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HALF LIGHTLIKE SUBMANIFOLDS OF AN INDEFINITE SASAKIAN MANIFOLD

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Abstract. We study half lightlike submanifolds of an indefinite Sasakian manifold. The aim of this paper is to prove the following result: If a locally symmetric half lightlike submanifold of an indefinite Sasakian manifold is totally umbilical, then it is of constant positive curvature 1. In addition to this result, we prove three characterization theorems for such a half lightlike submanifold.

0. Introduction

The class of lightlike submanifolds of codimension 2 is composed entirely of two classes by virtue of the rank of its radical distribution, named by half lightlike or coisotropic submanifolds [3]. Half lightlike submanifold is a special case of r-lightlike submanifold [2] such that $r = 1$ and its geometry is more general form than that of coisotrophic submanifolds. Much of the works on half lightlike submanifolds will be immediately generalized in a formal way to arbitrary r-lightlike submanifolds. Recently several authors have studied the geometry of lightlike submanifolds of an indefinite Sasakian manifold. Many works of such lightlike submanifolds assumed that M is totally umbilical (or totally geodesic), or M is screen conformal, or its screen distribution $S(TM)$ is totally umbilical in M.

In the theory of Sasakian manifolds, the following result is well-known: If a Sasakian manifold is locally symmetric, then it is of constant positive curvature 1. We proved lightlike hypersurface version of the above classical result: If a locally symmetric lightlike hypersurface of an indefinite Sasakian manifold is totally geodesic, then it is of constant positive curvature 1 [5].

The objective of this paper is the study of half lightlike version of the above classical result. We have the following result:

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• *If a locally symmetric half lightlike submanifold of an indefinite Sasakian manifold is totally umbilical, then it is of constant positive curvature* 1.

In addition to this result, we prove the following three characterization theorems for half lightlike submanifolds M of an indefinite Sasakian manifold $\bar{M}$:

- *There exist no totally umbilical half lightlike submanifolds M of an indefinite Sasakian manifold such that its structure vector field ζ is tangent to M.*
- *There exist no half lightlike submanifolds M of an indefinite Sasakian manifold such that the screen distribution $S(TM)$ is totally umbilical in M.*
- *There exist no screen conformal half lightlike submanifolds M of an indefinite Sasakian manifold $\bar{M}$.*

1. Half Lightlike Submanifolds

An odd dimensional semi-Riemannian manifold $(\bar{M}, \bar{g})$ is called an indefinite contact metric manifold if there exists a $(1,1)$-type tensor field J, a vector field ζ, called the characteristic vector field, and its 1-form θ satisfying

$$\begin{aligned} &J^2X = -X + \theta(X)\zeta,\ J\zeta = 0,\ \theta \circ J = 0,\ \theta(\zeta) = 1, \\ &\bar{g}(\zeta, \zeta) = \epsilon, \quad \bar{g}(JX, JY) = \bar{g}(X, Y) - \epsilon\,\theta(X)\theta(Y), \\ &\theta(X) = \epsilon\bar{g}(\zeta, X), \quad d\theta(X, Y) = \bar{g}(JX, Y),\ \epsilon = \pm 1, \end{aligned} \tag{1.1}$$

for any vector fields X, Y on $\bar{M}$. An indefinite contact manifold $\bar{M}$ is called an *indefinite Sasakian manifold* [5, 6] if

$$\bar{\nabla}_X\zeta = JX, \tag{1.2}$$

$$(\bar{\nabla}_X J)Y = \epsilon\,\theta(Y)X - \bar{g}(X, Y)\zeta. \tag{1.3}$$

Due to [5], we show that the structure vector field ζ of $\bar{M}$ is spacelike.

A submanifold M of an indefinite Sasakian manifold $\bar{M}$ of codimension 2 is called a *half lightlike submanifold* if the radical distribution

$$Rad(TM) = \{\xi \in \Gamma(TM) \mid \bar{g}(\xi,\, X) = 0, \quad X \in \Gamma(TM)\}$$

of M satisfies $Rad(TM) = TM \cap TM^\perp$, i.e., $Rad(TM)$ is a vector subbundle of the tangent bundle TM and the normal bundle $TM^\perp$ of rank 1, where we denote by $F(M)$ the algebra of smooth functions on M and by $\Gamma(E)$ the $F(M)$ module of smooth sections of any vector bundle E over M. Then there exist complementary non-degenerate distributions $S(TM)$ and $S(TM^\perp)$ of $Rad(TM)$ in TM and $TM^\perp$

