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GROUPOID ALGEBRAS ASSOCIATED WITH COVERING MAPS

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Abstract. For a compact Hausdorff space X with its p-fold covering map σ, we construct its corresponding topological groupoid Γ, and show that there is a strong relation between the dynamical structures of (X, σ) and the groupoid structures of Γ.

1. Preliminary

There is a long history of interrelation between topological dynamics and theory of C^*-algebras ([8]), and one of methods to connect these two fields is constructing groupoid algebras from single dynamical systems ([7]). Following Renault's argument, Deaconu constructed a class of groupoids associated with covering maps of compact Hausdorff spaces and C^*-algebras of these groupoids ([4]). We study some conditions for those C^*-algebras to be simple or prime, relations between covering map and groupoids, and Pimsner-Voiculescu six term exact sequence for K-groups of groupoid algebras.

For a compact Hausdorff space X and its p-fold covering map $\sigma : X \to X$, set

$$\Gamma = \{(x, n, y) \in X \times \mathbb{Z} \times X : \exists k, l \geq 0, n = l - k, \sigma^k x = \sigma^l y\}. \tag{1}$$

The pair $\{((x, n, y), (w, m, z))\} \in \Gamma^2$ is composable if $y = w$, and multiplication and inverse are defined by

$$(x, n, y)(y, m, z) = (x, n + m, z) \text{ and } (x, n, y)^{-1} = (y, -n, x).$$

For $(x, n, y) \in \Gamma$, $r(x, n, y) = (x, 0, x)$ is the range of (x, n, y) and $d(x, n, y) = (y, 0, y)$ is its domain. Γ^0, the unit space of Γ, is identified with X via the diagonal map, and the isotropy group bundle is given by $I = \{(x, n, x) \in \Gamma\}$.

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For $k \geq 0$, let

$$R_k = \{(x, 0, y) \in \Gamma : \sigma^k x = \sigma^k y\} \text{ and } R_\infty = \bigcup_{k \geq 0} R_k.$$

It is easy to check that R_k and R_∞ are subgroupoids of Γ, and their unit space is Γ^0.

Standing Assumption. Throughout this paper, X denotes a compact Hausdorff space, $\sigma : X \to X$ is a p-fold covering map, Γ is the groupoid defined in the formula (1), $C^*(\Gamma)$ is the groupoid C^*-algebra of Γ, and an ideal means a closed two-sided ideal.

Definition 1.1 ([4, 8])**.** Let (X, σ) be as above. For each $x \in X$, $\mathcal{O}_x = \bigcup_{k \geq 0} \sigma^{-k}(\sigma^k x)$ is called the orbit of x. And σ is called

(i) *minimal* if $\overline{\mathcal{O}}_x = X$ for every $x \in X$,
(ii) *irreducible* if for any nonempty open subsets U, V of X, $\sigma^n U \cap V \neq \emptyset$ for some $n \in \mathbb{N}_0$,and
(iii) *essentially free* if $\{x \in X : \sigma^k x = \sigma^l x$ for some $k, l \geq 0$ implies $k = l\}$ is dense in X.

Definition 1.2 ([7])**.** Let G be a topological groupoid with open range map and G^0 its unit space. We say that G is

(i) *minimal* if the only open invariant subsets of G^0 are the empty set $\emptyset$ and G^0 itself,
(ii) *irreducible* if every invariant nonempty open subset of G^0 is dense, and
(iii) *essentially principal* if G is locally compact and, for every closed invariant subset F of G^0, $\{u \in F : r^{-1}(u) \cap d^{-1}(u) = \{u\}\}$ is dense in F.

The following two theorems are basic properties of our groupoid algebras.

