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ON THE STABILITY OF A JENSEN TYPE FUNCTIONAL EQUATION IN MULTI-NORMED SPACES

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Abstract. In this paper we investigate the Hyers-Ulam stability of a Jensen type functional equation in multi-normed spaces and then extend the result to multi-normed left modules over a normed algebra $\mathcal{A}$.

1. Introduction

The study of stability problems originated from a question by S.M. Ulam [21] in 1940: *Under what condition does there exist a homomorphism near an approximate homomorphism?* In 1941, D.H. Hyers [8] gave a first affirmative answer to the question of Ulam for Banach spaces, which states that *if* $\delta > 0$ *and* $f : \mathcal{X} \to \mathcal{Y}$ *is a mapping with* $\mathcal{X}$ *a normed space,* $\mathcal{Y}$ *a Banach space such that*

$$||f(x+y) - f(x) - f(y)|| \leq \delta$$

for all $x, y \in \mathcal{X}$*, then there exists a unique additive mapping* $T : \mathcal{X} \to \mathcal{Y}$ *such that*

$$||f(x) - T(x)|| \leq \delta$$

for all $x \in \mathcal{X}$*.*

A generalized version of the theorem of Hyers for approximately additive mappings was first given by T. Aoki [1] in 1950. In 1978, Th.M. Rassias [18] independently introduced the unbounded Cauchy difference and was the first to prove the stability of the linear mapping between Banach spaces

During the past decades, a number of results concerning the stability have been obtained by various ways, and been applied to a number of functional equations and mappings [3, 7, 10, 19].

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The stability of the classical Jensen functional equation

$$2f\Big(\frac{x+y}{2}\Big) = f(x) + f(y) \tag{1.1}$$

and of its generalizations were studied by numerous researchers (cf., e.g., [9, 11, 12, 15]).

T. Trif [20] studied the generalized Hyers-Ulam stability of the Jensen type functional equation for normed spaces (or is called the Popoviciu functional equation from [17]):

$$\begin{aligned} &3f\Big(\frac{x+y+z}{3}\Big) + f(x) + f(y) + f(z) \\ &= 2\Big[f\Big(\frac{x+y}{2}\Big) + f\Big(\frac{y+z}{2}\Big) + f\Big(\frac{z+x}{2}\Big)\Big]. \end{aligned} \tag{1.2}$$

In view of [20], we note that the Popoviciu functional equation (1.2) is equivalent to the Jensen functional equation (1.1).

In this paper, using some ideas from the earlier works [14, 16], we investigate the stability of the Popoviciu functional equation in multi-normed spaces and further, in multi-normed left module over normed algebra.

The notion of multi-normed space was introduced by H.G. Dales and M.E. Polyakov (see [5, 6, 13, 14]). This concept is somewhat similar to operator sequence space and has some connections with operator spaces and Banach lattices. Motivations for the study of multi-normed spaces and many examples were given in [6].

Let $\mathbb{C}, \mathbb{R}$ and $\mathbb{N}$ be the sets of complex, real numbers and positive integers, respectively. Let $\mathcal{X}$ be a linear space over $\mathbb{C}$. For each $k \in \mathbb{N}$, we denote by $\mathcal{X}^k$ the linear space $\mathcal{X} \oplus \cdots \oplus \mathcal{X}$ consisting of k-tuples $(x_1, \cdots, x_k)$, where $x_1, \cdots, x_k \in \mathcal{X}$. The linear operations on $\mathcal{X}^k$ are defined coordinatewise. The zero element of either $\mathcal{X}$ or $\mathcal{X}^k$ is denoted by 0. We denote by $\mathbb{N}_k$ the set $\{1, 2, 3, \cdots, k\}$ and by G_k the group of permutations on k symbols.

Definition 1.1. A *multi-norm* on $\{\mathcal{X}^k : k \in \mathbb{N}\}$ is a sequence

$$(\|\cdot\|_k) = (\|\cdot\|_k : k \in \mathbb{N})$$

such that $\|\cdot\|_k$ is a norm on $\mathcal{X}^k$ for each $k \in \mathbb{N}$, such that $\|x\|_1 = \|x\|$ for each $x \in \mathcal{X}$, and such that for each $k \in \mathbb{N}$ $(k \geq 2)$, the following axioms are satisfied:

(i) $\|x_{\sigma(1)}, \cdots, x_{\sigma(k)}\|_k = \|(x_1, \cdots, x_k)\|_k \quad (\sigma \in G_k;\ x_1, \cdots, x_k \in \mathcal{X})$;

(ii) $\|(\alpha_1 x_1, \cdots, \alpha_k x_k)\|_k \leq \big(\max_{i \in \mathbb{N}_k} |\alpha_i|\big) \|(x_1, \cdots, x_k)\|_k$