respectively, called the *screen* and *co-screen distribution* on M, such that

$$TM = Rad(TM) \oplus_{orth} S(TM),\; TM^{\perp} = Rad(TM) \oplus_{orth} S(TM^{\perp}), \tag{1.4}$$

where the symbol $\oplus_{orth}$ denotes the orthogonal direct sum. We denote such a half lightlike submanifold by $M = (M, g, S(TM))$. Choose $L \in \Gamma(S(TM^{\perp}))$ as a unit vector field with $\bar{g}(L, L) = \pm 1$. In this paper we may assume that $\bar{g}(L, L) = 1$ without loss of generality. Consider the orthogonal complementary distribution $S(TM)^{\perp}$ to $S(TM)$ in $T\bar{M}$. Certainly $\xi \in \Gamma(Rad(TM))$ and $L \in \Gamma(S(TM^{\perp}))$ belong to $\Gamma(S(TM)^{\perp})$. Hence we have the following orthogonal decomposition

$$S(TM)^{\perp} = S(TM^{\perp}) \oplus_{orth} S(TM^{\perp})^{\perp},$$

where $S(TM^{\perp})^{\perp}$ is the orthogonal complementary to $S(TM^{\perp})$ in $S(TM)^{\perp}$. For any null section ξ of $Rad(TM)$ on a coordinate neighborhood $\mathcal{U} \subset M$, there exists a uniquely defined null vector field $N \in \Gamma(ltr(TM))$ satisfying

$$\bar{g}(\xi, N) = 1,\; \bar{g}(N, N) = \bar{g}(N, X) = \bar{g}(N, L) = 0,\; \forall\, X \in \Gamma(S(TM)).$$

We call N, $ltr(TM)$ and $tr(TM) = S(TM^{\perp}) \oplus_{orth} ltr(TM)$ the *lightlike transversal vector field, lightlike transversal vector bundle* and *transversal vector bundle* of M with respect to $S(TM)$ respectively. Therefore $T\bar{M}$ is decomposed as follows:

$$\begin{aligned} T\bar{M} &= TM \oplus tr(TM) = \{Rad(TM) \oplus tr(TM)\} \oplus_{orth} S(TM) \\ &= \{Rad(TM) \oplus ltr(TM)\} \oplus_{orth} S(TM) \oplus_{orth} S(TM^{\perp}). \end{aligned} \tag{1.5}$$

Let $\bar{\nabla}$ be the Levi-Civita connection of $\bar{M}$ and P the projection morphism of $\Gamma(TM)$ on $\Gamma(S(TM))$ with respect to the decomposition (1.4). Then the local Gauss and Weingarten formulas are given by

$$\bar{\nabla}_X Y = \nabla_X Y + B(X, Y)N + D(X, Y)L, \tag{1.6}$$

$$\bar{\nabla}_X N = -A_N X + \tau(X)N + \rho(X)L, \tag{1.7}$$

$$\bar{\nabla}_X L = -A_L X + \phi(X)N, \tag{1.8}$$

$$\nabla_X PY = \nabla^*_X PY + C(X, PY)\xi, \tag{1.9}$$

$$\nabla_X \xi = -A^*_{\xi} X - \tau(X)\xi, \tag{1.10}$$

for all $X, Y \in \Gamma(TM)$, where ∇ and ∇^* are the induced linear connections on TM and $S(TM)$ respectively, B and D are called the *local second fundamental forms* of M, C is called the *local second fundamental form* on $S(TM)$. A_N, A^*_{ξ} and A_L are linear operators on TM and τ, ρ and ϕ are 1-forms on TM. Since $\bar{\nabla}$ is torsion-free, ∇ is also torsion-free and both B and D are symmetric. From the facts

$B(X,Y) = \bar{g}(\bar{\nabla}_X Y, \xi)$ and $D(X,Y) = \bar{g}(\bar{\nabla}_X Y, L)$ for all $X,\, Y \in \Gamma(TM)$, we know that B and D are independent of the choice of a screen distribution and satisfy

$$B(X,\xi) = 0, \quad D(X,\xi) = -\phi(X),\ \forall\, X \in \Gamma(TM). \tag{1.11}$$

