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ASYMPTOTIC BEHAVIORS OF JENSEN TYPE FUNCTIONAL EQUATIONS IN HALF PLANES

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Abstract. Let $f : \mathbb{R} \to \mathbb{C}$. We consider the Hyers-Ulam stability of Jensen type functional inequality

$$|f(px + qy) - Pf(x) - Qf(y)| \le \epsilon$$

in the half planes $\{(x, y) : kx + sy \ge d\}$ for fixed $d, k, s \in \mathbb{R}$ with $k \ne 0$ or $s \ne 0$. As consequences of the results we obtain the asymptotic behaviors of f satisfying

$$|f(px + qy) - Pf(x) - Qf(y)| \to 0$$

as $kx + sy \to \infty$.

1. Introduction

The stability problems of functional equations have originated with S. M. Ulam in 1940 when he proposed the following problem [26]:

Let f be a mapping from a group G_1 to a metric group G_2 with metric $d(\cdot, \cdot)$ such that

$$d(f(xy),\, f(x)f(y)) \le \varepsilon.$$

Then does there exist a group homomorphism h and $\delta_\epsilon > 0$ such that

$$d(f(x),\, h(x)) \le \delta_\epsilon$$

for all $x \in G_1$?

As an answer for the question of Ulam, D. H. Hyers proved the following result.

Theorem 1.1. *Suppose that $\langle S, +\rangle$ is an additive semigroup, $\epsilon \ge 0$, and $f : S \to B$ with B a Banach space, satisfies the inequality*

$$\|f(x + y) - f(x) - f(y)\| \le \epsilon \tag{1.1}$$

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for all $x,\, y \in S$. Then there exists a unique function $A : S \to B$ satisfying

$$A(x+y) = A(x) + A(y) \tag{1.2}$$

for which

$$\|f(x) - A(x)\| \le \epsilon$$

for all $x \in S$.

We call the functions satisfying (1.2) *additive functions*. Generalizing the Hyers' result he proved that if a mapping $f : X \to Y$ between two Banach spaces satisfies

$$\|f(x+y) - f(x) - f(y)\| \le \Phi(x,y) \quad \text{for } \; x, y \in X$$

with $\Phi(x,y) = \epsilon(\|x\|^p + \|y\|^p)\,(\epsilon \ge 0,\, 0 \le p < 1)$, then there exists a unique additive function $A : X \to Y$ such that $\|f(x) - A(x)\| \le 2\epsilon|x|^p/(2-2^p)$ for all $x \in X$. In 1951, D. G. Bourgin[4] stated that if Φ is symmetric in $\|x\|$ and $\|y\|$ with $\sum_{j=1}^{\infty} \Phi(2^j x, 2^j x)/2^j < \infty$ for each $x \in X$, then there exists a unique additive function $A : X \to Y$ such that $\|f(x) - A(x)\| \le \sum_{j=1}^{\infty} \Phi(2^j x, 2^j x)/2^j$ for all $x \in X$. Unfortunately, there were no use of these results until 1978 when Th. M. Rassias [21] treated with the inequality of Aoki [1]. Following the Rassias' result, a great number of papers on the subject have been published concerning numerous functional equations in various directions [11, 13, 14, 15, 16, 18, 19, 20, 21, 25]. Among the results, stability problem in a restricted domain was investigated by F. Skof, who proved the stability problem of the inequality (1.1) in a restricted domain [25]. Developing this result, S.-M. Jung, J. M. Rassias and M. J. Rassias considered the stability problems in restricted domains for the Jensen functional equation [14] and Jensen type functional equations [19]. We also refer the reader to [2, 3, 6, 7, 8, 9, 22, 23, 24] for some related results on Hyers-Ulam stabilities in restricted conditions. Throughout this paper we denote by $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{C}$ the sets of real numbers, positive real numbers and complex numbers, respectively, $f : \mathbb{R} \to \mathbb{C}$ and p, q, P, Q be fixed nonzero real numbers. In this paper we prove the Hyers-Ulam stability of the Jensen type functional inequality

$$|f(px+qy) - Pf(x) - Qf(y)| \le \epsilon \tag{1.3}$$

in restricted domain $\Pi_{k,s,d} = \{(x,y) \in \mathbb{R}^2 : kx + sy \ge d\}$ for fixed $d, k, s \in \mathbb{R}$ with $k \ne 0$ or $s \ne 0$. As a consequence of the result we prove that if

$$|f(px+qy) - Pf(x) - Qf(y)| \to 0$$

as $kx + sy \to \infty$, then there exists a unique additive function $A : \mathbb{R} \to \mathbb{C}$ such that

$$f(x) = A(x) + f(0)$$

for all $x \in \mathbb{R}$.