$(\alpha_1, \cdots, \alpha_k \in \mathbb{C};\ x_1, \cdots, x_k \in \mathcal{X})$;

(iii) $\|(x_1, \cdots, x_{k-1}, 0)\|_k = \|(x_1, \cdots, x_{k-1})\|_{k-1} \quad (x_1, \cdots, x_{k-1} \in \mathcal{X})$;

(iv) $\|(x_1, \cdots, x_{k-1}, x_{k-1})\|_k = \|(x_1, \cdots, x_{k-1})\|_{k-1} \quad (x_1, \cdots, x_{k-1} \in \mathcal{X})$.

In this case, we say that $((\mathcal{X}^k, \|\cdot\|_k) : k \in \mathbb{N})$ is a *multi-normed space.*

Suppose that $((\mathcal{X}^k, \|\cdot\|_k) : k \in \mathbb{N})$ is a multi-normed space and $k \in \mathbb{N}$. It is easy to show that

(a) $\|(x, \cdots, x)\|_k = \|x\| \quad (x \in \mathcal{X})$;

(b) $\max_{i \in \mathbb{N}_k} \|x_i\| \leq \|(x_1, \cdots, x_k)\|_k \leq \sum_{i=1}^{k} \|x_i\| \leq k \max_{i \in \mathbb{N}_k} \|x_i\| \quad (x_1, \cdots, x_k \in \mathcal{X})$.

It follows from (b) that if $(\mathcal{X}, \|\cdot\|)$ is a Banach space, then $(\mathcal{X}^k, \|\cdot\|_k)$ is a Banach space for each $k \in \mathbb{N}$; in this case, $((\mathcal{X}^k, \|\cdot\|_k) : k \in \mathbb{N})$ is said to be a *multi-Banach space.*

Now we recall two important examples of multi-norms for an arbitrary normed space $\mathcal{X}$ (see, for details, [6]).

Example 1.2. The sequence $(\|\cdot\|_k : k \in \mathbb{N})$ on $\{\mathcal{X}^k : k \in \mathbb{N}\}$ defined by

$$\|(x_1, \cdots, x_k)\|_k := \max_{i \in \mathbb{N}_k} \|x_i\| \quad (x_1, \cdots, x_k \in \mathcal{X})$$

is a multi-norm called the *minimum multi-norm.* The terminology *minimum* is justified by (b).

Example 1.3. Let Λ be a non-empty set and let

$$\{(\|\cdot\|_k^\lambda : k \in \mathbb{N} \ \text{ and } \ \lambda \in \Lambda)\}$$

be the family of all multi-norms on $\{\mathcal{X}^k : k \in \mathbb{N}\}$. For $k \in \mathbb{N}$, we set

$$|||(x_1, \cdots, x_k)|||_k := \sup_{\lambda \in \Lambda} \|(x_1, \cdots, x_k)\|_k^\lambda \quad (x_1, \cdots, x_k \in \mathcal{X}).$$

Then the sequence $(|||\cdot|||_k : k \in \mathbb{N})$ is a multi-norm on $\{\mathcal{X}^k : k \in \mathbb{N}\}$, which is called the *maximum multi-norm.*

In the following, we need some fundamental ingredients which can be easily deduced from the triangle inequality for the norm $\|\cdot\|_k$ and (b).

Definition 1.4. Let $((\mathcal{X}^k, \|\cdot\|_k) : k \in \mathbb{N})$ be a multi-normed space. A sequence (x_n) in $\mathcal{X}$ is a *multi-null sequence* if, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\sup_{k \in \mathbb{N}} \|(x_n, \cdots, x_{n+k-1})\|_k < \varepsilon \quad (n \geq n_0).$$

Let $x \in \mathcal{X}$. We write that

$$\operatorname*{Lim}_{n\to\infty} x_n = x$$

if $(x_n - x)$ is a multi-null sequence; in this case, we say that the sequence (x_n) is *multi-convergent* to x in $\mathcal{X}$.

Definition 1.5. Let $((\mathcal{X}^k, \|\cdot\|_k) : k \in \mathbb{N})$ be a multi-normed space. A sequence $\{x_n\}$ in $\mathcal{X}$ is a *multi-Cauchy sequence* if, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\sup_{k\in\mathbb{N}} \|(x_m - x_n, \cdots, x_{m+k-1} - x_{n+k-1})\|_k < \varepsilon \quad (m, n \geq n_0).$$

We observe that if $((\mathcal{X}^k, \|\cdot\|_k) : k \in \mathbb{N})$ is a multi-Banach space, then a multi-Cauchy sequence is multi-convergent in $\mathcal{X}$.

