J. Korean Soc. Math. Educ. Ser. B: Pure Appl. Math. ISSN 1226-0657
Volume 18, Number 1 (February 2011), Pages 79–86

AN ENERGY DENSITY ESTIMATE OF HEAT EQUATION FOR HARMONIC MAP

HYUN JUNG KIM

ABSTRACT. Suppose that (M, g) is a complete and noncompact Riemannian manifold with Ricci curvature bounded below by $-K \leq 0$ and $(N, \overline{g})$ is a complete Riemannian manifold with nonpositive sectional curvature. Let $u : M \times [0, \infty) \to N$ be the solution of a heat equation for harmonic map with a bounded image. We estimate the energy density of u.

1. INTRODUCTION AND NOTATIONS

Let M and N be Riemannian manifolds of dimension m and n respectively, and let $\{x^\alpha\}$ and $\{y^i\}$ be the local coordinates of M and N, respectively. Let $u : M \times [0, \infty) \to N$ be a map that is represented by $u = (u^1, \ldots, u^n)$ in terms of the above local coordinates. We say that u satisfies the heat equation for harmonic map if it is a solution of the following nonlinear parabolic system:

$$(1.1) \qquad (\Delta - \frac{\partial}{\partial t})u^i(x,t) = g^{\alpha\beta}(x)\Gamma^i_{jk}(u(x,t))\frac{\partial u^j}{\partial x^\alpha}(x,t)\frac{\partial u^k}{\partial x^\beta}(x,t)$$

for $i = 1, \ldots, m$, where $(g^{\alpha\beta}) = (g_{\alpha\beta})^{-1}$ and $\Gamma^i_{jk}(y)$ is the Christoffel symbol at y in N.

This heat equation for harmonic map is a nonlinear parabolic system, which has been proved to be useful in the study of harmonic maps. The nonlinear terms, which are due to the curvature of the target manifold, give distinct geometric meaning to this problem.

In this paper, we provide a gradient estimate of the solution of (1.1). When the target manifold N is $\mathbb{R}$, we have several types of gradient estimate. In this case, $u : M \times [0, \infty) \to \mathbb{R}$ satisfies the linear parabolic equation of the type

Received by the editors October 15, 2010. Revised February 10, 2011. Accepted Feb. 17, 2011.
2000 *Mathematics Subject Classification.* 58G11, 58G20.
Key words and phrases. heat equation for harmonic map, energy density, Liouville Theorem.

$$(\Delta - \frac{\partial}{\partial t})u(x,t) = 0. \tag{1.2}$$

In [9], Li and Yau proved the gradient estimate of the positive solution of (1.2).

Theorem 1.1 ([9]). *Suppose that M is a complete Riemannian manifold with Ricci curvature bounded below by $-K < 0$. Let $u : M \times [t_0 - T, t_0) \to \mathbb{R}$ be a positive solution to the linear heat equation* (1.2). *Let $a > 0$ and $T > 0$. Then for any $(x,t) \in B_{\frac{a}{2}}(x_0) \times [t_0 - \frac{T}{2}, t_0]$ and $\alpha > 1$,*

$$\frac{|\nabla u|^2}{u^2} - \alpha \frac{u_t}{u} \leq C\left(\frac{1}{R^2} + \frac{1}{T} + K\right)$$

for a positive constant $C > 0$ depending only on the dimension n of M and α.

This gradient estimate induces a parabolic Harnack inequality, which exhibits the phenomenon wherein the temperature at a given point in space-time is controlled by the temperature of that point at a later time. In [7], Hamilton proved a new gradient estimate for the solution of the linear heat equation, that can compared the temperatures of two different points at the same time provided the temperature is bounded.

Theorem 1.2 ([7]). *Suppose that M is a compact Riemannian manifold with Ricci curvature bounded below by $-K < 0$. Let $u : M \times [t_0 - T, t_0) \to \mathbb{R}$ be a positive solution of* (1.2) *with $u(x,t) \leq L$ for all $(x,t) \in M \times (0, \infty)$ and for some constant $L > 0$. Then*

$$\frac{|\nabla u|^2}{u^2} \leq \left(\frac{1}{t} + 2K\right) \ln \frac{L}{u}.$$

In this study, M is an $m-$dimensional noncompact manifold and N is an $n-$dimensional manifold and a map $u : M \times [0, \infty) \to \mathbb{R}$ is a solution of (1.1), that is a nonlinear parabolic equation. We estimate the energy density of the solution of (1.1). These gradient estimates have been useful to derive the harnack equality, the existence of harmonic function and the Liouville theorem and so on. We modified the method in [2], where Cheng estimated the energy density of a harmonic map.