Theorem 1.3 ([4])**.** *Suppose that X, σ and Γ are as in the standing assumption. Then Γ carries a topology that makes Γ an r-discrete locally compact Hausdorff groupoid and R_∞ a principal r-discrete locally compact Hausdorff groupoid.*

For any open invariant subset U of Γ^0, let

$$I_c(U) = \left\{f \in C_c(\Gamma) : f(x, n, y) = 0 \text{ if } (x, n, y) \notin d^{-1}(U)\right\}$$

and $I(U)$ the closure of $I_c(U)$ in $C^*(\Gamma)$. Then $I(U)$ is an ideal of $C^*(\Gamma)$ [7, II.4.5].

Theorem 1.4 ([7, II. 4.5 and 4.6])**.** *Suppose that $\mathcal{O}(\Gamma)$ is the lattice of invariant* *open subsets of* Γ^0 *and that* $\mathcal{I}(C^*(\Gamma))$ *is the lattice of ideals of* $C^*(\Gamma)$. *The correspondence* $U \to I(U)$ *is a one-to-one order preserving relation from* $\mathcal{O}(\Gamma)$ *to* $\mathcal{I}(C^*(\Gamma))$. *Moreover, if* Γ *is essentially principal, then the correspondence is bijective.*

2. Main Results

We show that ynamical structures of (X, σ) are strongly related to the groupoid structures of Γ and its groupoid C^*-algebra $C^*(\Gamma)$.

Proposition 2.1. *Suppose that* X, σ *and* Γ *are as in the standing assumption. Then the followings are equivalent.*

(1) σ *is essentially free.*
(2) Γ *is essentially principal.*
(3) $Per_n(X) = \{x \in X : \sigma^k x = \sigma^{k+n} x$ *for every* $k \in \mathbb{N} \cup \{0\}\}$ *has empty interior for every positive integer* n.
(4) $C(X) \cong C^*(\Gamma^0)$ *is a maximal abelian subalgebra of* $C^*(\Gamma)$.
(5) Γ^0 *is the interior of* I *where* I *is the isotropy group bundle of* Γ.

Proof. (1) $\Longrightarrow$ (3) is trivial.

(3) $\Longrightarrow$ (1). Let $A = \{x \in X : \text{for any } k, l \geq 0, \sigma^k x = \sigma^l x \text{ implies } k = l\}$. If σ is not essentially free, then A is not dense in X, and we can find an open set $U \subset X$ such that $\overline{U} \cap \overline{A} = \emptyset$, for X is compact Hausdorff space. Since $X - A = \bigcup_{n=1}^{\infty} \bigcup_{k=0}^{\infty} \sigma^{-k}(Per_n(X))$, we have $\overline{U} = \bigcup_{n=1}^{\infty} \bigcup_{k=0}^{\infty} \overline{U} \cap \sigma^{-k}(Per_n(X))$, and by Baire Category theorem there exist integers $n \geq 1$ and $k \geq 0$ such that $\overline{U} \cap \sigma^{-k} Per_n(X)$ has nonempty interior.

We remind that $\sigma^{-k}(Per_n(X)) = \bigcup_{i=1}^{p^k} P_i$ where p is the index of the covering map $\sigma \colon X \to X$ and $P_i \simeq Per_n(X)$ with $P_i \cap P_j = \emptyset$ if $i \neq j$. Then $\overline{U} \cap \sigma^{-k}(Per_n(X))$ has nonempty interior implies $\operatorname{Int}\{\overline{U} \cap P_i\} \neq \emptyset$ for at least one i. Hence $\sigma^k(\overline{U} \cap P_i) \subset Per_n(X)$, and σ is open map implies $\operatorname{Int}\{Per_n(X)\} \neq \emptyset$.

(1) $\Longleftrightarrow$ (5). Let $B = X - A$ and define $B_n = \{x \in X : (x, n, x) \in I\}$ and $I_n = \{(x, n, x) \in I\}$ for every nonzero integer n. Then it is trivial that $B = \bigcup B_n$ and $I = \bigcup I_n \cup \Gamma^0$ with $I_n = \operatorname{Diag}\{B_n \times \{n\} \times B_n\}$. So $\operatorname{Int} I - \Gamma^0 = \bigcup \operatorname{Int} I_n \simeq B$, and A is dense in X if and only if $\operatorname{Int} I - \Gamma^0 = \emptyset$.