2. Hyers-Ulam Stability of Equation (1.2) in Multi-normed Spaces

In this section, $\mathcal{X}$ and $\mathcal{Y}$ will be a complex linear space and a complex Banach space, respectively. Given a function $f : \mathcal{X} \to \mathcal{Y}$ and $\alpha \in U = \{z \in \mathbb{C} : |z| = 1\}$, we set

$$\begin{aligned} D_\alpha f(x, y, z) := {} & 3f\Big(\frac{\alpha x + \alpha y + \alpha z}{3}\Big) + \alpha f(x) + \alpha f(y) + \alpha f(z) \\ & - 2\Big[\alpha f\Big(\frac{x+y}{2}\Big) + f\Big(\frac{\alpha y + \alpha z}{2}\Big) + f\Big(\frac{\alpha z + \alpha x}{2}\Big)\Big]. \end{aligned}$$

Theorem 2.1. *Let $((\mathcal{Y}^k, \|\cdot\|_k) : k \in \mathbb{N})$ be a multi-Banach space. If $\delta \geq 0$ and the function $f : \mathcal{X} \to \mathcal{Y}$ satisfies*

$$\sup_{k\in\mathbb{N}} \|(D_1 f(x_1, y_1, z_1), \cdots, D_1 f(x_k, y_k, z_k))\|_k \leq \delta \tag{2.1}$$

for all $x_1, \cdots, x_k, y_1, \cdots, y_k, z_1, \cdots, z_k \in \mathcal{X}$, then there exists a unique additive mapping $A : \mathcal{X} \to \mathcal{Y}$ such that

$$\sup_{k\in\mathbb{N}} \|(f(x_1) - f(0) - A(x_1), \cdots, f(x_k) - f(0) - A(x_k))\|_k \leq \frac{\delta}{3} \tag{2.2}$$

for all $x_1, \cdots, x_k \in \mathcal{X}$.

Proof. Let $g : \mathcal{X} \to \mathcal{Y}$ be the function defined by $g(x) := f(x) - f(0)$. Then $g(0) = 0$ and, since $D_1 g(x, y, z) = D_1 f(x, y, z)$ for all $x, y, z \in \mathcal{X}$, we have

$$\sup_{k\in\mathbb{N}} \|(D_1 g(x_1, y_1, z_1), \cdots, D_1 g(x_k, y_k, z_k))\|_k \leq \delta \tag{2.3}$$

for all $x_1, \cdots, x_k, y_1, \cdots, y_k, z_1, \cdots, z_k \in \mathcal{X}$.

For each $i = 1, 2, \cdots, k$, putting $y_i = x_i$ and $z_i = -2x_i$ in (2.3), we get

$$\sup_{k\in\mathbb{N}} \Big\| \Big(g(-2x_1) - 4g\Big(-\frac{x_1}{2}\Big), \cdots, g(-2x_k) - 4g\Big(-\frac{x_k}{2}\Big)\Big)\Big\|_k \le \delta$$

for all $x_1, \cdots, x_k \in \mathcal{X}$. Replacing x_i by $-2x_i$ for each $i = 1, 2, \cdots, k$ in the above relation yields

$$\sup_{k\in\mathbb{N}} \|(g(4x_1) - 4g(x_1), \cdots, g(4x_k) - 4g(x_k))\|_k \le \delta \tag{2.4}$$

for all $x_1, \cdots, x_k \in \mathcal{X}$.

Next we prove by induction on n that for all $x_1, \cdots, x_k \in \mathcal{X}$ it holds that

$$\sup_{k\in\mathbb{N}} \|(2^{-2n}g(2^{2n}x_1) - g(x_1), \cdots, 2^{-2n}g(2^{2n}x_k) - g(x_k))\|_k \le \delta \sum_{j=1}^{n} 2^{-2j} \tag{2.5}$$

for all $x_1, \cdots, x_k \in \mathcal{X}$. Dividing both sides of (2.4) by 2^2 ensures the validity of (2.5) for $n = 1$. Now, assume that the inequality (2.5) is true for some $n \in \mathbb{N}$. Replacing x_i in (2.4) by $2^{2n}x_i$ for each $i = 1, 2, \cdots, k$ and then dividing both sides of (2.4) by $2^{2(n+1)}$ yields

$$\begin{aligned}\sup_{k\in\mathbb{N}} &\|(2^{-2(n+1)}g(2^{2(n+1)}x_1) - 2^{-2n}g(2^{2n}x_1),\\ &\cdots, 2^{-2(n+1)}g(2^{2(n+1)}x_k) - 2^{-2n}g(2^{2n}x_k))\|_k \le \delta 2^{-2(n+1)}\end{aligned}$$