We say that $h(X,Y) = B(X,Y)N + D(X,Y)L$ is the *second fundamental tensor* of M. The induced connection ∇ of M is not metric and satisfies

$$(\nabla_X g)(Y,Z) = B(X,Y)\,\eta(Z) + B(X,Z)\,\eta(Y), \tag{1.12}$$

for all $X,\, Y,\, Z \in \Gamma(TM)$, where η is a 1-form on TM such that

$$\eta(X) = \bar{g}(X,N),\ \forall\, X \in \Gamma(TM). \tag{1.13}$$

But the connection ∇^* on $S(TM)$ is metric. The above three local second fundamental forms are related to their shape operators by

$$B(X,Y) = g(A^*_\xi X, Y), \qquad \bar{g}(A^*_\xi X, N) = 0, \tag{1.14}$$

$$C(X,PY) = g(A_N X, PY), \qquad \bar{g}(A_N X, N) = 0, \tag{1.15}$$

$$D(X,PY) = g(A_L X, PY), \qquad \bar{g}(A_L X, N) = \rho(X), \tag{1.16}$$

$$D(X,Y) = g(A_L X, Y) - \phi(X)\eta(Y),\ \forall\, X,\, Y \in \Gamma(TM). \tag{1.17}$$

By (1.14) and (1.15), we show that A^*_ξ and A_N are $\Gamma(S(TM))$-valued shape operators related to B and C respectively and A^*_ξ is self-adjoint on TM and

$$A^*_\xi \xi = 0. \tag{1.18}$$

But A_N and A_L are not self-adjoint on $S(TM)$ and TM respectively. Replacing Y by ξ to (1.6) and using (1.10) and (1.11), we have

$$\bar{\nabla}_X \xi = -A^*_\xi X - \tau(X)\xi - \phi(X)L,\ \forall\, X \in \Gamma(TM). \tag{1.19}$$

Denote by $\bar{R}$ and R the curvature tensors of the connections $\bar{\nabla}$ and ∇ respectively. Using the local Gauss-Weingarten formulas (1.6) $\sim$ (1.8) for M, we have the Gauss equation for M, for all $X,\, Y,\, Z \in \Gamma(TM)$:

$$\begin{aligned} \bar{R}(X,Y)Z = {} & R(X,Y)Z + B(X,Z)A_N Y - B(Y,Z)A_N X + D(X,Z)A_L Y \\ & - D(Y,Z)A_L X + \{(\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) + \tau(X)B(Y,Z) \\ & - \tau(Y)B(X,Z) + \phi(X)D(Y,Z) - \phi(Y)D(X,Z)\}N \\ & + \{(\nabla_X D)(Y,Z) - (\nabla_Y D)(X,Z) + \rho(X)B(Y,Z) - \rho(Y)B(X,Z)\}L. \end{aligned} \tag{1.20}$$

2. Characterization Theorems

In general, since the characteristic vector field ζ on $\bar{M}$ belongs to $T\bar{M}$, from the decomposition (1.5) of $T\bar{M}$, ζ is decomposed by

$$\zeta = P\zeta + a\xi + bN + eL, \tag{2.1}$$

where $a = \theta(N)$, $b = \theta(\xi)$ and $e = \theta(L)$ are smooth functions on $\bar{M}$.

Proposition 2.1. *Let M be a half lightlike submanifold of an indefinite contact metric manifold $\bar{M}$. Then there exists a screen distribution $S(TM)$ such that*