(5) $\Longleftrightarrow$ (4) is trivial by [7, II.4.7].

(1) $\Longleftrightarrow$ (2). Let $U = \{u = (x, 0, x) \in \Gamma^0 : \{u\} = r^{-1}(u) \cap d^{-1}(u) \subset \Gamma\}$. Then $x \in A$ if and only if $(x, 0, x) \in U$. So A is dense in X if and only if U is dense in Γ^0. □

We can obtain the following corollary from Theorem 1.4 and Proposition 2.1.

Corollary 2.2. *If σ is essentially free, then for any nonzero ideal J of $C^*(\Gamma)$, $J \cap C^*(\Gamma^0)$ and $J \cap C^*(R_\infty)$ are not $\{0\}$.*

Proposition 2.3. *Suppose that X, σ and Γ are as in the standing assumption. Then the followings are equivalent.*

(1) *σ is minimal.*
(2) *$C^*(R_\infty)$ is simple.*
(3) *R_∞ is a minimal groupoid.*

Proof. (1) $\Longrightarrow$ (2) is done in [4].

(2) $\Longrightarrow$ (3). Since R_∞ is a principal r-discrete groupoid, this is an easy consequence of Theorem 1.4.

(3) $\Longrightarrow$ (1). Assume that σ is not minimal. Then there exists an $x_0 \in X$ such that $\overline{\mathcal{O}}_{x_0} \subsetneq X$. Let $Y = X - \overline{\mathcal{O}}_{x_0}$ and $Z = \{(y, 0, y) : y \in Y\}$. Now we show that Z is an invariant open subset of Γ^0.

It is trivial that Z is an open subset of Γ and $Z \subset r(d^{-1}(Z))$ where r and d are range and domain maps of Γ. For an $(a, 0, a) \in r(d^{-1}(Z))$, there exists a $y \in Y$ such that $(a, 0, y) \in d^{-1}(Z) \subset R_\infty$ with $\sigma^k a = \sigma^k y$ for some $k \geq 0$. Since $r(d^{-1}(Z))$ is open in Γ^0 ([7]), $r(d^{-1}(Z)) \cap (\Gamma^0 - Z) \neq \emptyset$ means that $r(d^{-1}(Z)) \cap \mathrm{Int}(\Gamma^0 - Z) \neq \emptyset$. So if there exists $(a, 0, a) \in r(d^{-1}(Z)) \cap \mathrm{Int}(\Gamma^0 - Z)$, then we have $a \in \mathcal{O}_{x_0}$ and $y \in \mathcal{O}_{x_0}$ as $\sigma^k a = \sigma^k y$ for some $k \geq 0$. This is a contradiction to the facts that $y \in Y$ and $Y = X - \overline{\mathcal{O}}_{x_0}$. Hence we obtain that $r(d^{-1}(Z)) \subset Z$ and that Z is a nontrivial invariant open subset of Γ_0. Therefore R_∞ is not a minimal groupoid. □

By Theorem 1.4, there exists an injective relation between the set of ideals in $C^*(R_\infty)$ and that of $C^*(\Gamma)$ by $I_\infty(U) = I(U) \cap C^*(R_\infty)$ where U is an open invariant subset of Γ^0. If σ is essentially free, then the relation is bijective. So the following corollary is trivial by Proposition 2.3.

Corollary 2.4. *If σ is essentially free and minimal, then $C^*(\Gamma)$ is simple. Conversely, if $C^*(\Gamma)$ is simple, then σ is minimal.*

Remark 2.5 ([7, I. 4.1])**.** Γ is irreducible if $\mathrm{Im}\,\Gamma$ is dense in $\Gamma^0 \times \Gamma^0$ by the map $(r, d) : \Gamma \to \Gamma^0 \times \Gamma^0$ where r is the range map, and d is the domain map.

