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A NOTE ON CONNECTEDNESS IM KLEINEN IN $C(X)$

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Abstract. In this paper, we investigate the relationships between the space X and the hyperspace $C(X)$ concerning admissibility and connectedness im kleinen. The following results are obtained: Let X be a Hausdorff continuum, and let $A \in C(X)$. (1) If for each open set U containing A there is a continuum K and a neighborhood V of a point of A such that $V \subset IntK \subset K \subset U$, then $C(X)$ is *connected im kleinen.* at A. (2) If $IntA \neq \emptyset$, then for each open set U containing A there is a continuum K and a neighborhood V of a point of A such that $V \subset IntK \subset K \subset U$. (3) If X is *connected im kleinen.* at A, then A is admissible. (4) If A is admissible, then for any open subset $\mathcal{U}$ of $C(X)$ containing A, there is an open subset V of X such that $A \subset V \subset \bigcup \mathcal{U}$. (5) If for any open subset $\mathcal{U}$ of $C(X)$ containing A, there is a subcontinuum $\mathcal{K}$ of X such that $A \in Int\mathcal{K} \subset \mathcal{K} \subset \mathcal{U}$ and there is an open subset V of X such that $A \subset V \subset \bigcup Int\mathcal{K}$, then A is admissible.

0. Introduction

Let X be a Hausdorff continuum, and let $2^X(C(X), \mathcal{K}(X), C_K(X))$ the hyperspace of nonempty closed subsets(connected closed subsets, compact subsets, continua) of X with the Vietoris topology. Throughout by a *continuum* we mean a compact connected Hausdorff space. For a continuum X, $C(X)$ is endowed with the Vietoris topology and, since X is a continuum, the hyperspace $C(X)$ is also a continuum [8].

Wojdyslawsk [13] established the conditions of local connectedness between a space X and its hyperspace $2^X(C(X))$. Goodykoontz [3, 4, 5] investigated local connectedness as a pointwise property in the hyperspace $2^X(C(X))$ of metric continua. And Goodykoontz and Rhee [6] investigated the relationships between the space X and the hyperspaces concerning the properties of local compactness and local connectedness. They proved that a Hausdorff space X is connected im kleinen

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at $x \in X$ if and only if $2^X(\mathcal{K}(X),\ C_K(X))$ is connected im kleinen at $\{x\}$ and a locally compact Hausdorff space X is connected im kleinen at $x \in X$ if and only if $2^X(C(X),\ \mathcal{K}(X),\ C_K(X))$ is connected im kleinen at $\{x\}$. In 2003, Makuchowski [9, 10] investigated with respect to local connectedness at a subcontinuum of continua.

The purpose of this paper is to investigate the relationships between the space X and the hyperspace $C(X)$ concerning admissibility and connectedness im kleinen.

For notational purposes, small letters will denote elements of X, capital letters will denote subsets of X and elements of 2^X, and script letters are reserved for subsets of 2^X. If $\mathcal{B} \subset 2^X$, $\cup\mathcal{B} = \{A : A \in \mathcal{B}\}$. If $A \subset X$, the symbol $IntA(\overline{A}, Bd(A))$ will denote the interior(closure, boundary) of the set A.

1. Preliminaries

Let X be a topological space. Let $2^X = \{E \subset X : E \text{ is nonempty and closed}\}$, $\mathcal{K}(X) = \{E \in 2^X : E \text{ is compact}\}$, $C(X) = \{E \in 2^X : E \text{ is connected}\}$, and $C_K(X) = \mathcal{K}(X) \cap C(X)$, and endow each with the Vietoris topology. A basis for 2^X consists of all elements of the form

$$< U_1, U_2, \cdots, U_n >= \{A \in 2^X : A \cap U_i \neq \emptyset$$

for each i and $A \subset \bigcup_{i=1}^n U_i\}$, where $U_1, U_2, \cdots, U_n$ are open sets in X.

Let $T(x) = \{A \in C(X) : x \in A\}$. An element $A \in T(x)$ is said to be *admissible* at x in X if, for each basic open set $< U_1, U_2, ..., U_n > \cap C(X)$ containing A, there is a neighborhood V_x of x in X such that whenever $y \in V_x$ there is an element $B \in T(y)$ such that $B \in < U_1, U_2, ..., U_n > \cap C(X)$ [11].

