J. Korean Soc. Math. Educ. Ser. B: Pure Appl. Math.
http://dx.doi.org/10.7468/jksmeb.2015.22.2.169
Volume 22, Number 2 (May 2015), Pages 169–178

ISSN(Print) 1226-0657
ISSN(Online) 2287-6081

CARATHÉODORY'S INEQUALITY ON THE BOUNDARY

Bülent Nafi Örnek

Abstract. In this paper, a boundary version of Carathéodory's inequality is investigated. Also, new inequalities of the Carathéodory's inequality at boundary are obtained and the sharpness of these inequalities is proved.

1. Introduction

In recent years, boundary version of Schwarz lemma was investigated in D. M. Burns and S. G. Krantz [6], R. Osserman [8], V. N. Dubinin [2], M. Jeong [4, 5], H. P. Boas [1] and other's studies. On the other hand, in the book [7], Sharp Real-Parts Theorem's (in particular Carathéodory's inequalities), which are frequently used in the theory of entire functions and analytic function theory, have been studied.

The classical Schwarz lemma states that an holomorphic function f mapping the unit disc $D = \{z : |z| < 1\}$ into itself, with $f(0) = 0$, satisfies the inequality $|f(z)| \leq |z|$ for any point $z \in D$ and $|f'(0)| \leq 1$. Equality in these inequalities (in the first one, for $z \neq 0$) occurs only if $f(z) = \lambda z$, $|\lambda| = 1$ [3, p.329]. It is an elementary consequence of Schwarz lemma that if f extends continuously to some boundary point z_0 with $|z_0| = 1$, and if $|f(z_0)| = 1$ and $f'(z_0)$ exists, then $|f'(z_0)| \geq 1$, which is known as the Schwarz lemma on the boundary.

In this paper, we studied "boundary Carathéodory's inequalities" as analog the Schwarz lemma at the boundary [8].

The Carathéodory's inequality states that, if the function f is holomorphic on the unit disc D with $f(0) = 0$ and $\Re f \leq A$ in D, then the inequality

$$|f(z)| \leq \frac{2A\,|z|}{1-|z|}, \tag{1.1}$$

Received by the editors January 30, 2015. Revised April 21, 2015. Accepted May 01, 2015.
2010 *Mathematics Subject Classification.* 30C80.
Key words and phrases. Schwarz lemma on the boundary, Carathéodory's inequality, Angular limit and derivative, Julia-Wolff-Lemma.

holds for all $z \in D$, and moreover

$$\left|f'(0)\right| \leq 2A. \tag{1.2}$$

Equality is achieved in (1.1) (for some nonzero $z \in D$) or in (1.2) if and only if $f(z)$ is the function of the form

$$f(z) = \frac{2Aze^{i\theta}}{1 + ze^{i\theta}},$$

where θ is a real number [7, pp.3-4].

Robert Osserman considered the case that only one boundary fixed point of f is given and obtained a sharp estimate based on the values of the function. He has first showed that

$$\left|f'(z_0)\right| \geqslant \frac{2}{1 + |f'(0)|} \tag{1.3}$$

and

$$\left|f'(z_0)\right| \geqslant 1, \tag{1.4}$$

under the assuumption $f(0) = 0$ where f is a holomorphic function mapping the unit disc into itself and z_0 is a boundary point to which f extends continuously and $|f(z_0)| = 1$. In addition, the equality in (1.3) holds if and only if f is of the form

$$f(z) = ze^{i\theta}\frac{z - \alpha}{1 - \overline{\alpha}z},$$

where θ is a real number and $\alpha \in D$ satisfies $\arg\alpha = \arg z_0$. Also, the equality in (1.4) holds if and only if $f(z) = ze^{i\theta}$, where θ is a real number.