$$J(S(TM)^{\perp}) \subset S(TM).$$

Proof. Using (1.1), we have $\bar{g}(J\xi, \xi) = 0$. Thus $J\xi$ belongs to $TM \oplus_{orth} S(TM^{\perp})$. If $Rad(TM) \cap J(Rad(TM)) \neq \{0\}$, then there exists a non-vanishing smooth real valued function f such that $J\xi = f\xi$. Applying J to this equation and using (1.1), we have $(f^2 + 1)\xi = b\zeta$. Taking the scalar product with ξ and N in this equation by turns, we get $b = 0$ and $f^2 + 1 = 0$ respectively. It is an impossible case for real M. Thus $Rad(TM) \cap J(Rad(TM)) = \{0\}$. Moreover, if $S(TM^{\perp}) \cap J(Rad(TM)) \neq \{0\}$, then there exists a non-vanishing smooth real valued function h such that $J\xi = hL$. In this case we have $-b^2 = \bar{g}(J\xi, J\xi) = h^2\bar{g}(L, L) = h^2$. This implies $b = 0$ and $h = 0$. It is a contradiction to $h \neq 0$. Thus we have $S(TM^{\perp}) \cap J(Rad(TM)) = \{0\}$. This enables one to choose a screen distribution $S(TM)$ such that it contains $J(Rad(TM))$ as a vector subbundle. From the facts $\bar{g}(JN, N) = 0$ and $\bar{g}(JN, \xi) = -\bar{g}(N, J\xi) = 0$, using the above method, we also show that $J(ltr(TM))$ is a vector subbundle of $S(TM)$ of rank 1. On the other hand, from the facts $\bar{g}(JL, L) = 0$, $\bar{g}(JL, \xi) = -\bar{g}(L, J\xi) = 0$ and $\bar{g}(JL, N) = -\bar{g}(L, JN) = 0$, we show that $J(S(TM^{\perp}))$ is also a vector subbundle of $S(TM)$ of rank 1. Thus $J(S(TM)^{\perp})$ is a vector subbundle of $S(TM)$, where $S(TM)^{\perp} = \{Rad(TM) \oplus ltr(TM)\} \oplus_{orth} S(TM^{\perp})$ by (1.5). □

Note 1. Although $S(TM)$ is not unique, it is canonically isomorphic to the factor vector bundle $S(TM)^* = TM/Rad(TM)$ considered by Kupeli [7]. Thus all screen distributions are mutually isomorphic. For this reason, we consider only half lightlike submanifold M of $\bar{M}$ equipped with a $S(TM)$ such that $J(S(TM)^{\perp}) \subset S(TM)$.

Proposition 2.2. *Let M be a half lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Then ζ does not belong to $Rad(TM)$, $ltr(TM)$ and $S(TM^{\perp})$.*

Proof. If ζ belongs to $Rad(TM)$ or $ltr(TM)$, then (2.1) deduce $\zeta = a\xi$ or $\zeta = bN$ respectively. Using this and $\bar{g}(\zeta,\zeta) = 1$, we have the following impossible results:

$$1 = \bar{g}(\zeta,\zeta) = a^2\bar{g}(\xi,\xi) = 0 \text{ or } 1 = \bar{g}(\zeta,\zeta) = b^2\bar{g}(N,N) = 0.$$

Thus ζ does not belong to $Rad(TM)$ and $ltr(TM)$. If ζ belongs to $S(TM^\perp)$, then (2.1) deduce to $\zeta = eL$. Thus we have $1 = \bar{g}(\zeta,\zeta) = e^2\bar{g}(L,L) = e^2$. Thus $e = \pm 1$. Applying $\bar{\nabla}_X$ to $\zeta = eL$ and using (1.2) and (1.8), we have

$$JX = -eA_LX + e\phi(X)N, \quad \forall\, X \in \Gamma(TM).$$

Taking the scalar product with ξ and JN in this equation by turns and using (1.16) and the facts $\theta(N) = 0$ and $\bar{g}(JN, N) = 0$, we have

$$e\phi(X) = -g(X, J\xi),\ \eta(X) = -eD(X, JN),\ \forall\, X \in \Gamma(TM),$$

due to $JN \in \Gamma(S(TM))$. From this two equations and (1.11), we have

$$1 = \eta(\xi) = -eD(\xi, JN) = e\phi(JN) = -g(JN, J\xi) = -1.$$

It is a contradiction. Thus ζ also does not belong to $S(TM^\perp)$. □

Definition 1. We say that M is *totally umbilical* [3] if, on any coordinate neighborhood $\mathcal{U}$, there is a smooth vector field $\mathcal{H} \in \Gamma(tr(TM))$ such that

$$h(X,Y) = \mathcal{H}\, g(X,Y),\ \forall\, X,\, Y \in \Gamma(TM).$$

In case $\mathcal{H} = 0$ on $\mathcal{U}$, we say that M is *totally geodesic.*

It is easy to see that M is totally umbilical if and only if, on each coordinate neighborhood $\mathcal{U}$, there exist smooth functions β and δ such that

$$B(X,Y) = \beta g(X,Y),\ D(X,Y) = \delta g(X,Y),\ \forall\, X,\, Y \in \Gamma(TM). \tag{2.2}$$

Theorem 2.3. *Let M be a totally umbilical half lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Then ζ is not tangent to M.*