2. Hyers-Ulam Stability of Jensen Type Equation in Restricted Domains

We first consider the usual Cauchy functional inequality in the restricted domain $\Pi_{k,s,d} = \{(x, y) \in \mathbb{R}^2 : kx + sy \geq d\}$ for fixed $k, s, d \in \mathbb{R}$ with $k \neq 0$ or $s \neq 0$.

Theorem 2.1. *Let $\epsilon \geq 0$, $d, k, s \in \mathbb{R}$ with $k \neq 0$ or $s \neq 0$. Suppose that $f : \mathbb{R} \to \mathbb{C}$ satisfies*

$$|f(x + y) - f(x) - f(y)| \leq \epsilon \tag{2.1}$$

for all $x, y \in \mathbb{R}$, with $kx + sy \geq d$. Then there exists a unique additive function $A : \mathbb{R} \to \mathbb{C}$ such that

$$|f(x) - A(x)| \leq 3\epsilon \tag{2.2}$$

for all $x \in \mathbb{R}$.

Proof. From the symmetry of the inequality we may assume that $s \neq 0$. For given $x, y \in \mathbb{R}$, choose a $z \in \mathbb{R}$ such that $kx+ky+sz \geq d$, $kx+sy+sz \geq d$ and $ky+sz \geq d$. Then we have

$$\begin{aligned} &|f(x + y) - f(x) - f(y)| \\ &\leq |-f(x + y + z) + f(x + y) + f(z)| \\ &\quad + |f(x + y + z) - f(x) - f(y + z)| \\ &\quad + |f(y + z) - f(y) - f(z)| \\ &\leq 3\epsilon. \end{aligned} \tag{2.3}$$

Now by Theorem 1.1, there exists a unique additive function $A : \mathbb{R} \to \mathbb{C}$ such that

$$|f(x) - A(x)| \leq 3\epsilon$$

for all $x \in \mathbb{R}$. This completes the proof. □

Now we consider the Hyers-Ulam stability of the Jensen type functional inequality (1.3) in the restricted domains $\Pi_{k,s,d}$.

Theorem 2.2. *Let $\epsilon \geq 0$, $d, k, s \in \mathbb{R}$, $\frac{k}{p} \neq \frac{s}{q}$. Suppose that $f : \mathbb{R} \to \mathbb{C}$ satisfies*

$$|f(px+qy) - Pf(x) - Qf(y)| \leq \epsilon \tag{2.4}$$

for all $x, y \in \mathbb{R}$, with $kx + sy \geq d$. Then there exists a unique additive function $A : \mathbb{R} \to \mathbb{C}$ such that

$$|f(x) - A(x) - f(0)| \leq 4\epsilon \tag{2.5}$$

for all $x \in \mathbb{R}$.

Proof. Replacing x by $\frac{1}{p}x$, y by $\frac{1}{q}y$ in (2.4) we have

$$|f(x+y) - Pf(\frac{x}{p}) - Qf(\frac{y}{q})| \leq \epsilon \tag{2.6}$$

for all $x, y \in \mathbb{R}$, with $\frac{k}{p}x + \frac{s}{q}y \geq d$. For given $x, y \in \mathbb{R}$, choose a $z \in \mathbb{R}$ such that $\frac{k}{p}x + \frac{s}{q}y + (\frac{s}{q} - \frac{k}{p})z \geq d$, $\frac{k}{p}x + (\frac{s}{q} - \frac{k}{p})z \geq d$, $\frac{s}{q}y + (\frac{s}{q} - \frac{k}{p})z \geq d$, and $(\frac{s}{q} - \frac{k}{p})z \geq d$. Replacing x by $x - z$, y by $y + z$; x by $x - z$, y by z; x by $-z$, y by $y + z$; x by z^{-1}, y by z in (2.6) we have

$$\begin{aligned} &|f(x+y) - f(x) - f(y) + f(0)| \\ &\leq \left|f(x+y) - Pf\left(\frac{x-z}{p}\right) - Qf\left(\frac{y+z}{q}\right)\right| \\ &\quad + \left|-f(x) + Pf\left(\frac{x-z}{p}\right) + Qf\left(\frac{z}{q}\right)\right| \\ &\quad + \left|-f(y) + Pf\left(-\frac{z}{p}\right) + Qf\left(\frac{y+z}{q}\right)\right| \\ &\quad + \left|f(0) - Pf\left(-\frac{z}{p}\right) - Qf\left(\frac{z}{q}\right)\right| \\ &\leq 4\epsilon. \end{aligned} \tag{2.7}$$