Now we present some notations and equations that have been used in our proof. Let $u : M \times [0, \infty) \longrightarrow N$ be a smooth map. Choose a local orthonormal frame $\{e_\alpha, \frac{\partial}{\partial t}\}$ in a neighborhood of $(x,t) \in M \times [0, \infty)$ and a local orthonormal frame $\{f_i\}$ in a neighborhood of $u(x,t) \in N$. Let $\{\theta_\alpha, dt\}$ and $\{\omega_i\}$ be the dual coframes of $\{e_\alpha, \frac{\partial}{\partial t}\}$ and $\{f_i\}$ respectively. Let $\{\theta_{\alpha\beta}\}$ and $\{\omega_{ij}\}$ be the connection forms of M and N respectively.

Let $d = d_M + \frac{\partial}{\partial t}dt$ denote the exterior differentiation on $M \times [0, \infty)$, where d_M is the exterior differentiation on M. Then $u_{i\alpha}$ is defined as

$$u^*\omega_i = \sum_\alpha u_{i\alpha}\theta_\alpha + u_{it}dt.$$

The covariant derivative $u_{i\alpha\beta}$ of $u_{i\alpha}$ is defined as

$$\sum_\beta u_{i\alpha\beta}\theta_\beta + u_{i\alpha t}dt = du_{i\alpha} - \sum_j u_{j\alpha}u^*\omega_{ji} - \sum_\beta u_{i\beta}\theta_{\beta\alpha}.$$

Since $du_{i\alpha} = d_M u_{i\alpha} + u_{i\alpha t}dt$, we have that

$$\sum_\beta u_{i\alpha\beta}\theta_\beta = d_M u_{i\alpha} - \sum_j u_{j\alpha}u^*\omega_{ji} - \sum_\beta u_{i\beta}\theta_{\beta\alpha}.$$

Using the orthonormal frame, the solution of (1.1), that is the heat equation for harmonic map, can be simply expressed as

$$u_{it} = u_{i\alpha\alpha},$$

for $i = 1, \ldots, n$. We define the energy density $e(u)$ of u by $e(u) = \sum_{i\alpha} u_{i\alpha}^2$. Then the Bochner-type formula for the solution of (1.1), which will be of use in our proof, is given as

$$(1.3) \quad \frac{1}{2}(\Delta - \frac{\partial}{\partial t})e(u) = \sum_{i,\alpha,\beta} u_{i\alpha\beta}^2 - \sum_{i,j,k,l,\alpha,\beta} R_{ijkl}u_\alpha u_{j\beta}u_\alpha u_{l\beta} + \sum_{\alpha,\beta,i} K_{\alpha\beta}u_{i\alpha}u_{i\beta}$$

where R_{ijkl} is the curvature tensor of N and $K_{\alpha\beta}$ is the Ricci Curvature of M.

Consider the function $\rho^2(u(x,t))$ on $M \times [0, \infty)$ where ρ is the distance function from a fixed point $p \in N$. If the sectional curvature of N is nonnegative and $u : M \times [0, \infty) \to N$ is the solution of (1.1), we have that

$$(1.4) \quad (\Delta - \frac{\partial}{\partial t})\rho^2(u) = (\rho^2)_i(u_{i\alpha\alpha}) - u_{it} + D^2(\rho^2)(u_* e_\alpha, u_* e_\alpha)) \geq 2e(u).$$

This property of ρ will be used in main theorem.

2. Energy Density Estimate

In this section we present an estimate of the energy density of the heat equation for harmonic map. Our proof is a modification of the method used by Cheng [2] to the parabolic type.

Theorem 1. *Suppose that (M, g) is a complete Riemannian manifold with Ricci curvature bounded below by $-K < 0$ and $(N, \bar{g})$ is a complete and simply connected Riemannian manifold with nonpositive sectional curvature. Let $u : M \times [0, \infty) \to N$*

be a solution of (1.1). *Let* $a > 0$ *and* $T > 0$. *Let* $b_1 = \sup_{M\times[0,T]}(\rho \circ u)(x,t)$. *Then for any* $(x,t) \in B_{\frac{a}{2}}(x_0) \times (0,T]$

$$e(u)(x,t) \leq C\left(\frac{b_1^2 T}{a^2 t} + \frac{b_1^2}{t} + \frac{(aK+\sqrt{K})b_1^2 T}{at}\right)$$

for some constant $C > 0$.