Proof. Assume that ζ is tangent to M. Using (1.2) and (1.6), we obtain

$$JX = \nabla_X\zeta + B(X,\zeta)N + D(X,\zeta)L,\ \forall\, X \in \Gamma(TM).$$

Taking the scalar product with ξ and L in this equation by turns, we have

$$B(X,\zeta) = -g(X, J\xi),\ D(X,\zeta) = -g(X, JL),\ \forall\, X \in \Gamma(TM).$$

As M is totally umbilical, from this equations and (2.2), we obtain

$$\beta g(X,\zeta) = -g(X, J\xi),\ \delta g(X,\zeta) = -g(X, JL),\ \forall\, X \in \Gamma(TM). \tag{2.3}$$

Replacing X by JN in the first equation of (2.3) and X by JL in the second equation of (2.3), we deduce the following two impossible results:

$$\begin{aligned} 0 = \beta 0 = \beta g(JN, \zeta) &= -g(JN, J\xi) = -1, \\ 0 = \delta 0 = \delta g(JL, \zeta) &= -g(JL, JL) = -1, \end{aligned}$$

respectively. Thus the vector field ζ is not tangent to M. □

Corollary 1. *There exist no totally umbilical half lightlike submanifold M of an indefinite Sasakian manifold $\bar{M}$ such that ζ is tangent to M.*

Applying the operator $\bar{\nabla}_X$ to $\bar{g}(JN, L) = 0$ and $\bar{g}(J\xi, N) = 0$ by turns and using (1.3), (1.7), (1.8), (1.14)∼(1.16) and (1.19), we have respectively

$$D(X, JN) = C(X, JL) - e\eta(X), \;\forall\, X \in \Gamma(TM), \tag{2.4}$$

$$B(X, JN) = C(X, J\xi) - b\eta(X), \;\forall\, X \in \Gamma(TM). \tag{2.5}$$

Using this two equations we have the following two important results:

Definition 2. We say that $S(TM)$ is *totally umbilical* [3] in M if, on any coordinate neighborhood $\mathcal{U}$, there is a smooth function γ such that

$$C(X, PY) = \gamma\, g(X, Y), \;\forall\, X,\, Y \in \Gamma(TM). \tag{2.6}$$

In case $\gamma = 0$ on $\mathcal{U}$, we say that $S(TM)$ is *totally geodesic* in M.

Theorem 2.4. *There exist no half lightlike submanifolds of an indefinite Sasakian manifold $\bar{M}$ such that $S(TM)$ is totally umbilical in M.*

Proof. Assume that $S(TM)$ is totally umbilical in M. Then, from (2.4), (2.5) and (2.6), for all $X \in \Gamma(TM)$, we have respectively

$$B(X, JN) = \gamma\, g(X, J\xi) - b\eta(X), \;\; D(X, JN) = \gamma\, g(X, JL) - e\eta(X). \tag{2.7}$$

Replacing X by ξ to (2.7) and using (1.11), we have respectively

$$b = 0 \qquad \text{and} \qquad e = \phi(JN).$$

Applying the operator $\bar{\nabla}_X$ to $\bar{g}(\zeta, \xi) = 0$ and using (1.2) and (1.19), we have

$$\bar{g}(A^*_\xi X, \zeta) + g(X, J\xi) + e\phi(X) = 0, \;\forall\, X \in \Gamma(TM).$$

Replacing X by JN to this equation and using $g(J\xi, JN) = 1$, we have

$$B(JN, P\zeta) = -(e^2 + 1).$$

On the other hand, from the first equation of (2.7) with $b = 0$, we have

$$B(JN, P\zeta) = \gamma g(P\zeta, J\xi) = \gamma g(\zeta - a\xi - eL, J\xi) = 0.$$

Thus we have $e^2 = -1$. It is a contradiction. Thus we have our assertion. □

Definition 3. A half lightlike submanifold M of $\bar{M}$ is called *screen conformal* [4] if there exist a non-vanishing smooth function φ on a neighborhood $\mathcal{U}$ in M such that $A_N = \varphi A^*_\xi$, or equivalently,

$$C(X, PY) = \varphi B(X, Y), \ \forall X, Y \in \Gamma(TM). \tag{2.8}$$

Theorem 2.5. *There exist no screen conformal half lightlike submanifolds M of an indefinite Sasakian manifold $\bar{M}$.*

Proof. Assume that M is screen conformal. Replacing X by ξ to (2.4) and (2.5) and using (1.11) and (2.8), we have respectively

$$b = 0 \quad \text{and} \qquad e = \phi(JN).$$

By the same method of Theorem 2.4 and using the above equations, we have an impossible result: $e^2 = -1$. Thus we have our theorem. □

Note 2. If ζ is tangent to M, by Proposition 2.2, ζ does not belong to $Rad(TM)$. This enables one to choose a $S(TM)$ which contains ζ. This implies that *if ζ is tangent to M, then it belongs to $S(TM)$.* Călin also proved this result in his book [1] which many authors assumed in their papers. In this case, by Corollary 1 of Theorem 2.3, M cannot be totally umbilical.