Now by Theorem 1.1, there exists a unique additive function $A : \mathbb{R} \to \mathbb{C}$ such that

$$|f(x) - A(x) - f(0)| \leq 4\epsilon$$

for all $x \in \mathbb{R}$. This completes the proof. □

Theorem 2.3. *Let $\epsilon \geq 0$, $d, k, s \in \mathbb{R}$ with $k \neq 0$ or $s \neq 0$. Suppose that $f : \mathbb{R} \to \mathbb{C}$ satisfies*

$$|f(px+qy) - Pf(x) - Qf(y)| \leq \epsilon \tag{2.8}$$

for all $x, y \in \mathbb{R}$, with $kx + sy \geq d$. Then there exists a unique additive function $A : \mathbb{R} \to \mathbb{C}$ such that

$$|f(x) - A(x) - f(0)| \le \frac{4\epsilon}{|P|} \tag{2.9}$$

for all $x \in \mathbb{R}$ *if* $s \neq 0$, *and*

$$|f(x) - A(x) - f(0)| \le \frac{4\epsilon}{|Q|} \tag{2.10}$$

for all $x \in \mathbb{R}$ *if* $k \neq 0$.

Proof. Assume that $s \neq 0$. For given $x, y \in \mathbb{R}$, choose a $z \in \mathbb{R}$ such that $kx+ky+sz \ge d$, $kx + \frac{ps}{q}y + sz \ge d$, $ky + sz \ge d$ and $\frac{ps}{q}y + sz \ge d$. Replacing x by $x + y$, y by z; x by x, y by $\frac{p}{q}y + z$; x by y, y by z; x by 0, y by $\frac{p}{q}y + z$ in (2.8) we have

$$\begin{aligned} &|Pf(x+y) - Pf(x) - Pf(y) + Pf(0)| \\ &\le |-f(px+py+qz) + Pf(x+y) + Qf(z)| \\ &\quad + \left|f(px+py+qz) - Pf(x) - Qf\left(\frac{p}{q}y + z\right)\right| \\ &\quad + |f(py+qz) - Pf(y) - Qf(z)| \\ &\quad + \left|-f(py+qz) + Pf(0) + Qf\left(\frac{p}{q}y + z\right)\right| \\ &\le 4\epsilon. \end{aligned} \tag{2.11}$$

Dividing (2.11) by $|P|$ and using Theorem 1.1, we obtain that there exists a unique additive function $A : \mathbb{R} \to \mathbb{C}$ such that

$$|f(x) - A(x) - f(0)| \le \frac{4\epsilon}{|P|}$$

for all $x \in \mathbb{R}$. Assume that $k \neq 0$. For given $x, y \in \mathbb{R}$, choose a $z \in \mathbb{R}$ such that $sx + sy + kz \ge d$, $\frac{qk}{p}x + sy + kz \ge d$, $sx + kz \ge d$ and $\frac{qk}{p}x + kz \ge d$. Replacing y by $x + y$, x by z; y by y, x by $\frac{q}{p}x + z$; y by x, x by z; y by 0, x by $\frac{q}{p}x + z$ in (2.8) we have

$$\begin{aligned} &|Qf(x+y) - Qf(x) - Qf(y) + Qf(0)| \\ &\le |-f(px+py+qz) + Pf(z) + Qf(x+y)| \\ &\quad + \left|f(qx+qy+pz) - Pf\left(\frac{q}{p}x + z\right) - Qf(y)\right| \\ &\quad + |f(qx+pz) - Pf(z) - Qf(x)| \\ &\quad + \left|-f(qx+pz) + Pf\left(\frac{q}{p}x + z\right) + Qf(0)\right| \\ &\le 4\epsilon. \end{aligned} \tag{2.12}$$

Dividing (2.12) by $|Q|$ and using Theorem 1.1, we obtain that there exists a unique additive function $A : \mathbb{R} \to \mathbb{C}$ such that

$$|f(x) - A(x) - f(0)| \leq \frac{4\epsilon}{|Q|}$$

for all $x \in \mathbb{R}$. This completes the proof. □

We obtain that $A = 0$ in Theorem 2.2 and Theorem 2.3 provided that $p \neq P$ and p or P is a rational number, or $q \neq Q$ and q or Q is a rational number. As a matter of fact we have the followings.