for all $x_1, \cdots, x_k \in \mathcal{X}$ and so

$$\begin{aligned}&\sup_{k\in\mathbb{N}} \|(2^{-2(n+1)}g(2^{2(n+1)}x_1) - g(x_1), \cdots, 2^{-2(n+1)}g(2^{2(n+1)}x_k) - g(x_k))\|_k\\ &\le \sup_{k\in\mathbb{N}} \|(2^{-2(n+1)}g(2^{2(n+1)}x_1) - 2^{-2n}g(2^{2n}x_1),\\ &\qquad \cdots, 2^{-2(n+1)}g(2^{2(n+1)}x_k) - 2^{-2n}g(2^{2n}x_k))\|_k\\ &\quad + \sup_{k\in\mathbb{N}} \|(2^{-2n}g(2^{2n}x_1) - g(x_1), \cdots, 2^{-2n}g(2^{2n}x_k) - g(x_k))\|_k\\ &\le \delta 2^{-2(n+1)} + \delta \sum_{j=1}^{n} 2^{-2j} = \delta \sum_{j=1}^{n+1} 2^{-2j}\end{aligned}$$

for all $x_1, \cdots, x_k \in \mathcal{X}$. This completes the proof of the inequality (2.5).

Let $x_1, x_2, \cdots, x_k$ be any points in $\mathcal{X}$. By virtue of (2.5), we have

$$\begin{aligned}&\sup_{k\in\mathbb{N}} \|(2^{-2n}g(2^{2n}x_1) - 2^{-2m}g(2^{2m}x_1), \cdots, 2^{-2n}g(2^{2n}x_k) - 2^{-2m}g(2^{2m}x_k))\|_k\\ &\le 2^{-2m} \sup_{k\in\mathbb{N}} \|(2^{-2(n-m)}g(2^{2(n-m)} \cdot 2^{2m}x_1) - g(2^{2m}x_1),\\ &\qquad \cdots, 2^{-2(n-m)}g(2^{2(n-m)} \cdot 2^{2m}x_k) - g(2^{2m}x_k))\|_k\end{aligned}$$

$$\leq 2^{-2m}\,\delta \sum_{j=1}^{n-m} 2^{-2j} \leq 2^{-2m}\,\frac{\delta}{3} \quad (m<n),$$

that is,

$$\begin{aligned}(2.6)\quad &\sup_{k\in\mathbb{N}} \|(2^{-2n}g(2^{2n}x_1)-2^{-2m}g(2^{2m}x_1),\cdots,2^{-2n}g(2^{2n}x_k)-2^{-2m}g(2^{2m}x_k))\|_k\\ &\leq 2^{-2m}\,\frac{\delta}{3} \quad (m<n).\end{aligned}$$

Let us fix $x \in \mathcal{X}$. Then it follows from (2.6) that

$$\begin{aligned}&\sup_{k\in\mathbb{N}} \|(2^{-2n}g(2^{2n}x)-2^{-2m}g(2^{2m}x),\\ &\qquad \cdots,2^{-2(n+k-1)}g(2^{2(n+k-1)}x)-2^{-2(m+k-1)}g(2^{2(m+k-1)}x))\|_k\\ &\leq \sup_{k\in\mathbb{N}} \Big\|\Big(2^{-2n}g(2^{2n}x)-2^{-2m}g(2^{2m}x),\\ &\qquad \cdots,\frac{1}{2^{2(k-1)}}\Big(2^{-2n}g(2^{2n}\cdot 2^{2(k-1)}x)-2^{-2m}g(2^{2m}\cdot 2^{2(k-1)}x)\Big)\Big)\Big\|_k\\ &\leq \sup_{k\in\mathbb{N}} \|(2^{-2n}g(2^{2n}x)-2^{-2m}g(2^{2m}x),\\ &\qquad \cdots,2^{-2n}g(2^{2n}\cdot 2^{2(k-1)}x)-2^{-2m}g(2^{2m}\cdot 2^{2(k-1)}x))\|_k \leq 2^{-2m}\,\frac{\delta}{3} \quad (m<n).\end{aligned}$$

This inequality implies that $\{2^{-2n}g(2^{2n}x)\}$ is a multi-Cauchy sequence and so it is multi-convergent in $\mathcal{Y}$. Consequently, we can define the mapping $A:\mathcal{X}\to\mathcal{Y}$ by

$$A(x) := \operatorname*{Lim}_{n\to\infty} 2^{-2n}g(2^{2n}x).$$

Therefore, for any $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\sup_{k\in\mathbb{N}} \|(2^{-2n}g(2^{2n}x)-A(x),\cdots,2^{-2(n+k-1)}g(2^{2(n+k-1)}x)-A(x))\|_k < \varepsilon$$

for all $n \geq n_0$. In particular, by (b), we have

$$\lim_{n\to\infty} \|2^{-2n}g(2^{2n}x)-A(x)\| = 0,$$

say,

$$A(x) = \lim_{n\to\infty} 2^{-2n}g(2^{2n}x)$$

for all $x \in \mathcal{X}$.