Lemma 2.6. *Let G be a topological groupoid with open range map and G^0 its unit space. If U is an invariant subset of Γ^0, then $V = \Gamma^0 - U$ and $W = IntU$ are also invariant subsets of Γ^0.*

Proof. It is trivial that $W \subset r(d^{-1}(W))$. Since r is an open map, $r \circ d^{-1}W$ is an open subset of $U = r(d^{-1}(U))$. So we have $r(d^{-1}(W)) \subset \operatorname{Int} U$, and $W = r(d^{-1}(W))$.

To show that V is an invariant subset of Γ^0, we only need to show $r(d^{-1}(V)) \subset V$. If $r(d^{-1}(V)) \cap U \neq \emptyset$, then there exists $(x, 0, x) \in r(d^{-1}(V)) \cap U$. So we can choose $(v, 0, v) \in V$ such that $(x, n, v) \in d^{-1}(V) \subset \Gamma$ and $(v, -n, x) \in \Gamma$. Since we assumed $(x, 0, x) \in U$ where U is an invariant subset of Γ^0, we have $(v, -n, x) \in d^{-1}(U)$ and $(v, 0, v) \in r(d^{-1}(U)) = U$, a contradiction. Hence $r(d^{-1}(V))$ is a subset of V, and V is an invariant subset of Γ^0. □

Remark 2.7. It is an easy consequence of Lemma 2.6 that, for any subset U of Γ^0 and $V = \Gamma^0 - U$, one of $\operatorname{Int} U$, $\overline{U}$, $\operatorname{Int} V$ and $\overline{V}$ is an invariant subset of Γ^0 implies that the remaining three subsets are also invariant subsets of Γ^0.

Proposition 2.8. *Suppose that X, σ and Γ are as in the standing assumption. Then the followings are equivalent.*

(1) *σ is irreducible.*
(2) *Γ is irreducible.*
(3) *$C^*(R_\infty)$ is a prime algebra.*

Proof. (1) $\Longrightarrow$ (2). Let U be a nonempty open invariant subset of $\Gamma^0 \simeq X$, and show that U is dense in X. We remind that $\sigma^n U \subseteq U$. If $V = X - \overline{U} \neq \emptyset$, then there exists $n \in \mathbb{N}$ such that $\sigma^n U \cap V \neq \emptyset$ as σ is irreducible. So $U \cap (X - \overline{U}) \neq \emptyset$, a contradiction. Therefore we have $V = \emptyset$, and U is dense in X.

(2) $\Longrightarrow$ (1). If σ is not irreducible, then there exist nonempty open subsets $U, V \subset X$ such that $\sigma^n U \cap V = \emptyset$ for every $n \in \mathbb{N} \cup \{0\}$. Note that, for any $x \in U$ and $y \in V$, $(x, 0, x) \times (y, 0, y) \notin \operatorname{Im}(r, d)$ because $(x, 0, x) \times (y, 0, y) \in \operatorname{Im}(r, d)$ implies that there exists an $n \in \mathbb{N} \cup \{0\}$ such that $(x, n, y) \in \Gamma$, and we have $\sigma^k x = \sigma^{k+n} y \in \sigma^k U \cap \sigma^{k+n} V$. Hence $U \times V$ is a nonempty subset of $\Gamma^0 \times \Gamma^0 - \operatorname{Im}(r, d)$, and $\operatorname{Im}(r, d)$ is not dense in $\Gamma^0 \times \Gamma^0$.

(1) $\Longrightarrow$ (3). As R_∞ is a principal groupoid, by Theorem 1.3 and Theorem 1.4, every ideal is of the form $I(U)$ where U is an open invariant subset of Γ^0. For any nonzero ideals $I(U)$ and $I(V)$ of $C^*(R_\infty)$, we have $I(U) \cap I(V) = I(U \cap V)$. Since σ is irreducible, we showed in the above that U and V are dense in Γ^0. So $U \cap V$ is nonempty dense subset of Γ^0. Hence $I(U) \cap I(V) \neq \{0\}$, and $C^*(R_\infty)$ is a prime algebra.