Theorem 2.4. *Let $\epsilon \geq 0, d, k, s \in \mathbb{R}$, $\frac{k}{p} \neq \frac{s}{q}$. Suppose that $p \neq P$ and p or P is a rational number, or $q \neq Q$ and q or Q is a rational number, and $f : \mathbb{R} \to \mathbb{C}$ satisfies*

$$|f(px + qy) - Pf(x) - Qf(y)| \leq \epsilon \tag{2.13}$$

for all $x, y \in \mathbb{R}$, with $kx + sy \geq d$. Then we have

$$|f(x) - f(0)| \leq 4\epsilon \tag{2.14}$$

for all $x \in \mathbb{R}$.

Proof. We prove (2.14) only for the case that $p \neq P$ and p or P is a rational number since the other case is similarly proved. From (2.5) and (2.13), using the triangle inequality we have

$$|A(px + qy) - PA(x) - QA(y)| \leq M \tag{2.15}$$

for all $x, y \in \mathbb{R}$, with $kx + sy \geq d$, where $M = \epsilon(5 + 4|P| + 4|Q|) + |f(0)(1 - P - Q)|$. If $k \neq 0$, putting $y = 0$ in (2.15) we have

$$|A(px) - PA(x)| \leq M \tag{2.16}$$

for all $x \in \mathbb{R}$, with $kx \geq d$. Since A is additive and p is rational, it follows from (2.16) that

$$|A(x)| \leq \frac{M}{|p - P|} \tag{2.17}$$

for all $x \in \mathbb{R}$, with $kx \geq d$. If there exists $x_0 \in \mathbb{R}$ such that $A(x_0) \neq 0$, we can choose a rational number r such that $rkx_0 \geq d$ and $|rA(x_0)| > \frac{M}{|p-P|}$(it is realized when r is large if $kx_0 > 0$, and when $-r$ is large if $kx_0 < 0$). Now we have

$$\frac{M}{|p - P|} < |rA(x_0)| = |A(rx_0)| \leq \frac{M}{|p - P|}. \tag{2.18}$$

Thus it follows that $A = 0$. If P is a rational number, it follows (2.16) that

$$|A((p - P)x)| \leq M$$

for all $x \in \mathbb{R}$, with $kx \geq d$, which implies

$$|A(x)| \leq M \tag{2.19}$$

for all $x \in \mathbb{R}$, with $\frac{kx}{p-P} \geq d$. Similarly, using (2.19) we can show that $A = 0$. If $k = 0$, choosing $y_0 \in \mathbb{R}$ such that $sy_0 \geq d$, putting $y = y_0$ in (2.15) and using the triangle inequality we have

$$|A(px) - PA(x)| \leq M + |A(qy_0) - QA(y_0)| \tag{2.20}$$

for all $x \in \mathbb{R}$. Similarly, using (2.20) we can show that $A = 0$. Now the inequality (2.14) follows from (2.5). This completes the proof. □

From Theorem 2.3, using the same approach in the proof of Theorem 2.4 we have the following.

Theorem 2.5. *Let ϵ, $d, k, s \in \mathbb{R}$ with $k \neq 0$ or $s \neq 0$. Suppose that $p \neq P$ and p or P is a rational number, or $q \neq Q$ and q or Q is a rational number, and $f : \mathbb{R} \to \mathbb{C}$ satisfies*

$$|f(px + qy) - Pf(x) - Qf(y)| \leq \epsilon \tag{2.21}$$

for all $x, y \in \mathbb{R}$, with $kx + sy \geq d$. Then we have

$$|f(x) - f(0)| \leq \frac{4\epsilon}{|P|} \tag{2.22}$$

for all $x \in \mathbb{R}$ if $s \neq 0$, and

$$|f(x) - f(0)| \leq \frac{4\epsilon}{|Q|} \tag{2.23}$$

for all $x \in \mathbb{R}$ if $k \neq 0$.

We call $L : \mathbb{R}_+ \to \mathbb{C}$ *a logarithmic function* provided that

$$L(xy) = L(x) + L(y)$$

for all $x, y > 0$. Using Theorem 2.2 we have the following.

Corollary 2.6. *Let $\epsilon, d > 0$, $k, s \in \mathbb{R}$, $\frac{k}{p} \neq \frac{s}{q}$. Suppose that $g : \mathbb{R}_+ \to \mathbb{C}$ satisfies*

$$|g(x^p y^q) - Pg(x) - Qg(y)| \leq \epsilon \tag{2.24}$$

for all $x, y > 0$, with $x^k y^s \geq d$. Then there exists a unique logarithmic function $L : \mathbb{R}_+ \to \mathbb{C}$ such that

$$|g(x) - L(x) - g(1)| \leq 4\epsilon \tag{2.25}$$

for all $x \in \mathbb{R}_+$.