Let x, y and z be any points in $\mathcal{X}$. Putting $x_1 = \cdots = x_k = 2^{2n}x$, $y_1 = \cdots = y_k = 2^{2n}y$ and $z_1 = \cdots = z_k = 2^{2n}z$ in (2.3) and dividing both sides by 2^{2n}, we obtain

$$\begin{aligned}\|D_1 A(x,y,z)\| &= \lim_{n\to\infty} 2^{-2n}\|D_1 g(2^{2n}x, 2^{2n}y, 2^{2n}z)\|\\ &\le \lim_{n\to\infty} 2^{-2n}\delta = 0.\end{aligned}$$

Hence A satisfies (1.2) for all $x, y, z \in \mathcal{X}$. Since $A(0) = 0$, it follows that A is additive. Moreover, by passing to the limit in (2.5) when $n \to \infty$, we see that

$$\sup_{k\in\mathbb{N}} \|(A(x_1) - g(x_1), \cdots, A(x_k) - g(x_k))\|_k \le \frac{\delta}{3}$$

which means the inequality (2.2) for all $x_1, \cdots, x_k \in \mathcal{X}$.

Let $A' : \mathcal{X} \to \mathcal{Y}$ be another additive mapping satisfying (2.2). Using (2.2) and (a), we get

$$\begin{aligned}\|A(x) - A'(x)\| &= 2^{-n}\|A(2^n x) - A'(2^n x)\|\\ &\le 2^{-n}(\|A(2^n x) - f(2^n x) - f(0)\| + \|f(2^n x) - f(0) - A'(2^n x)\|)\\ &\le 2^{-n}\,\frac{2}{3}\,\delta.\end{aligned}$$

for all $x \in \mathcal{X}$ and $n \in \mathbb{N}$. Thus we conclude that $A(x) = A'(x)$ for all $x \in \mathcal{X}$. This proves the uniqueness of A. □

Theorem 2.2. *Let* $((\mathcal{Y}^k, \|\cdot\|_k) : k \in \mathbb{N})$ *be a multi-Banach space. If* $\delta \ge 0$ *and the function* $f : \mathcal{X} \to \mathcal{Y}$ *satisfies*

$$\sup_{k\in\mathbb{N}} \|(D_\alpha f(x_1, y_1, z_1), \cdots, D_\alpha f(x_k, y_k, z_k))\|_k \le \delta \tag{2.7}$$

for all $x_1, \cdots, x_k, y_1, \cdots, y_k, z_1, \cdots, z_k \in \mathcal{X}$ *and for all* $\alpha \in U$*, then there exists a unique* $\mathbb{C}$*-linear mapping* $A : \mathcal{X} \to \mathcal{Y}$ *satisfying the inequality* (2.2).

Proof. Put $\alpha = 1$ in (2.7). Then it follows from Theorem 2.1 that there exists a unique additive mapping $A : \mathcal{X} \to \mathcal{Y}$ satisfying the inequality (2.2).

For each $i = 1, 2, \cdots, k$, setting $y_i = x_i = x$ and $z_i = -2x$ in (2.7) and then considering (a), we get

$$\Big\|3f(0) + \alpha f(-2x) - 4f\Big(-\frac{\alpha}{2}x\Big)\Big\| \le \delta \tag{2.8}$$

for all $x \in \mathcal{X}$. Put $\varepsilon := \delta + 3\|f(0)\|$. From (2.8) and the triangle inequality, it follows that

$$\Big\|\alpha f(-2x) - 4f\Big(-\frac{\alpha}{2}x\Big)\Big\| \le \varepsilon \tag{2.9}$$

for all $x \in \mathcal{X}$. Substituting $-2x$ for x in (2.9) yields

$$\|\alpha f(4x) - 4f(\alpha x)\| \le \varepsilon \tag{2.10}$$

for all $x \in \mathcal{X}$. Using induction on $n \in \mathbb{N}$ with (2.10), we see that

$$\|\alpha f(2^{2n}x) - 4f(2^{2(n-1)}\alpha x)\| \leq \varepsilon \tag{2.11}$$

for all $x \in \mathcal{X}$.