The space X is said to be *locally connected* at x in X, if for each neighborhood U of x there is a connected neighborhood V of x such that $V \subset U$ [7]. The space X is said to be *connected im kleinen* at x, if for each neighborhood U of x there is a component of U which contains x in its interior [7, 9]. The space X is said to be *locally connected* provided that X is locally connected at each of its points. If a space X is connected im kleinen at each of its points, then X is locally connected.
The space X is said to be *locally arcwise connected at x*, if for each neighborhood U of x there is an arcwise connected neighborhood V of x such that $V \subset U$. The space X is said to be *locally arcwise connected*, if X is locally arcwise connected at each of its points. The space X is said to be *arcwise connected im kleinen* at x, if for each neighborhood U of x there is an arcwise connected, component of U which

contains x in its interior. If a space X is arcwise connected im kleinen at each of its points, then X is locally arcwise connected.

A continuum X is said to be *connected im kleinen* at a subcontinuum A, if for each open subset U of X containing A, there is a subcontinuum K such that $A \subset IntK \subset K \subset U$ [10]. A continuum X is said to be *locally connected* at a subcontinuum A, if for each open subset U of X containing A, there is an open connected subset V such that $A \subset V \subset U$ [1]. Obviously, if a subcontinuum is degenerate, then the notion of connectedness im kleinen(local connectedness) at a subcontinuum is the same as the notion of connectedness im kleinen(local connectedness) at a point. Note that if X is connected im kleinen(locally connected) at each point of A, then X is connected im kleinen(locally connected) at a subcontinuum A, but not conversely,

Result 1.1 ([12])**.** (Boundary Bumpping Theorem) *Let X be a Hausdorff continuum, and let $A \in C(X)$. Then for each open set U in X containing A, the component C_A of $\overline{U}$ containing A intersects $Bd(U)$.*

2. Connectedness im Kleinen and Admissibility

Theorem 2.1. *Let X be a Hausdorff continuum, and let $A \in C(X)$. If for each open set U containing A there is a continuum K and a neighborhood V of a point of A such that $V \subset IntK \subset K \subset U$, then $C(X)$ is connected im kleinen at A.*

Proof. Let $\mathcal{U} =< U_1, \cdots, U_n > \cap C(X)$ be an open subset of $C(X)$ containing A. Then $U = \bigcup_{i=1}^{n} U_i$ is an open subset of X containing A. And, there is a continuum K and a neighborhood V of a point x of A such that

$$V \subset IntK \subset K \subset U.$$

And

$$\begin{aligned} A = A \cup \{x\} &\in< U_1, \cdots, U_n, V > \cap C(X) \\ &\subset< U_1, \cdots, U_n, IntK > \cap C(X) \\ &\subset< U_1, \cdots, U_n, K > \cap C(X) \subset \mathcal{U}. \end{aligned}$$

Let $L_1, L_2 \in< U_1, \cdots, U_n, V > \cap C(X)$. Then $L_1 \cap K \neq \emptyset$ and $L_2 \cap K \neq \emptyset$. It follows that $L_1 \cup L_2 \cup K \in< U_1, \cdots, U_n, V > \cap C(X)$. Hence there is order arcs $\mathcal{L}_1$ and $\mathcal{L}_2$ in $< U_1, \cdots, U_n, K > \cap C(X)$ from L_1 to $L_1 \cup L_2 \cup K$ and from L_2 to $L_1 \cup L_2 \cup K$. It follows that there is an arc in $\mathcal{L}_1 \cup \mathcal{L}_2$ from L_1 to L_2, and it is clear that $\mathcal{L}_1 \cup \mathcal{L}_2 \subset < U_1, \cdots, U_n, V > \cap C(X)$. Therefore $C(X)$ is *locally arcwise connected* at A. □

Theorem 2.2. Let $A \in C(X)$. If $IntA \neq \emptyset$, then for each open set U containing A there is a continuum K and a neighborhood V of a point of A such that $V \subset IntK \subset K \subset U$.