Proof. Replacing x by e^u, y by e^v in (2.24) and setting $f(x) = g(e^x)$ we have

$$|f(pu + qv) - Pf(u) - Qf(v)| \le \epsilon \tag{2.26}$$

for all $u, v \in \mathbb{R}$, with $ku + sv \ge \ln d$. Using Theorem 2.2 we have

$$|f(x) - A(x) - f(0)| \le 4\epsilon$$

for all $x \in \mathbb{R}$, which implies

$$|g(x) - A(\ln x) - g(1)| \le 4\epsilon \tag{2.27}$$

for all $x > 0$. Letting $L(x) = A(\ln x)$ we get the result. □

3. Asymptotic Behavior of the Inequality

In this section, we consider asymptotic behaviors of the functional inequalities (1.3) and (2.1).

Theorem 3.1. *Let $k, s \in \mathbb{R}$ satisfy one of the conditions; $k \ne 0$, $s \ne 0$. Suppose that $f : \mathbb{R} \to \mathbb{C}$ satisfies the asymptotic condition*

$$|f(x + y) - f(x) - f(y)| \to 0 \tag{3.1}$$

as $kx + sy \to \infty$. Then f is an additive function.

Proof. By the condition (3.1), for each $n \in \mathbb{N}$, there exists $d_n \in \mathbb{R}$ such that

$$|f(x + y) - f(x) - f(y)| \le \frac{1}{n}$$

for all $x, y \in \mathbb{R}$, with $kx + sy \ge d_n$. By Theorem 2.1, there exists a unique additive function $A_n : \mathbb{R} \to \mathbb{C}$ such that

$$|f(x) - A_n(x)| \le \frac{3}{n} \tag{3.2}$$

for all $x \in \mathbb{R}$. From (3.2), using triangle inequality we have

$$|A_n(x) - A_m(x)| \le \frac{3}{n} + \frac{3}{m} \le 6 \tag{3.3}$$

for all $x \in \mathbb{R}$ and all positive integers n, m. Now, the inequality (3.3) implies $A_n = A_m$. Indeed, for all $x \in \mathbb{R}$ and rational numbers $r > 0$ we have

$$|A_n(x) - A_m(x)| = \frac{1}{r}|A_n(rx) - A_m(rx)| \le \frac{6}{r}. \tag{3.4}$$

Letting $r \to \infty$ in (3.4) we have $A_n = A_m$. Thus, letting $n \to \infty$ in (3.2) we get the result. □

Theorem 3.2. *Let $k, s \in \mathbb{R}$ satisfy one of the conditions; $k \neq 0$, $s \neq 0$, $\frac{k}{p} \neq \frac{s}{q}$. Suppose that $f : \mathbb{R} \to \mathbb{C}$ satisfies the asymptotic condition*

$$|f(px + qy) - Pf(x) - Qf(y)| \to 0 \tag{3.5}$$

as $kx + sy \to \infty$. Then there exists a unique additive function $A : \mathbb{R} \to \mathbb{C}$ such that

$$f(x) = A(x) + f(0) \tag{3.6}$$

for all $x \in \mathbb{R}$.

Proof. By the condition (3.5), for each $n \in \mathbb{N}$, there exists $d_n \in \mathbb{R}$ such that

$$|f(px + qy) - Pf(x) - Qf(y)| \le \frac{1}{n} \tag{3.7}$$

for all $x, y \in \mathbb{R}$, with $kx + sy \ge d_n$. By Theorem 2.2 and Theorem 2.3, there exists a unique additive function $A_n : \mathbb{R} \to \mathbb{C}$ such that

$$|f(x) - A_n(x) - f(0)| \le \frac{4}{n} \tag{3.8}$$

if $\frac{k}{p} \neq \frac{s}{q}$,

$$|f(x) - A_n(x) - f(0)| \le \frac{4}{n|P|} \tag{3.9}$$

if $s \neq 0$, and

$$|f(x) - A_n(x) - f(0)| \le \frac{4}{n|Q|} \tag{3.10}$$

if $k \neq 0$. For all cases (3.8), (3.9) and (3.10), there exists $M > 0$ such that

$$|A_n(x) - A_m(x)| \le M \tag{3.11}$$

for all $x \in \mathbb{R}$ and all positive integers n, m. Similarly as in the proof of Theorem 3.1, it follows from (3.11) that $A_n = A_m$ for all $n, m \in \mathbb{N}$. Letting $n \to \infty$ in (3.8), (3.9) and (3.10) we get the result. □

Similarly using Theorem 2.4 and Theorem 2.5 we have the following.