Moreover, if $f(z) = c_p z^p + c_{p+1} z^{p+1} + ...$, then

$$\left|f'(z_0)\right| \geqslant p + \frac{1 - |c_p|}{1 + |c_p|}. \tag{1.5}$$

It follows that

$$\left|f'(z_0)\right| \geqslant p, \tag{1.6}$$

with equality only if f is of the form $f(z) = z^p e^{i\theta}$, θ real [8].

If, in addition, the function f has an angular limit $f(z_0)$ at $z_0 \in \partial D$, $|f(z_0)| = 1$, then by the Julia-Wolff lemma the angular derivative $f'(z_0)$ exists and $1 \leq |f'(z_0)| \leq \infty$ (see [11]).

The inequality (1.5) is a particular case of a result due to Vladimir N. Dubinin in (see [2]), who strengthened the inequality $|f'(z_0)| \geq 1$ by involving zeros of the function f. Some other types of strengthening inequalities are obtained in (see [9], [10]).

We have following results, which can be offered as the boundary refinement of the Carathéodory's inequality.

Theorem 1.1. *Let f be a holomorphic function in the unit disc D, $f(0)=0$ and $\Re f \leqslant A$ for $|z|<1$. Further assume that, for some $z_0 \in \partial D$, f has an angular limit $f(z_0)$ at z_0, $\Re f(z_0)=A$. Then*

$$\left|f'(z_0)\right| \geqslant \frac{A}{2}. \tag{1.7}$$

Moreover, the equality in (1.7) *holds if and if*

$$f(z)=2A\frac{ze^{i\theta}}{1+ze^{i\theta}},$$

wehere θ is a real number.

Proof. The function

$$\varphi(z)=\frac{f(z)}{f(z)-2A} \tag{1.8}$$

is holomorphic in the unit disc D, $|\varphi(z)|<1$, $\varphi(0)=0$ and $|\varphi(z_0)|=1$ for $z_0 \in \partial D$. That is,

$$\begin{aligned} |f(z)-2A|^2 &= |f(z)|^2-2\Re\left(f(z)2A\right)+4A^2 \\ &= |f(z)|^2-4A\Re\left(f(z)\right)+4A^2. \end{aligned}$$

From the hypothesis, since $\Re f(z) \leqslant A$ and $4A\Re f(z) \leqslant 4A^2$, we take

$$|2A-f(z)|^2 \geq |f(z)|^2-4A\Re f(z)+4A\Re f(z)=|f(z)|^2.$$

Therefore, we obtain

$$\left|\frac{f(z)}{f(z)-2A}\right|<1.$$

From (1.4), we obtain

$$1 \leq \left|\varphi'(z_0)\right| = \frac{2A\left|f'(z_0)\right|}{|f(z_0)-2A|^2} \leq \frac{2A\left|f'(z_0)\right|}{A^2} = \frac{2\left|f'(z_0)\right|}{A}. \tag{1.9}$$

So, we take

$$\left|f'(z_0)\right| \geqslant \frac{A}{2}.$$

If $|f'(z_0)|=\frac{A}{2}$ from (1.9) and $|\varphi'(z_0)|=1$, we obtain

$$f(z)=2A\frac{ze^{i\theta}}{1+ze^{i\theta}}.$$

□

Theorem 1.2. *Let f be a holomorphic function in the unit disc D, $f(0)=0$ and $\Re f \leqslant A$ for $|z|<1$. Further assume that, for some $z_0 \in \partial D$, f has an angular limit $f(z_0)$ at z_0, $\Re f(z_0)=A$. Then*

$$\left|f'(z_0)\right| \geq \frac{2A^2}{2A+|f'(0)|}. \tag{1.10}$$

The inequality (1.10) *is sharp, with equality for the function*

$$f(z)=2Az\frac{z+a}{1+2az+z^2},$$

where $a=\frac{|f'(0)|}{2A}$ is an arbitrary number on $[0,1]$ (see (1.2)).

