valid.

Theorem 2.6. *Every falling fuzzy commutative ideal is a falling fuzzy ideal.*

Proof. Let \tilde{H} be a falling fuzzy commutative ideal of X . Then $\xi(\omega)$ is a commutative ideal of X , and hence it is an ideal of X . Thus \tilde{H} is a falling fuzzy ideal of X . \square

The converse of Theorem 2.6 is not true in general as shown by the following example.

Example 2.7. Consider a BCK-algebra $X = \{0, a, b, c, d\}$ with a Cayley table which is given by Table 3 (see [5, p.260]). Let $(\Omega, \mathcal{A}, P) = ([0, 1], \mathcal{A}, m)$ and let $\xi : [0, 1] \rightarrow \mathcal{P}(X)$ be defined by

$$\xi(t) := \begin{cases} \{0, c\} & \text{if } t \in [0, 0.3), \\ \{0, a, b, d\} & \text{if } t \in [0.3, 1]. \end{cases}$$

Then $\xi(t)$ is an ideal of X for all $t \in [0, 1]$. Hence $\tilde{H}(x) = P(t \mid x \in \xi(t))$ is a falling fuzzy ideal of X , and

$$\tilde{H}(x) = \begin{cases} 0.3 & \text{if } x = c, \\ 0.7 & \text{if } x \in \{a, b, d\}, \\ 1 & \text{if } x = 0. \end{cases}$$

In this case, we can easily check that \tilde{H} is a fuzzy ideal of X (see [3]). If $t \in [0, 0.3)$, then $\xi(t) = \{0, c\}$ which is not a commutative ideal of X since $(b * d) * c \in \xi(t)$ and

$c \in \xi(t)$, but $b * (d * (d * b)) \notin \xi(t)$. Therefore \tilde{H} is not a falling fuzzy commutative ideal of X .

Theorem 2.8. *In a commutative BCK-algebra, every falling fuzzy ideal is a falling fuzzy commutative ideal.*

Proof. Let \tilde{H} be a falling fuzzy ideal of a commutative BCK-algebra X . Then $\xi(\omega)$ is an ideal of X , and so it is a commutative ideal of X since every ideal of a commutative BCK-algebra is commutative. Therefore \tilde{H} is a falling fuzzy commutative ideal of X . \square

Corollary 2.9. *If a BCK-algebra X satisfies one of the following conditions:*

- (1) $(\forall x, y \in X) (x * y = 0 \implies x * (y * (y * x)) = 0)$,
- (2) $(\forall x, y \in X) (x \leq y \implies x = y * (y * x))$,
- (3) $(\forall x, y \in X) (x * (x * y) = y * (y * (x * (x * y))))$,
- (4) $(\forall x, y, z \in X) (x, y \leq z, z * y \leq z * x \implies x \leq y)$,
- (5) $(\forall x, y, z \in X) (x \leq z, z * y \leq z * x \implies x \leq y)$,

then every falling fuzzy ideal is a falling fuzzy commutative ideal.

Let (Ω, \mathcal{A}, P) be a probability space and \tilde{H} a falling shadow of a random set $\xi : \Omega \rightarrow \mathcal{P}(X)$. For any $x \in X$, let

$$(2.3) \quad \Omega(x; \xi) := \{\omega \in \Omega \mid x \in \xi(\omega)\}.$$

Then $\Omega(x; \xi) \in \mathcal{A}$.

Proposition 2.10. *If a falling shadow \tilde{H} of a random set $\xi : \Omega \rightarrow \mathcal{P}(X)$ is a falling fuzzy commutative ideal of X , then*

- (1) $(\forall x, y, z \in X) (\Omega((x * y) * z; \xi) \cap \Omega(z; \xi) \subseteq \Omega(x * (y * (y * x)); \xi))$.
- (2) $(\forall x, y, z \in X) (\Omega(x * (y * (y * x)); \xi) \subseteq \Omega((x * y) * z; \xi))$.

Proof. (1) Let $\omega \in \Omega((x * y) * z; \xi) \cap \Omega(z; \xi)$. Then $(x * y) * z \in \xi(\omega)$ and $z \in \xi(\omega)$. Since $\xi(\omega)$ is a commutative ideal of X , it follows from (b3) that $x * (y * (y * x)) \in \xi(\omega)$ so that $\omega \in \Omega(x * (y * (y * x)); \xi)$. Therefore

$$\Omega((x * y) * z; \xi) \cap \Omega(z; \xi) \subseteq \Omega(x * (y * (y * x)); \xi)$$

for all $x, y, z \in X$.

- (2) If $\omega \in \Omega(x * (y * (y * x)); \xi)$, then $x * (y * (y * x)) \in \xi(\omega)$. Note that

$$\begin{aligned}
& ((x * y) * z) * (x * (y * (y * x))) = ((x * y) * (x * (y * (y * x)))) * z \\
& \leq ((y * (y * x)) * y) * z = ((y * y) * (y * x)) * z \\
& = (0 * (y * x)) * z = 0 * z = 0.
\end{aligned}$$

This yields $((x * y) * z) * (x * (y * (y * x))) = 0 \in \xi(\omega)$. Since $\xi(\omega)$ is a commutative ideal and hence an ideal of X , it follows that $(x * y) * z \in \xi(\omega)$ and so $\omega \in \Omega((x * y) * z; \xi)$. Hence $\Omega(x * (y * (y * x)); \xi) \subseteq \Omega((x * y) * z; \xi)$ for all $x, y, z \in X$. \square

Theorem 2.11. *If \tilde{H} is a falling fuzzy commutative ideal of X , then*

$$(2.4) \quad (\forall x, y, z \in X) (\tilde{H}(x * (y * (y * x))) \geq T_m(\tilde{H}((x * y) * z), \tilde{H}(z)))$$

where $T_m(s, t) = \max\{s + t - 1, 0\}$ for any $s, t \in [0, 1]$.

Proof. By Definition 2.2, $\xi(\omega)$ is a commutative ideal of X for any $\omega \in \Omega$. Hence $\{\omega \in \Omega \mid (x * y) * z \in \xi(\omega)\} \cap \{\omega \in \Omega \mid z \in \xi(\omega)\} \subseteq \{\omega \in \Omega \mid x * (y * (y * x)) \in \xi(\omega)\}$, and thus

$$\begin{aligned}
\tilde{H}(x * (y * (y * x))) &= P(\omega \mid x * (y * (y * x)) \in \xi(\omega)) \\
&\geq P(\{\omega \mid (x * y) * z \in \xi(\omega)\} \cap \{\omega \mid z \in \xi(\omega)\}) \\
&\geq P(\omega \mid (x * y) * z \in \xi(\omega)) + P(\omega \mid z \in \xi(\omega)) \\
&\quad - P(\omega \mid (x * y) * z \in \xi(\omega) \text{ or } z \in \xi(\omega)) \\
&\geq \tilde{H}((x * y) * z) + \tilde{H}(z) - 1.
\end{aligned}$$

Hence

$$\begin{aligned}
\tilde{H}(x * (y * (y * x))) &\geq \max\{\tilde{H}((x * y) * z) + \tilde{H}(z) - 1, 0\} \\
&= T_m(\tilde{H}((x * y) * z), \tilde{H}(z)).
\end{aligned}$$

This completes the proof. \square

Theorem 2.11 means that every falling fuzzy commutative ideal of X is a T_m -fuzzy commutative ideal of X .

3. ACKNOWLEDGEMENTS

The authors wish to thank the anonymous reviewers for their valuable suggestions. The first author, Y.B. Jun, is an Executive Research Worker of Educational Research Institute in Gyeongsang National University.

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