Theorem 3.3. *Let $k, s \in \mathbb{R}$ satisfy one of the conditions; $k \neq 0$, $s \neq 0$, $\frac{k}{p} \neq \frac{s}{q}$. Suppose that $p \neq P$ and p or P is a rational number, or $q \neq Q$ and q or Q is a rational number, and $f : \mathbb{R} \to \mathbb{C}$ satisfies the asymptotic condition*

$$|f(px + qy) - Pf(x) - Qf(y)| \to 0 \tag{3.12}$$

as $kx + sy \to \infty$. Then f is a constant function.

4. Stability of Pexider Equation in Restricted Domains

Let $f, g, h : \mathbb{R} \to \mathbb{C}$. We prove the Hyers-Ulam stability of the Pexider functional inequality

$$|f(x+y) - g(x) - h(y)| \le \epsilon$$

in the restricted domains $\Pi_{k,s,d}$. Throughout this section s, k, d and $\epsilon \ge 0$ are fixed real numbers.

Lemma 4.1. *Suppose that $k \neq s$ and $f, g, h : \mathbb{R} \to \mathbb{C}$ satisfy*

$$|f(x+y) - g(x) - h(y)| \le \epsilon \tag{4.1}$$

for all $x, y \in \mathbb{R}$, with $kx + sy \ge d$. Then there exists a unique additive function $A_1 : \mathbb{R} \to \mathbb{C}$ such that

$$|f(x) - A_1(x) - f(0)| \le 4\epsilon \tag{4.2}$$

for all $x \in \mathbb{R}$.

Proof. For given $x, y \in \mathbb{R}$, choose a $z \in \mathbb{R}$ such that $kx + sy + (s-k)z \ge d$, $kx + (s-k)z \ge d$, $sy + (s-k)z \ge d$ and $(s-k)z \ge d$. Then we have

$$\begin{aligned} &|f(x+y) - f(x) - f(y) + f(0)| \\ &\le |f(x+y) - g(x-z) - h(y+z)| \\ &\quad + |-f(x) + g(x-z) + h(z)| \\ &\quad + |-f(y) + g(-z) + h(y+z)| \\ &\quad + |f(0) - g(-z) - h(z)| \\ &\le 4\epsilon. \end{aligned} \tag{4.3}$$

Now by Theorem 1.1, there exists a unique additive function $A_1 : \mathbb{R} \to \mathbb{C}$ such that

$$|f(x) - A_1(x) - f(0)| \le 4\epsilon$$

for all $x \in \mathbb{R}$. This completes the proof. □

Lemma 4.2. *Suppose that $s \neq 0$ and $f, g, h : \mathbb{R} \to \mathbb{C}$ satisfy*

$$|f(x+y) - g(x) - h(y)| \le \epsilon \tag{4.4}$$

for all $x, y \in \mathbb{R}$, with $kx + sy \ge d$. Then there exists a unique additive function $A_2 : \mathbb{R} \to \mathbb{C}$ such that

$$|g(x) - A_2(x) - g(0)| \le 4\epsilon \tag{4.5}$$

for all $x \in \mathbb{R}$.

Proof. For given $x, y \in \mathbb{R}$, choose a $z \in \mathbb{R}$ such that $kx+ky+sz \geq d$, $kx+sy+sz \geq d$, $ky+sz \geq d$ and $sy+sz \geq d$. Then we have

$$\begin{aligned}
&|g(x+y) - g(x) - g(y) + g(0)| \\
&\quad \leq |-f(x+y+z) + g(x+y) + h(z)| \\
&\qquad + |f(x+y+z) - g(x) - h(y+z)| \\
&\qquad + |f(y+z) - g(y) - h(z)| \\
&\qquad + |-f(y+z) + g(0) + h(y+z)| \\
&\quad \leq 4\epsilon.
\end{aligned} \tag{4.6}$$

Now by Theorem 1.1, there exists a unique additive function $A_2 : \mathbb{R} \to \mathbb{C}$ such that

$$|g(x) - A_2(x) - g(0)| \leq 4\epsilon$$

for all $x \in \mathbb{R}$. This completes the proof. □

Lemma 4.3. *Suppose that $k \neq 0$ and $f, g, h : \mathbb{R} \to \mathbb{C}$ satisfy*

$$|f(x+y) - g(x) - h(y)| \leq \epsilon \tag{4.7}$$

for all $x, y \in \mathbb{R}$, with $kx + sy \geq d$. Then there exists a unique additive function $A_3 : \mathbb{R} \to \mathbb{C}$ such that

$$|h(x) - A_3(x) - h(0)| \leq 4\epsilon \tag{4.8}$$

for all $x \in \mathbb{R}$.