3. Totally Umbilical Submanifolds

Theorem 3.1. *Let M be a totally umbilical half lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Then $b \neq 0$ and M is totally geodesic.*

Proof. First we prove that if M is totally umbilical, then $b = \bar{g}(\zeta, \xi) \neq 0$: Assume that $b = \theta(\xi) = 0$. Then we have the following relations.

$$g(J\xi, JN) = 1, \ g(J\xi, JL) = g(J\xi, J\xi) = 0.$$

Applying the operator $\bar{\nabla}_X$ to $\bar{g}(\zeta, \xi) = 0$ and using (1.2) and (1.19), we have

$$\bar{g}(A^*_\xi X, \zeta) + g(X, J\xi) + e\phi(X) = 0, \ \forall X \in \Gamma(TM).$$

Substituting (2.1) in this equation and using (1.14), we have

$$B(X, P\zeta) + g(X, J\xi) + e\phi(X) = 0, \ \forall X \in \Gamma(TM).$$

As M is totally umbilical, we get $\phi = 0$ due to (1.11). Thus we have

$$\beta \bar{g}(X, \zeta) + g(X, J\xi) = 0, \quad \forall\, X \in \Gamma(TM).$$

Replacing X by JN in this and using (1.1) and $g(J\xi, JN) = 1$, we have

$$0 = \beta \bar{g}(\zeta, JN) + g(J\xi, JN) = 1.$$

It is a contradiction. This implies $b \neq 0$.

Next we prove that if M is totally umbilical, then M is totally geodesic: Applying $\bar{\nabla}_X$ to $\bar{g}(J\xi, L) = 0$ and using (1.3), (1.8), (1.14), (1.16) and (1.19), we have

$$B(X, JL) = D(X, J\xi), \;\forall\, X \in \Gamma(TM).$$

As M is totally umbilical, from the last equation and (2.2), we have

$$\beta g(X, JL) = \delta g(X, J\xi), \;\forall\, X \in \Gamma(TM).$$

Replacing X by $J\xi$, JN and JL in this equation by turns, we have

$$e\beta = b\delta, \quad ae\beta = (ab - 1)\delta, \quad (e^2 - 1)\beta = be\delta, \tag{3.1}$$

due to $b \neq 0$. Substituting the first equation of (3.1) into the second and third equations of (3.1) by turns, we get $\beta = \delta = 0$. Thus we have $\mathcal{H} = 0$. □

From (1.12) and Theorem 3.1, we have the following result:

Corollary 2. *Let M be a totally umbilical half lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Then there exists a unique torsion-free metric connection ∇ induced by the Levi-Civita connection $\bar{\nabla}$ on M.*

Assume that M is totally umbilical. We get $b \neq 0$ and $B = D = A^*_\xi = \phi = 0$. Consider a locally unit timelike vector field V on M and its 1-form v defined by

$$V = -b^{-1} J\xi, \quad v(X) = -g(X, V), \;\forall\, X \in \Gamma(TM). \tag{3.2}$$

Theorem 3.2. *Let M be a totally umbilical half lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Then the 1-form v is closed, i.e., $dv = 0$ on TM.*

Proof. Applying the operator $\bar{\nabla}_X$ to $\bar{g}(\zeta, \xi) = b$ with $X \in \Gamma(TM)$ and using (1.2), (1.6), (1.10), (1.11) and the facts $A^*_\xi = \phi = 0$, we obtain

$$Xb + b\tau(X) \;=\; -bv(X), \;\forall\, X \in \Gamma(TM). \tag{3.3}$$

Applying the operator $\bar{\nabla}_X$ to $bV = -J\xi$ with $X \in \Gamma(TM)$ and using (1.3), (1.6), (1.10), (1.11) and (3.3) and the fact $b \neq 0$, we get

$$\nabla_X V = -X + v(X)V, \;\forall\, X \in \Gamma(TM). \tag{3.4}$$

Using (1.20), (3.4) and the facts ∇ is torsion-free and M is totally geodesic, we have

$$\bar{R}(X,Y)V = R(X,Y)V = 2dv(X,Y)V + v(X)Y - v(Y)X,\ \forall\, X,\, Y \in \Gamma(TM).$$