(3) $\Longrightarrow$ (1). If σ is not irreducible, then Γ^0 has two nonempty disjoint open invariant subsets U, V by Lemma 2.6. For for these open invariant sets, their corre-

sponding ideals satisfy $I(U) \cap I(V) = I(U \cap V) = \{0\}$. Therefore $C^*(R_\infty)$ is not a prime algebra. □

We have the following corollary from Theorem 1.4 and Proposition 2.8.

Corollary 2.9. *If σ is essentially free and irreducible, then $C^*(\Gamma)$ is prime. Conversely, $C^*(\Gamma)$ is prime implies that σ is irreducible.*

3. K-theory of $C^*(\Gamma)$

We show that it is possible to compute K-theory of groupoid algebras from dynamical properties of (X, σ) using Paschke's argument ([6]).

Definition 3.1. Let X, σ and Γ be as in the standing assumption. Define $\alpha : S^1 \to Aut(C^*(\Gamma))$ by $\lambda \mapsto \alpha_\lambda$ such that

$$\alpha_\lambda(f)(x, n, y) = \lambda^{-n} f(x, n, y) \text{ for } f \in C_c(\Gamma).$$

Remark 3.2. $(C^*(\Gamma), S^1, \alpha)$ is a C^*-dynamical system by [7, II.5.1]. The fixed point algebra of α is $C^*(R_\infty)$, which is an inductive limit of $C^*(R_n)$, $n \geq 0$ ([4]).

Definition 3.3 ([6])**.** Let β be a continuous action of a compact abelian group G on a C^*-algebra $\mathcal{A}$, and $\mathcal{B}$ its fixed point algebra. For a character χ in the dual group $\hat{G}$, set

$$E_\chi = \{a \in \mathcal{A} : \beta_s(a) = \chi(s)a \text{ for every } s \in G\}.$$

We say that β has *large spectral subspaces* if $\overline{E_\chi^* E_\chi} = \mathcal{B}$ for each $\chi \in \hat{G}$.

Proposition 3.4. *The action α on $C^*(\Gamma)$ has large spectral subspaces.*

Proof. It suffices to show that $\overline{E_1^* E_1} = C^*(R_\infty) = \overline{E_1 E_1^*}$ (see [6] for details). As E_1 is generated by continuous functions supported on $\{(x, -1, y) \in \Gamma\}$, it is not difficult to obtain $\overline{E_1^* E_1} \subseteq C^*(R_\infty)$ and $\overline{E_1 E_1^*} \subseteq C^*(R_\infty)$.

For any $f \in C_c(R_n)$, we define $\tilde{g}, \tilde{h} \in C_c(\Gamma^0)$ by

$$\tilde{g}(x, 0, x) = f(x, 0, y)|f(x, 0, y)|^{-\frac{1}{2}} \text{ and } \tilde{h}(y, 0, y) = |f(x, 0, y)|^{\frac{1}{2}}$$

so that we have $f(x, 0, y) = \tilde{g}(x, 0, x)\tilde{h}(y, 0, y)$ ([5]). Denote $g = p^{\frac{n}{2}}\tilde{g}$ and $h = p^{\frac{n}{2}}\tilde{h}$ where p is the index of the covering map σ. Then, for $v \in E_1$ defined by

$$v(x, n, y) = \begin{cases} 1/\sqrt{p} & \text{if } n = -1 \text{ and } y = \sigma x, \\ 0 & \text{otherwise}, \end{cases}$$

we have

$$\begin{aligned} f = \tilde{g}\tilde{h} &= gv^n v^{*n} h \\ &= (gv^n v^{*n-1})(v^* h) && \in E_1 E_1^* \\ &= (gv^n v^{*n+1})(vh) && \in E_1^* E_1. \end{aligned}$$