Proof. For given $x, y \in \mathbb{R}$, choose a $z \in \mathbb{R}$ such that $sx+sy+kz \geq d$, $kx+sy+kz \geq d$, $sx+kz \geq d$ and $kx+kz \geq d$. Then we have

$$\begin{aligned}
&|h(x+y) - h(x) - h(y) + h(0)| \\
&\quad \leq |-f(x+y+z) + g(z) + h(x+y)| \\
&\qquad + |f(x+y+z) - g(x+z) - h(y)| \\
&\qquad + |f(z+x) - g(z) - h(x)| \\
&\qquad + |-f(x+z) + g(x+z) + h(0)| \\
&\quad \leq 4\epsilon.
\end{aligned} \tag{4.9}$$

Now by Theorem 1.1, there exists a unique additive function $A_3 : \mathbb{R} \to \mathbb{C}$ such that

$$|h(x) - A_3(x) - h(0)| \leq 4\epsilon$$

for all $x \in \mathbb{R}$. This completes the proof. □

Now we state and prove the main theorem of this section.

Theorem 4.4. *Suppose that $k, s \neq 0$, $k \neq s$ and $f, g, h : \mathbb{R} \to \mathbb{C}$ satisfy*

$$|f(x+y) - g(x) - h(y)| \leq \epsilon \tag{4.10}$$

for all $x, y \in \mathbb{R}$, with $kx + sy \geq d$. Then there exists a unique additive function $A : \mathbb{R} \to \mathbb{C}$ such that

$$|f(x) - A(x) - f(0)| \leq 4\epsilon,$$
$$|g(x) - A(x) - g(0)| \leq 4\epsilon,$$
$$|h(x) - A(x) - h(0)| \leq 4\epsilon,$$

for all $x \in \mathbb{R}$.

Proof. In view of Lemma 4.1, Lemma 4.2 and Lemma 4.3, it suffices to prove that $A_1 = A_2 = A_3$. For given $x, y \in \mathbb{R}$, choosing a $z \in \mathbb{R}$ such that $kx + sy + (s-k)z \geq d$, $(s-k)z \geq d$ and replacing x by $x - z$, y by $y + z$, and x by $-z$, y by $-z$ in (4.10) we have

$$|f(x+y) - g(x-z) - h(y+z)| \leq \epsilon, \tag{4.11}$$
$$|-f(0) + g(-z) + h(z)| \leq \epsilon. \tag{4.12}$$

The inequalities (4.6) and (4.9) imply

$$|g(x+y) - g(x) - g(y) + g(0)| \leq 4\epsilon, \tag{4.13}$$
$$|h((x+y) - h(x) - h(y) + h(0)| \leq 4\epsilon \tag{4.14}$$

for all $x, y \in \mathbb{R}$. Replacing y by $-z$ in (4.13) we have

$$|g(x-z) - g(x) - g(-z) + g(0)| \leq 4\epsilon \tag{4.15}$$

for all $x, y, z \in \mathbb{R}$. Replacing x by z in (4.14) we have

$$|h(y+z) - h(z) - h(y) + h(0)| \leq 4\epsilon \tag{4.16}$$

for all $x, y, z \in \mathbb{R}$. From (4.11), (4.12), (4.15) and (4.16), using the triangle inequality we have

$$|f(x+y) - g(x) - h(y) - f(0) + g(0) + h(0)| \leq 10\epsilon \tag{4.17}$$

for all $x, y \in \mathbb{R}$. Using the triangle inequality and (4.2), (4.5), (4.8) and (4.17) we have

$$\begin{aligned}
&|A_1(x+y) - A_2(x) - A_3(y)| \\
&\le |-f(x+y) + A_1(x+y) + f(0)| \\
&\quad + |g(x) - A_2(x) - g(0)| + |h(y) - A_3(y) - h(0)| \\
&\quad + |f(x+y) - g(x) - h(y) - f(0) + g(0) + h(0)| \\
&\le 22\epsilon.
\end{aligned} \tag{4.18}$$

Putting $y = 0$ and $x = 0$ in (4.18) separately, and using the fact that every nonzero additive function is unbounded as the same method of the proof in Theorem 2.4 we have $A_1 = A_2$ and $A_1 = A_3$. Letting $A := A_1 = A_2 = A_3$ we complete the proof. □

Now we consider asymptotic behaviors of the inequality (4.1).