Taking the scalar product with V in this equation and using (3.2) and the fact $\bar{g}(\bar{R}(X,Y)V,\, V) = 0$, we have $dv = 0$ on TM, i.e., v is closed on TM. □

Theorem 3.3. *Let M be a totally umbilical half lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. If M is locally symmetric, i.e., its curvature tensor R satisfies $\nabla_X R = 0$ for all $X \in \Gamma(TM)$, then M has a constant curvature* 1.

Proof. As $dv = 0$ on TM, the last equation reduce to the equation

$$R(X,Y)V = v(X)Y - v(Y)X,\ \forall\, X,\, Y \in \Gamma(TM). \tag{3.5}$$

Differentiating (3.2) with $Y \in \Gamma(TM)$ and using (3.2) and (3.4), we have

$$(\nabla_X v)(Y) = g(X,Y) + v(X)v(Y), \quad \forall\, X,\, Y \in \Gamma(TM). \tag{3.6}$$

Differentiating (3.5) with $Z \in \Gamma(TM)$ and using (3.5) and the fact that M is locally symmetric, i.e., $\nabla_Z R = 0$ for any $Z \in \Gamma(TM)$, we have

$$R(X,Y)\nabla_Z V = (\nabla_Z v)(X)Y - (\nabla_Z v)(Y)X,\ \forall\, X,\, Y \in \Gamma(TM).$$

Substituting (3.4) and (3.6) in this equation and using (3.5), we obtain

$$R(X,Y)Z = g(Y,Z)X - g(X,Z)Y, \quad \forall\, X,\, Y,\, Z \in \Gamma(TM).$$

Thus M is a space of constant positive curvature 1. □

The induced Ricci type tensor $R^{(0,\,2)}$ of M is defined by

$$R^{(0,\,2)}(X,Y) = trace\{Z \to R(Z,X)Y\},\ \forall\, X,\, Y \in \Gamma(TM).$$

In general, $R^{(0,\,2)}$ is not symmetric [2, 3, 4]. A tensor field $R^{(0,\,2)}$ of M is called its *induced Ricci tensor* of M, denoted by Ric, if it is symmetric [4].

Definition 4. Define the curvature tensor R^{ℓ} of $ltr(TM)$ by

$$R^{\ell}(X,Y)N = \nabla^{\ell}_X \nabla^{\ell}_Y N - \nabla^{\ell}_Y \nabla^{\ell}_X N - \nabla^{\ell}_{[X,\,Y]} N, \quad \nabla^{\ell}_X N = \tau(X)N,$$

for all $X,\, Y \in \Gamma(TM)$. If R^{ℓ} vanishes identically, then the lightlike transversal connection ∇^{ℓ} of M is said to be *flat* (or *trivial*) [6].

Theorem 3.4 ([6])**.** *Let M be a half lightlike submanifold of a semi-Riemannian manifold $\bar{M}$. The following assertions are equivalent:*

(i) *The lightlike transversal connection of M is flat, i.e., $R^{\ell} = 0$.*
(ii) *The 1-form τ is closed, i.e., $d\tau = 0$, on any $\mathcal{U} \subset M$.*

(iii) *The Ricci type tensor $R^{(0,2)}$ is an induced Ricci tensor of M.*

Theorem 3.5. *Let M be a totally umbilical half lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Then M holds* (i) $\sim$ (iii) *in Theorem 3.4.*

Proof. Applying ∇_Y to (3.3) with $Y \in \Gamma(TM)$ and using (3.3), we get

$$XYb = b\{\tau(X) + v(X)\}\{\tau(Y) + v(Y)\} - b\{X(\tau(Y)) + X(v(Y))\}.$$

for all $X, Y \in \Gamma(TM)$. Using this equation and (3.3), we show that

$$b\{d\tau(X,Y) + dv(X,Y)\} = \{XY - YX - [X,Y]\}b = 0,$$

Thus we have $d\tau(X,Y) = dv(X,Y)$ for all $X, Y \in \Gamma(TM)$, due to $b \neq 0$. Since $dv = 0$, we have $d\tau = 0$. Thus M holds (i) $\sim$ (iii) in Theorem 3.4. □

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