Proof. Using the inequality (1.3) for the function (1.8), we obtain

$$\left|\varphi'(z_0)\right| \geq \frac{2}{1+|\varphi'(0)|},$$

$$\frac{2\,|f'(z_0)|}{A} \geq \frac{2A\,|f'(z_0)|}{|f(z_0)-2A|^2} \geq \frac{4A}{2A+|f'(0)|}$$

and

$$\left|f'(z_0)\right| \geq \frac{2A^2}{2A+|f'(0)|}.$$

Now, we shall show that the inequality (1.10) is sharp. Choose arbitrary $a \in [0,1]$. Let

$$f(z)=2Az\frac{z+a}{1+2az+z^2}.$$

Then

$$f'(z)=2A\frac{az^2+2z+a}{(1+2az+z^2)^2}$$

and

$$f'(1)=\frac{A}{1+a}.$$

Since $|f'(0)|=2Aa$, (1.10) is satisfied with equality. □

An interesting special case of Theorem1.2 is when $f'(0)=0$, in which case inequality (1.10) implies $|f'(z_0)| \geqslant A$. Clearly equality holds for $f(z)=\frac{2Az^2e^{i\theta}}{1+z^2e^{i\theta}}$, θ real.

Now, if $f(z)=c_pz^p+c_{p+1}z^{p+1}+....$, is a holomorphic function in the unit disc D and $\Re f \leqslant A$ for $|z|<1$, it can be seen that Carathéodory's inequality can be obtained with standard methods as follows:

$$|f(z)| \leq \frac{2A\,|z|^p}{1-|z|^p}$$

and

$$|c_p| \leq 2A. \tag{1.11}$$

The following result is a generalization of Theorem1.1.

Theorem 1.3. *Let* $f(z) = c_p z^p + c_{p+1} z^{p+1} + ...,\ c_p \neq 0,\ p \geq 1$ *be a holomorphic function in the unit disc* D *and* $\Re f \leqslant A$ *for* $|z| < 1$. *Further assume that, for some* $z_0 \in \partial D$, f *has an angular limit* $f(z_0)$ *at* z_0, $\Re f(z_0) = A$. *Then*

$$\left|f'(z_0)\right| \geqslant \frac{A}{2} p. \tag{1.12}$$

In addition, the equality in (1.12) *holds if and if*

$$f(z) = 2A \frac{z^p e^{i\theta}}{1 + z^p e^{i\theta}}.$$

wehere θ *is a real number.*

Proof. Using the inequality (1.6) for the function (1.8), we obtain

$$p \leq \left|\varphi'(z_0)\right| = \frac{2A\left|f'(z_0)\right|}{\left|f(z) - 2A\right|^2} \leq \frac{2\left|f'(z_0)\right|}{A}. \tag{1.13}$$

Therefore, we take

$$\left|f'(z_0)\right| \geqslant \frac{A}{2} p.$$

If $|f'(z_0)| = \frac{A}{2} p$ from (1.13) and $|\varphi'(z_0)| = p$, we obtain

$$f(z) = 2A \frac{z^p e^{i\theta}}{1 + z^p e^{i\theta}}.$$

□

Theorem 1.4. *Under hypotheses of Theorem*1.3, *we have*

$$\left|f'(z_0)\right| \geqslant \frac{A}{2} \left(p + \frac{2A - |c_p|}{2A + |c_p|}\right). \tag{1.14}$$

The inequality (1.14) *is sharp, with equality for the function*

$$f(z) = 2Az^p \frac{b + z}{1 + bz + bz^p + z^{p+1}},$$

where $b = \frac{|c_p|}{2A}$ *is arbitrary number from* $[0, 1]$ (*see* (1.11)).