Theorem 4.5. *Let $k, s \in \mathbb{R}$, $k \neq s$. Suppose that $f, g, h : \mathbb{R} \to \mathbb{C}$ satisfy the asymptotic condition*

$$|f(x+y) - g(x) - h(y)| \to 0 \tag{4.19}$$

as $kx + sy \to \infty$. Then there exists a unique additive function $A : \mathbb{R} \to \mathbb{C}$ such that

$$f(x) = A(x) + f(0) \tag{4.20}$$

for all $x \in \mathbb{R}$.

Proof. By the condition (4.19), for each $n \in \mathbb{N}$, there exists $d_n \in \mathbb{R}$ such that

$$|f(x+y) - g(x) - h(y)| \le \frac{1}{n} \tag{4.21}$$

for all $x, y \in \mathbb{R}$, with $kx + sy \ge d_n$. By Theorem 2.1, there exists a unique additive function $A_n : \mathbb{R} \to \mathbb{C}$ such that

$$|f(x) - A_n(x) - f(0)| \le \frac{4}{n} \tag{4.22}$$

for all $x \in \mathbb{R}$. From (4.22), using the triangle inequality we have

$$|A_n(x) - A_m(x)| \le \frac{4}{n} + \frac{4}{m} \le 8 \tag{4.23}$$

for all $x \in \mathbb{R}$. Thus it follows from (4.23) that $A_n = A_m$ for all $n, m \in \mathbb{N}$. Letting $n \to \infty$ in (4.22), we get the result. □

Using Theorem 4.2 we obtain the results.

Theorem 4.6. *Let $s \neq 0$. Suppose that $f, g, h : \mathbb{R} \to \mathbb{C}$ satisfy the asymptotic condition*

$$|f(x+y) - g(x) - h(y)| \to 0 \tag{4.24}$$

as $kx+sy\to\infty$. Then there exists a unique additive function $A:\mathbb{R}\to\mathbb{C}$ such that

$$g(x) = A(x) + g(0) \tag{4.25}$$

for all $x\in\mathbb{R}$.

Using Theorem 4.3 we obtain the following.

Theorem 4.7. *Let $k\neq 0$. Suppose that $f,g,h:\mathbb{R}\to\mathbb{C}$ satisfy the asymptotic condition*

$$|f(x+y)-g(x)-h(y)|\to 0 \tag{4.26}$$

as $kx+sy\to\infty$. Then there exists a unique additive function $A:\mathbb{R}\to\mathbb{C}$ such that

$$h(x) = A(x) + h(0) \tag{4.27}$$

for all $x\in\mathbb{R}$.

Using Theorem 4.4 we obtain the following.

Theorem 4.8. *Let $k\neq 0, s\neq 0$ and $k\neq s$. Suppose that $f,g,h:\mathbb{R}\to\mathbb{C}$ satisfy the asymptotic condition*

$$|f(x+y)-g(x)-h(y)|\to 0 \tag{4.28}$$

as $kx+sy\to\infty$. Then there exists a unique additive function $A:\mathbb{R}\to\mathbb{C}$ such that

$$f(x) = A(x) + f(0),$$
$$g(x) = A(x) + g(0),$$
$$h(x) = A(x) + h(0)$$

for all $x\in\mathbb{R}$.

Proof. By the condition (4.28), for each $n\in\mathbb{N}$, there exists $d_n\in\mathbb{R}$ such that

$$|f(x+y)-g(x)-h(y)|\leq \frac{1}{n} \tag{4.29}$$

for all $x,y\in\mathbb{R}$, with $kx+sy\geq d_n$. By Theorem 4.4, there exists a unique additive function $\mathbb{R}\to\mathbb{C}$ such that

$$|f(x)-A_n(x)-f(0)|\leq \frac{4}{n}, \tag{4.30}$$

$$|g(x)-A_n(x)-g(0)|\leq \frac{4}{n}, \tag{4.31}$$

$$|h(x)-A_n(x)-h(0)|\leq \frac{4}{n} \tag{4.32}$$

for all $x\in\mathbb{R}$. Similarly as in the proof of Theorem 4.5, we can show that $A_n=A_m$ for all $n,m\in\mathbb{N}$. Letting $n\to\infty$ in (4.30), (4.31) and (4.32) we get the result. □

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