Hence $C^*(R_n) \subseteq \overline{E_1 E_1^*}$ and $C^*(R_n) \subseteq \overline{E_1^* E_1}$, for $C^*(R_n)$ is the norm closure of $C_c(R_n)$. Since $C^*(R_\infty)$ is an inductive limit of $C^*(R_n)$ by Remark 3.2, we have $C^*(R_\infty) \subseteq \overline{E_1^* E_1}$ and $C^*(R_\infty) \subseteq \overline{E_1 E_1^*}$, and this completes the proof. □

Corollary 3.5. (1) *$C^*(R_\infty)$ is a full-corner of $C^*(\Gamma) \times_\alpha S^1$.*

(2) *$C^*(R_\infty) \otimes \mathcal{K} \cong (C^*(\Gamma) \times_\alpha S^1) \otimes \mathcal{K}$, and*

(3) *$C^*(\Gamma) \otimes \mathcal{K} \cong (C^*(R_\infty) \otimes \mathcal{K}) \times_{\hat{\alpha}} \mathbb{Z}$.*

Proof. (1) is trivial by Proposition 3.4 and [6, 2.1]. (2) comes from statement (1) and [1, 2.6], and (3) is a consequence of [6, 2.3]. □

Proposition 3.6. *We have the following six-term exact sequence.*

$$\begin{array}{ccccc} K_0(C^*(R_\infty)) & \xrightarrow{\sigma^* - id} & K_0(C^*(R_\infty)) & \xrightarrow{i^*} & K_0(C^*(\Gamma)) \\ \uparrow & & & & \downarrow \\ K_1(C^*(\Gamma)) & \xleftarrow{i^*} & K_1(C^*(R_\infty)) & \xleftarrow{\sigma^* - id} & K_1(C^*(R_\infty)). \end{array}$$

Proof. By Proposition 3.4, the action α has large spectral subspaces, and the fixed point algebra $C^*(R_\infty)$ is a unital algebra. So we can obtain the six term exact sequence by applying Theorem 3.1 of [6]. □

Example 3.7 ([2, 3, 4])**.** Let X be the infinite product space $\prod_{i\geq 0} X_i$ where $X_i = \{1, 2, \ldots, p\}$ for each i, and $\sigma : X \to X$ be the unilateral shift given by $(\sigma(x))_i = x_{i+1}, i \geq 0$. Let $A = A(i,j)$ be a $p \times p$ matrix with $\{0,1\}$-entries, define

$$X_A = \{x = (x_i) : A(x_i, x_{i+1}) = 1\},$$

and denote $\sigma|_{X_A}$ by σ. As in Example 2 of [4], we assume

$$\sum_i A(i,j) = q \ \forall j \text{ for some } q \geq 2.$$

Then σ is a q-fold covering, and we have $C^*(\Gamma^0) = \mathcal{D}_A$, $C^*(R_\infty) = \mathcal{F}_A$, and $C^*(\Gamma) = \mathcal{O}_A$, the Cuntz-Krieger algebra of A ([3, 4]).

If A satisfies the Cuntz-Krieger condition (I), then $\mathcal{D}_A$ is a maximal abelian subalgebra of $\mathcal{O}_A$ by Remark 2.18 of [3]. So σ is essentially free by Proposition 2.1.

If A is irreducible, then σ is minimal. Therefore we obtain by Corollary 2.4 that if A satisfies (I) and A is irreducible, then $\mathcal{O}_A$ is simple [3, 2.14].

Since $\mathcal{F}_A$ is an AF-algebra, we can apply the exact sequence in Proposition 3.6 to compute the K-theory of $\mathcal{O}_A$, (see [2]), that

$$K_0(\mathcal{O}_A) \cong \mathbb{Z}^p/(1-A^t)\mathbb{Z}^p,$$
$$K_1(\mathcal{O}_A) \cong \ker(1-A^t : \mathbb{Z}^p \to \mathbb{Z}^p).$$

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