Proof. Using the inequality (1.5) for the function (1.8), we obtain

$$\left|\varphi'(z_0)\right| \geq p + \frac{1 - |a_p|}{1 + |a_p|},$$

where $|a_p| = \frac{\left|\varphi^{(p)}(0)\right|}{p!}$. Since

$$|a_p| = \frac{\left|\varphi^{(p)}(0)\right|}{p!} = \frac{\left|f^{(p)}(0)\right|}{p!2A} = \frac{|c_p|}{2A},$$

we may write

$$p + \frac{1 - \frac{|c_p|}{2A}}{1 + \frac{|c_p|}{2A}} \leq \frac{2A\,|f'(z_0)|}{|f(z) - 2A|^2}.$$

Thus, we take

$$\left|f'(z_0)\right| \geq \frac{A}{2}\left(p + \frac{2A - |c_p|}{2A + |c_p|}\right).$$

The equality in (1.14) is obtained for function

$$f(z) = 2Az^p \frac{z + b}{1 + bz + bz^p + z^{p+1}},\ \ 0 \leq b \leq 1,$$

as show simple calculations. □

Consider the following product:

$$B(z) = \prod_{k=1}^{n} \frac{z - a_k}{1 - \overline{a_k} z}$$

$B(z)$ is called a finite Blaschke product, where $a_1, a_2, ..., a_n \in \mathbb{C}$. Let the function $f(z) = c_p z^p + ...$ satisfy the conditions of Carathéodory's inequality and also have zeros $a_1, a_2, ..., a_n$ with order $k_1, k_2, \ldots, k_n$, respectively. Thus, one can see that Carathéodory's inequality can be strengthened with the standard methods as follows:

$$|f(z)| \leq \frac{2A\,|z|^p\,|B(z)|}{1 - |z|^p\,|B(z)|}, \tag{1.15}$$

and

$$|c_p| \leq 2A \prod_{k=1}^{n} |a_k|. \tag{1.16}$$

The inequalities (1.15) and (1.16) show that the inequalities (1.1) and (1.2) will be able to be strengthened, if the zeros of function which are different from origin of $f(z)$ in the (1.12) and (1.14) are taken into account.

Theorem 1.5. *Let $f(z) = c_p z^p + c_{p+1} z^{p+1} + ...,\ c_p \neq 0,\ p \geq 1$ be a holomorphic function in the unit disc D, and $\Re f \leqslant A$ for $|z| < 1$. Assume that for some $z_0 \in \partial D$, f has an angular limit $f(z_0)$ at z_0, $\Re f(z_0) = A$. Let $a_1, a_2, ..., a_n$ be zeros of the* *function f in D that are different from zero. Then we have the inequality*

$$\left|f'(z_0)\right| \geqslant \frac{A}{2}\left\{p+\sum_{k=1}^{n}\frac{1-|a_k|^2}{|z_0-a_k|^2}+\frac{2A\prod\limits_{k=1}^{n}|a_k|-|c_p|}{2A\prod\limits_{k=1}^{n}|a_k|+|c_p|}\right\}. \tag{1.17}$$

In addition, the equality in (1.17) *occurs for the function*

$$f(z)=2A\frac{z^p\prod\limits_{k=1}^{n}\frac{z-a_k}{1-\overline{a_k}z}}{1+z^p\prod\limits_{k=1}^{n}\frac{z-a_k}{1-\overline{a_k}z}},$$

where $a_1, a_2, ..., a_n$ are positive real numbers.

Proof. Let $\varphi(z)$ be as in the proof of Theorem1.1 and $a_1, a_2, ..., a_n$ be zeros of the function f in D that are different from zero.

$$B(z)=\prod_{k=1}^{n}\frac{z-a_k}{1-\overline{a_k}z}$$

is a holomorphic functions in D, and $|B(z)|<1$ *for* $|z|<1$. By the maximum principle for each $z\in D$, we have

$$|\varphi(z)|\leq|B(z)|.$$

The auxiliary function

$$\phi(z)=\frac{\varphi(z)}{B(z)}=\frac{f(z)}{f(z)-2A}\frac{1}{\prod\limits_{k=1}^{n}\frac{z-a_k}{1-\overline{a_k}z}}$$

is holomorphic in D, and $|\phi(z)|<1$ for $|z|<1$, $\phi(0)=0$ and $|\phi(z_0)|=1$ for $\mathrm{z}_0\in\partial D$. Moreover, it can be seen that

$$\frac{z_0\varphi'(z_0)}{\varphi(z_0)}=\left|\varphi'(z_0)\right|\geq\left|B'(z_0)\right|=\frac{z_0B'(z_0)}{B(z_0)}.$$