Proof. The proof given here is a modification of the method of [4] and [2], that Let $B_a(x_0)$ be the closed geodesic ball of radius $a > 0$ and center x_0 in M and $\overline{B_a(x_0)}$ be the closed ball, that is $\overline{B_a(x_0)} = B_a(x_0) \cup \partial B_a(x_0)$. Let γ be the distance function in M from x_0 and let ρ denote the distance function in N from a point $p \in N$. Then we can choose a constant $b > 0$ such that $\sup_{M\times[0,T)}(\rho \circ u)(x,t) = b_1 < b$.

Let $a > 0$ and $T > 0$. Consider the function $\Phi : \overline{B_a(x_0)} \times [0,T] \to \mathbb{R}$ defined by

$$\Phi = \frac{t\,(a^2-\gamma^2)^2 e(u)}{(b^2-\rho^2\circ u)^2}.$$

Let

$$\Phi(x_1,t_1) = \max_{B_a(x_0)\times[0,T]} \frac{t\,(a^2-\gamma^2)^2 e(u)}{(b^2-\rho^2\circ u)^2}.$$

Then $\Phi(x_1,0) = 0$ and $\Phi(x_1,t) = 0$ for $x_1 \in \partial B_a(x_0)$. And the maximum of Φ should occur at $(x_1,t_1) \in B_a(x_0) \times (0,T]$. At $(x_1,t_1) \in B_a(x_0) \times (0,T]$, Φ has the following properties :

$$\Delta \log \Phi(x_1,t_1) \leq 0, \quad d\log\Phi(x_1,t_1) = 0 \quad \text{and} \quad \frac{\partial}{\partial t}\log\Phi(x_1,t_1) \geq 0.$$

Rewriting these at (x_1,t_1), we have

$$0 = \frac{-2d\gamma^2}{(a^2-\gamma^2)} + \frac{de(u)}{e(u)} + \frac{2d(\rho^2\circ u)}{(b^2-\rho^2\circ u)} \tag{2.1}$$

$$\begin{aligned} 0 \geq {} & \frac{-2\Delta\gamma^2}{(a^2-\gamma^2)} + \frac{-2|d\gamma^2|^2}{(a^2-\gamma^2)^2} + \frac{(\Delta - \frac{\partial}{\partial t})e(u)}{e(u)} - \frac{|de(u)|^2}{e(u)^2} \\ & + \frac{2(\Delta - \frac{\partial}{\partial t})(\rho^2\circ u)}{(b-\rho^2\circ u)} + \frac{2|d(\rho^2\circ u)|^2}{(b^2-\rho^2\circ u)^2} - \frac{1}{t}. \end{aligned} \tag{2.2}$$

Schwartz's inequality implies

$$|de(u)|^2 \leq 4\left(\sum_{i\alpha,\beta} u_{i\alpha\beta}^2\right)e(u). \tag{2.3}$$

Using the Bochner type formula (1.3) and our assumption of the curvatures, we get that

$$(\Delta - \frac{\partial}{\partial t})e(u) \geq \frac{1}{2}\frac{|de(u)|^2}{e(u)} - 2Ke(u). \tag{2.4}$$

By the Gauss lemma and the Schwartz inequality, we get that $|d\rho^2(u)| \leq 2\rho(u)\sqrt{e(u)}$ and $|d\gamma| = 1$. And the Hessian comparison theorem in [12] implies that

$$\Delta\gamma^2 \;=\; 2|d\gamma|^2 + 2\gamma\Delta\gamma \;\geq\; 2 + 2(n-1)(1+\sqrt{\gamma}) \geq C_0(1+\sqrt{K}\gamma)$$

for some constant $C_0 > 0$. Putting (1.4), (2.1) and (2.4) to (2.2), we have that

$$\begin{aligned} 0 &\geq -\frac{1}{t} + \frac{-2\Delta\gamma^2}{a^2-\gamma^2} - \frac{4|d\gamma^2|^2}{(a^2-\gamma^2)^2} - \frac{4|d\gamma^2||d\rho^2\circ u|}{(a^2-\gamma^2)(b^2-\rho^2\circ u)} - 2K + \frac{4e(u)}{(b^2-\rho^2\circ u)} \\ &\geq -\frac{1}{t} - 2K - \frac{-2C_0(1+\sqrt{K}\gamma)}{a^2-\gamma^2} - \frac{16\gamma^2}{(a^2-\gamma^2)^2} - \frac{16\gamma(\rho^2\circ u)}{(a^2-\gamma^2)(b^2-\rho^2\circ u)}\sqrt{e(u)} \\ &\quad + \frac{4e(u)}{(b^2-\rho^2\circ u)}. \end{aligned}$$