Proof. Let U be an open set containing A. Let $x \in IntA$. Then there is an open set V such that $x \in V \subset IntA$, and hence $x \in IntA \subset A \subset U$. In this case A is a continuum which satisfies the condition of the continuum K in this theorem. □

We get the below Corollary from Theorem 2.1 and Theorem 2.2.

Corollary 2.3 ([Theorem 3 of [4]])**.** *Let $A \in C(X)$. If $IntA \neq \emptyset$, then $C(X)$ is locally arcwise connected at A.*

Proof. Let $A \in C(X)$ and let $< U_1, \cdots, U_n > \cap C(X)$ be a basic open set containing A. Let $x \in IntA$ and let V be an open set such that $x \in V \subset IntA$ and such that $V \subset \bigcap\{U_i \mid x \in U_i\}$. Then $A \in < U_1, \cdots, U_n, V > \subset < U_1, \cdots, U_n >$. Let $L_1, L_2 \in < U_1, \cdots, U_n, V > \cap C(X)$. Then $L_1 \cap V \neq \emptyset$ and $L_2 \cap V \neq \emptyset$, so $L_1 \cap A \neq \emptyset$ and $L_2 \cap A \neq \emptyset$. It follows that $L_1 \cup L_2 \cup A \in < U_1, \cdots, U_n, V > \cap C(X)$. Hence there is order arcs $\mathcal{L}_1$ and $\mathcal{L}_2$ in $< U_1, \cdots, U_n, V > \cap C(X)$ from L_1 to $L_1 \cup L_2 \cup A$ and from L_2 to $L_1 \cup L_2 \cup A$. It follows that there is an arc in $\mathcal{L}_1 \cup \mathcal{L}_2$ from L_1 to L_2, and it is clear that $\mathcal{L}_1 \cup \mathcal{L}_2 \subset < U_1, \cdots, U_n, V > \cap C(X)$. □

Theorem 2.4. *Let X be a Hausdorff continuum, and let $A \in C(X)$. If X is connected im kleinen at A, then A is admissible.*

Proof. Let $x \in A \in C(X)$ and X is *connected im kleinen* at A. Let $< U_1, \cdots, U_n > \cap C(X)$ be a basic open set containing A, and let $U = \bigcup_{i=1}^{n} U_i$. Then $A \subset U$ and there is a continuum K such that $A \subset IntK \subset K \subset U$. Set $V_x = IntK$. Then for every $y \in V_x$, y is an element of K. And since $A \subset K$ and $K \subset U$, $K \in < U_1, \cdots, U_n > \cap C(X)$. Thus A is admissible. □

Theorem 2.5. *Let X be a Hausdorff continuum, and let $A \in C(X)$. If A is admissible, then for any open subset $\mathcal{U}$ of $C(X)$ containing A, there is an open subset V of X such that $A \subset V \subset \bigcup \mathcal{U}$.*

Proof. Let $\mathcal{U}$ be an open set containing A in $C(X)$, and let $x \in A$. Then by the definition of admissibility there is an open set V_x containing x in X such that for every $y \in V_x$ there is a continuum B in $C(X)$ such that $y \in B \in \mathcal{U}$. Set $V = \bigcup_{x \in A} V_x$. Then $A \subset V \subset \bigcup \mathcal{U}$. □

Theorem 2.6. *Let X be a Hausdorff continuum, and let $A \in C(X)$. If for any open subset $\mathcal{U}$ of $C(X)$ containing A, there is a subcontinuum $\mathcal{K}$ of X such that $A \in Int\mathcal{K} \subset \mathcal{K} \subset \mathcal{U}$ and there is an open subset V of X such that $A \subset V \subset \bigcup Int\mathcal{K}$, then A is admissible.*

Proof. Let $\mathcal{U} =< U_1, \cdots, U_n > \cap C(X)$ be a basic open subset of $C(X)$ containing A, let $\mathcal{K}$ a continuum in $C(X)$ contains A in its interior, let V an open subset of X such that $A \subset V \subset \bigcup Int\mathcal{K} \subset \bigcup \mathcal{K}$. Then for any element y of V, $K = \bigcup \mathcal{K}$ is a continuum in $\mathcal{U}$ containing y. □

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