Besides, with the simple calculations, we take

$$\left|B'(z_0)\right|=\frac{z_0B'(z_0)}{B(z_0)}=\sum_{k=1}^{n}\frac{1-|a_k|^2}{|z_0-a_k|^2}.$$

From (1.5), we obtain

$$p+\frac{1-|a_p|}{1+|a_p|}\leq\left|\phi'(z_0)\right|=\left|\frac{z_0\varphi'(z_0)}{\varphi(z_0)}-\frac{z_0B'(z_0)}{B(z_0)}\right|=\left\{\left|\varphi'(z_0)\right|-\left|B'(z_0)\right|\right\},$$

where $|a_p| = \frac{\left|\phi^{(p)}(0)\right|}{p!}$. Since

$$|a_p| = \frac{\left|\phi^{(p)}(0)\right|}{p!} = \frac{\left|f^{(p)}(0)\right|}{p!2A\prod\limits_{k=1}^{n}|a_k|} = \frac{|c_p|}{2A\prod\limits_{k=1}^{n}|a_k|},$$

we may write

$$p + \frac{2A\prod\limits_{k=1}^{n}|a_k| - |c_p|}{2A\prod\limits_{k=1}^{n}|a_k| + |c_p|} \leq \left\{\frac{2A\left|f'(z_0)\right|}{\left|f(z_0)-2A\right|^2} - \sum_{k=1}^{n}\frac{1-|a_k|^2}{|z_0-a_k|^2}\right\}.$$

Therefore, we have

$$\left|f'(z_0)\right| \geqslant \frac{A}{2}\left\{p + \sum_{k=1}^{n}\frac{1-|a_k|^2}{|z_0-a_k|^2} + \frac{2A\prod\limits_{k=1}^{n}|a_k| - |c_p|}{2A\prod\limits_{k=1}^{n}|a_k| + |c_p|}\right\}.$$

Now, we shall show that the inequality (1.17) is sharp. Let

$$f(z) = 2A\frac{z^p\prod\limits_{k=1}^{n}\frac{z-a_k}{1-\overline{a_k}z}}{1+z^p\prod\limits_{k=1}^{n}\frac{z-a_k}{1-\overline{a_k}z}}.$$

Then

$$\begin{aligned} f'(z) &= 2A\frac{\left(pz^{p-1}\prod\limits_{k=1}^{n}\frac{z-a_k}{1-\overline{a_k}z} + \sum\limits_{k=1}^{n}\frac{1-|a_k|^2}{(1-\overline{a_k}z)^2}\prod\limits_{\substack{k\neq i\\ i=1}}^{n}\frac{z-a_i}{1-\overline{a_i}z}z^p\right)\left(1+z^p\prod\limits_{k=1}^{n}\frac{z-a_k}{1-\overline{a_k}z}\right)}{\left(1+z^p\prod\limits_{k=1}^{n}\frac{z-a_k}{1-\overline{a_k}z}\right)^2} \\ &\quad -2A\frac{\left(pz^{p-1}\prod\limits_{k=1}^{n}\frac{z-a_k}{1-\overline{a_k}z} + \sum\limits_{k=1}^{n}\frac{1-|a_k|^2}{(1-\overline{a_k}z)^2}\prod\limits_{\substack{k\neq i\\ i=1}}^{n}\frac{z-a_i}{1-\overline{a_i}z}z^p\right)z^p\prod\limits_{k=1}^{n}\frac{z-a_k}{1-\overline{a_k}z}}{\left(1+z^p\prod\limits_{k=1}^{n}\frac{z-a_k}{1-\overline{a_k}z}\right)^2} \end{aligned}$$