As in [4], we have that,

$$\begin{aligned} e(u)(x_1,t_1) \leq \max\Bigg\{ &\frac{64\gamma^2(\rho^2\circ u)}{(a^2-\gamma^2)^2}, \quad \frac{(b^2-\rho^2\circ u)}{2\,t_1} + K\,(b^2-\rho^2\circ u) \\ &+ \frac{C_0(1+\sqrt{K}\,\gamma)(b^2-\rho^2\circ u)}{(a^2-\gamma^2)} + \frac{8\gamma^2(b^2-\rho^2\circ u)}{(a^2-\gamma^2)^2}\Bigg\}\Bigg|_{x=x_1} \end{aligned}$$

If $(x,t) \in B_{\frac{a}{2}}(x_0) \times (0,T]$, then

$$\begin{aligned} &\left\{\frac{t\,(a^2-\gamma^2)^2 e(u)}{(b^2-\rho^2\circ u)^2}\right\}(x,t) \\ &\leq \left\{\frac{t_1\,(a^2-\gamma^2)^2 e(u)}{(b^2-\rho^2\circ u)^2}\right\}(x_1,t_1) \\ &\leq \max\Bigg\{\frac{64\gamma^2(\rho^2\circ u)t_1}{(b^2-\rho^2\circ u)^2}, \quad \frac{(a^2-\gamma^2)^2}{2(b^2-\rho^2\circ u)} + \frac{t_1\,K(a^2-\gamma^2)^2}{(b^2-\rho^2\circ u)} \\ &\qquad + \frac{C_0 t_1(1+\sqrt{K}\gamma)(a^2-\gamma^2)}{(b^2-\rho^2\circ u)} + \frac{8t_1\gamma^2}{(b^2-\rho^2\circ u)}\Bigg\}\Bigg|_{x=x_1} \\ &\leq \max\left\{\frac{16a^2b_1^2T}{\beta^2}, \ \frac{a^4}{\beta} + \frac{T\,K\,a^4}{\beta} + \frac{C_0(1+a\sqrt{K})a^2\,T}{\beta} + \frac{2T\,a^2}{\beta}\right\} \end{aligned}$$

where $\beta = \inf\{(b^2 - \rho^2 \circ u)(x) : x \in B_a(x_0)\}$. Therefore for any $(x,t) \in B_{\frac{a}{2}}(x_0) \times (0,T]$, we have that

$$\begin{aligned} e(u)(x,t) \le & \max\Bigg\{ \frac{16a^2b_1^2T}{\beta^2 t}\frac{(b^2-\rho^2\circ u)^2}{(a^2-\gamma^2)^2}, \frac{a^4}{4\beta^2 t}\frac{(b^2-\rho^2\circ u)^2}{(a^2-\gamma^2)^2} + \frac{Ka^4T}{2\beta t}\frac{(b^2-\rho^2\circ u)^2}{(a^2-\gamma^2)^2} \\ & + \frac{C_0(1+a\sqrt{K})a^2T}{\beta t}\frac{(b^2-\rho^2\circ u)^2}{(a^2-\gamma^2)^2} + \frac{2a^2T}{\beta t}\frac{(b^2-\rho^2\circ u)^2}{(a^2-\gamma^2)^2}\Bigg\} \\ \le & \max\left\{ \frac{256b_1^2b^4T}{9\beta^2a^2t}, \frac{8b^4}{9\beta t} + \frac{16Kb^4T}{9\beta t} + \frac{16C_0(1+a\sqrt{K})b^4t_1}{9a^2\beta} + \frac{32b^4T}{9a^2\beta t}\right\}. \end{aligned}$$

Letting $b = \sqrt{2}b_1$, we get that $2\beta = b^2 - b_1^2$ and $\beta = b_1^2$. Therefore we can have a constant $C > 0$ that

$$\begin{aligned} e(u)(x,t) \le & \max\left\{ \frac{256b_1^2T}{9a^2t}, \quad \frac{8b_1^2}{9t} + \frac{64Kb_1^2T}{t} + \frac{64C_0(1+a\sqrt{K})b_1^2T}{a^2t} + \frac{128b_1^2T}{9a^2t}\right\} \\ \le & C\left(\frac{b_1^2T}{a^2t} + \frac{b_1^2}{t} + \frac{(aK+\sqrt{K})b_1^2T}{at}\right). \end{aligned}$$

□

When the manifold (M,g) has nonnegative Ricci curvature, we have the simple gradient estimate which is shown below.