Now letting $\alpha = 1$ in (2.11) and then replacing x by αx in the result, we obtain

$$\|f(2^{2n}\alpha x) - 4f(2^{2(n-1)}\alpha x)\| \leq \varepsilon \tag{2.12}$$

for all $x \in \mathcal{X}$. By (2.11) and (2.12), we get

$$\begin{aligned}\|f(2^{2n}\alpha x) - \alpha f(2^{2n}x)\| &\leq \|f(2^{2n}\alpha x) - 4f(2^{2(n-1)}\alpha x)\| \\ &\quad + \|\alpha f(2^{2n}x) - 4f(2^{2(n-1)}\alpha x)\| \leq 2\varepsilon,\end{aligned}$$

that is,

$$\|f(2^{2n}\alpha x) - \alpha f(2^{2n}x)\| \leq 2\varepsilon. \tag{2.13}$$

for all $x \in \mathcal{X}$ which implies

$$\lim_{n\to\infty} 2^{-2n}\|f(2^{2n}\alpha x) - \alpha f(2^{2n}x)\| = 0,$$

for all $x \in \mathcal{X}$. Hence we conclude that

$$A(\alpha x) = \lim_{n\to\infty} 2^{-2n} f(2^{2n}\alpha x) = \lim_{n\to\infty} 2^{-2n}\alpha f(2^{2n}x) = \alpha A(x)$$

for all $\alpha \in U$ and $x \in \mathcal{X}$.

Clearly, $A(0x) = 0 = 0A(x)$ for all $x \in \mathcal{X}$. Now, let $\lambda \in \mathbb{C}$ ($\lambda \neq 0$), and let $N \in \mathbb{N}$ greater than $|\lambda|$. By applying a geometric argument, we see that there exists $\lambda_1, \lambda_2 \in U$ such that $2\frac{\lambda}{N} = \lambda_1 + \lambda_2$. By the additivity of A, we get $A(\frac{1}{2}x) = \frac{1}{2}A(x)$ for all $x \in \mathcal{X}$. Therefore

$$\begin{aligned}A(\lambda x) &= A\left(\frac{N}{2}\cdot 2\cdot\frac{\lambda}{N}x\right) = NA\left(\frac{1}{2}\cdot 2\cdot\frac{\lambda}{N}x\right) = \frac{N}{2}A((\lambda_1+\lambda_2)x) \\ &= \frac{N}{2}(\lambda_1+\lambda_2)A(x) = \frac{N}{2}\cdot 2\cdot\frac{\lambda}{N}A(x) = \lambda A(x)\end{aligned}$$

for all $x \in \mathcal{A}$, so that A is $\mathbb{C}$-linear. □

3. Stability of Equation (1.2) in Multi-normed Modules

In this section, we extend the Hyers-Ulam stability of the Popoviciu's functional equation to multi-normed left modules over a normed algebra and obtain some related results. For the sake of convenience, we use the same symbol $\|\cdot\|$ in order to represent the norms on a normed algebra and a normed module.

Consider first some definitions and examples:

Definition 3.1. Let $(\mathcal{A}, \|\cdot\|)$ be a normed algebra such that $((\mathcal{A}^k, \|\cdot\|_k) : k \in \mathbb{N})$ is a multi-normed space. $((\mathcal{A}^k, \|\cdot\|_k) : k \in \mathbb{N})$ is called a *multi-normed algebra* if

$$\|(a_1b_1, \cdots, a_kb_k)\|_k \leq \|(a_1, \cdots, a_k)\|_k \, \|(b_1, \cdots, b_k)\|_k$$

for $k \in \mathbb{N}$ and $a_1, \cdots, a_k, b_1, \cdots, b_k \in \mathcal{A}$. Furthermore, the multi-normed algebra $((\mathcal{A}^k, \|\cdot\|_k) : k \in \mathbb{N})$ is said to be a *multi-Banach algebra* if $((\mathcal{A}^k, \|\cdot\|_k) : k \in \mathbb{N})$ is a multi-Banach space.

Example 3.2. Let $p, q \in \mathbb{R}$ with $1 \leq p \leq q < \infty$ and $\mathcal{A} = \ell^p$. The algebra $\mathcal{A}$ is a Banach sequence algebra with respect to coordinatewise multiplication of sequences (see [4, Example 4.2.42]). Let $(\|\cdot\|_k : k \in \mathbb{N})$ be the standard (p, q)-multi-norm on $\{A^k : n \in \mathbb{N}\}$ (see [6]). Then $((\mathcal{A}^k, \|\cdot\|_k) : k \in \mathbb{N})$ is a multi-Banach algebra.