and

$$f'(1) = 2A\frac{\left(p\prod\limits_{k=1}^{n}\frac{1-a_k}{1-\overline{a_k}}+\sum\limits_{k=1}^{n}\frac{1-|a_k|^2}{(1-\overline{a_k})^2}\prod\limits_{\substack{k\neq i\\ i=1}}^{n}\frac{1-a_i}{1-\overline{a_i}}\right)\left(1+\prod\limits_{k=1}^{n}\frac{1-a_k}{1-\overline{a_k}}\right)}{\left(1+\prod\limits_{k=1}^{n}\frac{1-a_k}{1-\overline{a_k}}\right)^2}$$

$$-2A\frac{\left(p\prod\limits_{k=1}^{n}\frac{1-a_k}{1-\overline{a_k}}+\sum\limits_{k=1}^{n}\frac{1-|a_k|^2}{(1-\overline{a_k})^2}\prod\limits_{\substack{k\neq i\\ i=1}}^{n}\frac{1-a_i}{1-\overline{a_i}}\right)\prod\limits_{k=1}^{n}\frac{1-a_k}{1-\overline{a_k}}}{\left(1+\prod\limits_{k=1}^{n}\frac{1-a_k}{1-\overline{a_k}}\right)^2}.$$

Since $a_1, a_2, ..., a_n$ are positive real numbers, we take

$$f'(1) = \frac{A}{2}\left(p+\sum_{k=1}^{n}\frac{1-|a_k|^2}{|1-a_k|^2}\right).$$

Since $|c_p| = 2A\prod\limits_{k=1}^{n}|a_k|$, (1.17) is satisfied with equality. □

References

1. H.P. Boas: Julius and Julia: Mastering the Art of the Schwarz lemma. *Amer. Math. Monthly* **117** (2010), 770-785.
2. V.N. Dubinin: The Schwarz inequality on the boundary for functions regular in the disc. *J. Math. Sci.* **122** (2004), no. 6, 3623-3629.
3. G.M. Golusin: *Geometric Theory of Functions of Complex Variable [in Russian]*, 2nd edn., Moscow 1966.
4. M. Jeong: The Schwarz lemma and its applications at a boundary point. *J. Korean Soc. Math. Educ. Ser. B: Pure Appl. Math.* **21** (2014), no. 3, 275-284.
5. ———: The Schwarz lemma and boundary fixed points. *J. Korean Soc. Math. Educ. Ser. B: Pure Appl. Math.* **18** (2011), no. 3, 219-227.
6. D.M. Burns & S.G. Krantz: Rigidity of holomorphic mappings and a new Schwarz Lemma at the boundary. *J. Amer. Math. Soc.* **7** (1994), 661-676.
7. G. Kresin & V. Maz'ya: *Sharp real-part theorems. A unified approach.* Translated from the Russian and edited by T. Shaposhnikova. Lecture Notes in Mathematics, 1903. Springer, Berlin, 2007.
8. R. Osserman: A sharp Schwarz inequality on the boundary. *Proc. Amer. Math. Soc.* **128** (2000), 3513-3517.
9. T. Aliyev Azeroğlu & B.N. Örnek: A refined Schwarz inequality on the boundary. *Complex Variables and Elliptic Equations* **58** (2013), no. 4, 571-577.

10. B. N. Örnek: Sharpened forms of the Schwarz lemma on the boundary. *Bull. Korean Math. Soc.* **50** (2013), no. 6, 2053-2059.
11. Ch. Pommerenke: *Boundary Behaviour of Conformal Maps*. Springer-Verlag, Berlin, 1992.

DEPARTMENT OF MATHEMATICS, GEBZE TECHNICAL UNIVERSITY, GEBZE-KOCAELI 41400, TURKEY
Email address: nornek@gtu.edu.tr