Corollary 1. *Suppose that M is a complete noncompact Riemannian manifold with nonnegative Ricci curvature and $(N,\overline{g})$ is a complete and simply connected Riemannian manifold with nonpositive sectional curvature. Let $u : M \times [0,\infty) \to N$ be a solution of* (1.1) *with the bounded image . Let T be a positive constant and $b_1 = \sup_{M\times[0,\infty)}(\rho\circ u)(x,t) < \infty$. Then we have that for any $(x,t) \in M \times (0,T]$*

$$e(u)(x,t) \le C\frac{b_1^2}{t},$$

for some constant $C > 0$.

Proof. Since M has nonnegative Ricci curvature, the lower bound of Ricci curvature of M is $K = 0$. By Theorem 1, for any $a > 0$ and for any $(x,t) \in B_{\frac{a}{2}}(x_0) \times (0,T]$,

$$e(u)(x,t) \le C\left(\frac{b_1^2T}{a^2t} + \frac{b_1^2}{t}\right).$$

As $a \to \infty$, our theorem is proved. □

We can deduce the Liouville theorem from the energy density estimate of the heat equation for harmonic map.

Corollary 2. *Suppose that* (M,g) *is a complete Riemannian manifold with nonnegative Ricci curvature and* $(N,\overline{g})$ *is a complete and simply connected Riemannian manifold with nonpositive sectional curvature. Let* $u : M \times [0,\infty) \to N$ *is a solution of* (1.1) *with bounded image. Then* $u(\cdot,t)$ *converges uniformly to a constant map that is also a harmonic map.*

Proof. Since the image of u from $M \times [0,\infty)$ is bounded, $u(x,t)$ converges on N as $t \to \infty$. By Theorem 4.3 in [10], $u(\cdot,t)$ converges uniformly to a harmonic map $u_\infty(\cdot)$ with their first and second derivatives as t goes to ∞. Using Corollary 1, we have that

$$\begin{aligned} e(u_\infty) &= \lim_{t\to\infty} e(u)(x,t) \\ &\leq \lim_{t\to\infty} C\frac{b_1^2}{t} = 0. \end{aligned}$$

Therefore, the energy density of u_∞ is zero and so the harmonic map u_∞ is constant. □

References

1. P. Aviles, H.I. Choi & M. Micallef: Boundary Behavior of Harmonic Maps on Nonsmooth Domains and Complete Negatively Curved Manifolds. *J. Funct. Anal.* **99** (1991), 293-331.
2. S. Cheng: The Liouville Theorem for Harmonic Maps. *Proc. Sym. Pure Math.* **36** (1980), 147-151.
3. H.I. Choi: On the Liouville Theorem for Harmonic Maps. *Proc. Amer. Math. Soc.* **85** (1982), 91-94.
4. D. Chi, H. Choi & H. Kim: Heat Equation for Harmonic Maps of the Compactificaion of Complete Manifolds. *J. Geom. Anal.* (1998).
5. H. Donnelly & P. Li: Heat Equation and Compactifications of Complete Riemannian Manifolds. *Duke Math. J.* **51** (1984), 667-673.
6. P. Eberlein & B. O'Neill: Visibility Manifolds. *Pacific J. Math.* **46** (1973), 45-109.
7. R.S. Hamilton: A Matrix Harnack Estimate for the Heat Equation. *Comm. Anal. Geom.* **1** (1993), 113-126.
8. P. Hsu: Heat Semigroup on a Complete Riemannian Manifold. *Annals of Prob.* **17** (1989), 1248-1254.
9. P. Li & S.T. Yau: On he Parabolic Kernel of the Schroedinger operator. *Acta Math.* **156** (1986), 153-201.

10. P. Li & L.F. Tam: The Heat Equation and Harmonic Maps of Complete Manifolds. *Invent. Math.* **105** (1991), 1-46.
11. G.G. Liao & L.-F. Tam: On the Heat Equation for Harmonic Maps from Non-compact Manifolds.
12. R. Scheon & S.T. Yau: *Lectures on Differential Geometry.* International Press, 1995, 1-23.

DEPARTMENT OF MATHEMATICS, HOSEO UNIVERSITY, 165 SECHUL-RI, ASAN, CHUNG-NAM, 336-795, KOREA
Email address: hjkim@hoseo.edu