Definition 3.3. Let $(\mathcal{A}, \|\cdot\|)$ be a normed algebra. Let $(\mathcal{M}, \|\cdot\|)$ be a normed left $\mathcal{A}$-module such that $((\mathcal{M}^k, \|\cdot\|_k) : k \in \mathbb{N})$ is a multi-normed space. $((\mathcal{M}^k, \|\cdot\|_k) : k \in \mathbb{N})$ is said to be a *multi-normed left $\mathcal{A}$-module* if there exists a positive constant K such that $\|(ax_1, \cdots, ax_k)\|_k \leq K\|a\| \, \|(x_1, \cdots, x_k)\|_k$ for all $a \in \mathcal{A}$ and $x_1, \cdots, x_k \in \mathcal{M}$. Moreover, we says that the multi-normed left $\mathcal{A}$-module $((\mathcal{M}^k, \|\cdot\|_k) : k \in \mathbb{N})$ is a *multi-Banach left $\mathcal{A}$-module* if $((\mathcal{M}^k, \|\cdot\|_k) : k \in \mathbb{N})$ is a multi-Banach space.

Example 3.4. Let $(\mathcal{A}, \|\cdot\|)$ be a normed algebra and $((\mathcal{A}^k, \|\cdot\|_k) : k \in \mathbb{N})$ a multi-normed algebra. Then $((\mathcal{A}^k, \|\cdot\|_k) : k \in \mathbb{N})$ is a multi-normed left $\mathcal{A}$-module.

Example 3.5. Let $(\mathcal{A}, \|\cdot\|)$ be a normed algebra and $((\mathcal{A}^k, \|\cdot\|_k) : k \in \mathbb{N})$ a multi-normed algebra. Let I be a closed left ideal of $\mathcal{A}$, let $\mathcal{M} = \mathcal{A}/I$, and let $a \mapsto a + I$ denote the canonical mapping of $\mathcal{A}$ onto $\mathcal{M}$. Then the normed linear space $(\mathcal{M}, \|\cdot\|)$ becomes a normed left $\mathcal{A}$-module with the module multiplication given by $ax = ab + I$, where $b \in x \in \mathcal{M}, a \in \mathcal{A}$. Now, it is easy to see that $((\mathcal{M}^k, \|\cdot\|_k) : k \in \mathbb{N})$ is a multi-normed left $\mathcal{A}$-module.

Definition 3.6. Let $\mathcal{A}$ be an algebra. A left $\mathcal{A}$-module $\mathcal{M}$ is said to be *unitary* if $\mathcal{A}$ has a unit element e and $ex = x$ for all $x \in \mathcal{M}$.

Throughout this section, let $(\mathcal{A}, \|\cdot\|)$ be a unital normed algebra with unit e, $\mathcal{M}_1$ a unitary left $\mathcal{A}$-module and $(\mathcal{M}_2, \|\cdot\|)$ a unitary Banach left $\mathcal{A}$-module.

Recall [16] that an additive mapping $f : \mathcal{M}_1 \to \mathcal{M}_2$ is said to be $\mathcal{A}$-linear if $f(ax) = af(x)$ for all $a \in \mathcal{A}$ and $x \in \mathcal{M}_1$.

Given a function $f : \mathcal{M}_1 \to \mathcal{M}_2$ and $a \in \mathcal{A}$ with $\|a\| = 1$, we put

$$D_a f(x,y,z) := 3f\left(\frac{ax+ay+az}{3}\right) + af(x) + af(y) + af(z) \\ - 2\left[af\left(\frac{x+y}{2}\right) + f\left(\frac{ay+az}{2}\right) + f\left(\frac{az+ax}{2}\right)\right]$$

for all $x, y, z \in \mathcal{M}_1$.

Theorem 3.7. *Let $((\mathcal{M}_2^k, \|\cdot\|_k) : k \in \mathbb{N})$ be a multi-Banach left $\mathcal{A}$-module. If $\delta \geq 0$ and the function $f : \mathcal{M}_1 \to \mathcal{M}_2$ satisfies*

$$\sup_{k\in\mathbb{N}} \|(D_a f(x_1,y_1,z_1), \cdots, D_a f(x_k,y_k,z_k))\|_k \leq \delta \tag{3.1}$$

for all $x_1, \cdots, x_k, y_1, \cdots, y_k, z_1, \cdots, z_k \in \mathcal{M}_1$ and for all $a \in \mathcal{A}$ with $\|a\| = 1$, then there exists a unique $\mathcal{A}$-linear mapping $A : \mathcal{M}_1 \to \mathcal{M}_2$ satisfying the inequality (2.2).

Proof. Using Theorem 2.2, it follows from the inequality (3.1) for $a = \alpha e$, $\alpha \in U$, that there exists a unique $\mathbb{C}$-linear mapping $A : \mathcal{M}_1 \to \mathcal{M}_2$ defined by

$$A(x) = \lim_{n\to\infty} 2^{-2n} f(2^{2n}x)$$

for all $x \in \mathcal{M}_1$ such that the inequality (2.2) is valid.

The substitution a for α in (2.8)$\sim$(2.13) and the same process yield

$$\|f(2^{2n}ax) - af(2^{2n}x)\| \leq 2\varepsilon \tag{3.2}$$

for all $x \in \mathcal{M}_1$ which gives

$$\lim_{n\to\infty} 2^{-2n}\|f(2^{2n}ax) - af(2^{2n}x)\| = 0,$$

for all $x \in \mathcal{M}_1$. Thus, we see that

$$A(ax) = \lim_{n\to\infty} 2^{-2n} f(2^{2n}ax) = \lim_{n\to\infty} 2^{-2n} af(2^{2n}x) = aA(x)$$

for all $a \in \mathcal{A}$ with $\|a\| = 1$ and all $x \in \mathcal{M}_1$. Since A is $\mathbb{C}$-linear and $A(ax) = aA(x)$ for each element $a \in \mathcal{A}$ with $\|a\| = 1$, we have, for all $a \in \mathcal{A} \setminus \{0\}$ and all $x \in \mathcal{M}_1$,

$$A(ax) = A\left(\|a\| \frac{a}{\|a\|}x\right) = \|a\| A\left(\frac{a}{\|a\|}x\right) = \|a\| \frac{a}{\|a\|} A(x) = aA(x).$$

Therefore, the unique $\mathbb{C}$-linear mapping $A : \mathcal{M}_1 \to \mathcal{M}_2$ is an $\mathcal{A}$-linear mapping, as desired. □

Theorem 3.8. *Let $\mathcal{A}$ be a Banach $*$-algebra, $pos(\mathcal{A})$ the set of positive elements of $\mathcal{A}$ and $((\mathcal{M}_2^k, \|\cdot\|_k) : k \in \mathbb{N})$ a multi-Banach left $\mathcal{A}$-module. If the function $f : \mathcal{M}_1 \to \mathcal{M}_2$ satisfies the inequality* (3.1) *for all $x_1, \cdots, x_k, y_1, \cdots, y_k, z_1, \cdots, z_k \in \mathcal{M}_1$ and all $a \in pos(\mathcal{A})$ with $\|a\| = 1$ and $a = i$, then there exists a unique $\mathcal{A}$-linear mapping $A : \mathcal{M}_1 \to \mathcal{M}_2$ satisfying the inequality* (2.2).

Proof. By the same reasoning as in the proof of Theorem 2.2, there exists a unique additive mapping $A : \mathcal{M}_1 \to \mathcal{M}_2$ defined by $A(x) = \lim_{n\to\infty} 2^{-2n} f(2^{2n}x)$ for all $x \in \mathcal{M}_1$ such that the inequality (2.2) holds. Following the same method as in the proof of Theorem 3.7, we see that

$$A(ax) = \lim_{n\to\infty} 2^{-2n} f(2^{2n}ax) = \lim_{n\to\infty} 2^{-2n} af(2^{2n}x) = aA(x)$$

for all $a \in pos(\mathcal{A})$ with $\|a\| = 1$ or $a = i$, and $x \in \mathcal{M}_1$. For any element $a \in \mathcal{A}$, $a = a_1 + ia_2$, where $a_1 = \frac{a+a^*}{2}$ and $a_2 = \frac{a-a^*}{2i}$ are self-adjoint elements, furthermore, $a = {a_1}^+ - {a_1}^- + i{a_2}^+ - i{a_2}^-$, where ${a_1}^+, {a_1}^-, {a_2}^+, {a_2}^- \in pos(\mathcal{A})$ (see [2, Lemma 38.8]). Therefore, we have

$$\begin{aligned}
A(ax) &= A({a_1}^+x - {a_1}^-x + i{a_2}^+x - i{a_2}^-x) \\
&= {a_1}^+A(x) - {a_1}^-A(x) + {a_2}^+A(ix) - {a_2}^-A(ix) \\
&= {a_1}^+A(x) - {a_1}^-A(x) + i{a_2}^+A(x) - i{a_2}^-A(x) \\
&= ({a_1}^+ - {a_1}^- + i{a_2}^+ - i{a_2}^-)A(x) \\
&= aA(x)
\end{aligned}$$

for all $a \in \mathcal{A}$ and all $x \in \mathcal{M}_1$ which completes the proof of the theorem. □

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