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THE DIVISOR CLASS GROUP OF SURFACES OVER FINITE FIELDS

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Abstract. We investigate the divisor class group of surfaces over finite fields. For some surfaces the divisor class group depends on the characteristic of the field. We calculate the determinant of a matrix which will provide an information about the divisor class group of the surfaces.

1. Introduction

Every ring in this paper is assumed to be commutative and noetherian.

Let $(R, \underline{m})$ be a local ring with a maximal ideal $\underline{m}$. An element $f \in R$ is called a *determinant* in R, if it is a determinant of an $n \times n$ matrix $(n \geq 2)$ with the entries in the maximal ideal $\underline{m}$.

It is due to Eisenbud that the unique factorization of a surface of embedding dimension 3 can be computed by determinants.

Theorem 1.1 ([5, p. 124]). *Let $(R, \underline{m})$ be a 3-dimensional regular local ring and $S = R/(f)$ with $f \in R$. Then S is factorial if and only if f is not a determinant in R.*

Suppose that f is a determinant of an $n \times n$ matrix A with entries in $\underline{m}$ and $n \geq 2$. Let B be the $(n-1) \times n$ matrix obtained from A by deleting the first row, then the ideal I of $(n-1) \times (n-1)$ minors of B is unmixed of height 2. Thus $I/(f)$ is an unmixed ideal of height 1 in S that is not principal. So S is not factorial.

Therefore, for any noetherian local ring $(R, \underline{m})$ and $f \in R$ if $S = R/(f)$ is factorial, then f is not a determinant in R . The converse holds only for a regular local ring with $\dim R \leq 3$. If $\dim R \geq 4$, then the converse is not true.

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Theorem 1.2 ([4, Theorem 2.4]). *Let k be a real closed field and R be the 4-dimensional regular local and $R = k[x,y,z,w]_{(x,y,z,w)}$. Then $f = x^2 + y^2 + z^2 + w^2$ is not a determinant in R.*

The divisor class group of $k[x,y,z,w]_{(x,y,z,w)}/(x^2+y^2+z^2+w^2)$ is infinite cyclic [6].

It is due to Gauss that if R is factorial, then so is $R[x]$. But this is not true for the formal power series ring $R[[x]]$. Gauss's method cannot be applied to $R[[x]]$ since the units of $R[x]$ and those of $R[[x]]$ are different. Also the content of a formal power series is not defined. Counterexamples have been found by Salmon ([10]), Zinn-Justine and Danilov.

Example 1.3 ([14, 15]). Let $R = F(u)[[x,y,z]]/(x^2 + y^i + uz^{2j})$, where F is a field and x, y, z and u are variables. Then R is factorial for all odd $(i,j) \neq (3,3)$, but $R[[t]]$ is not factorial.

Surfaces of embedding dimension 3 were focused on due to following results by Samuel and Scheja.

Theorem 1.4 ([11]). *Let S be a locally Cohen-Macaulay factorial domain. If $S_p[[x]]$ is factorial for any height 2 prime ideal p in S, $S[[x]]$ is factorial.*

Theorem 1.5 ([13]). *Let $(R, \underline{m})$ be a complete local factorial domain with depth $R \geq 3$, then $R[[x]]$ is factorial.*

If $(S, \underline{n})$ is a 2-dimensional local domain of embedding dimension 3, then the divisor class group $Cl(S)$ of S is generated by the classes of the height 1 prime ideals that is not contained in $\underline{n}^{e(R)}$ where $e(R)$ is the multiplicity of R [3]. Note that for the surfaces $S = R/(f)$ of a 3-dimensional regular local ring $(R, \underline{m})$, $e(R) = o(f)$ where $o(f) = e$ if $f \in m^e$, $f \notin m^{e+1}$. Hence $Cl(S)$ is generated by the classes of the height 1 prime ideals that is not contained in $\underline{n}^{o(f)}$. We calculate the divisor class group of some of the surfaces of embedding dimension 3.

This paper focuses on the surfaces of embedding dimension 3 over a finite field. Let F be a finite field and $R = F[[x,y,z]]$. We study surfaces satisfying one of the following equations : $x^2 + y^3 + az^4 = 0$, $x^2 + y^3 + az^3 = 0$, $x^2 + y^2 + az^n = 0$ ($n \geq 2$, $a \in F$).

2. THE DIVISOR CLASS GROUP OVER FINITE FIELDS

For a 3-dimensional regular local ring $(R, \underline{m})$ with $\underline{m} = (x,y,z)$, $R/(x^2+y^3+z^5)$

is factorial (cf. [9], [13], [3]). Also this is essentially the only nonregular factorial ring when R/m is algebraically closed.

Theorem 2.1 ([7]). *Let $(S, \underline{n})$ be a 2-dimensional nonregular local ring such that $S/\underline{n}$ is algebraically closed field of characteristic $\neq 2, 3, 5$. Then the completion $\hat{S}$ of S is factorial if and only if $S \cong R/(x^2 + y^3 + z^5)$ for some 3-dimensional regular local ring R with a regular system of parameters x, y, z of R.*

Let $(S, \underline{n})$ be a 2-dimensional non-regular local ring with $\underline{n} = (x, y, z)$ and $S/\underline{n}$ is real closed. Then Lipman further showed that S has a rational singularity and is factorial if and only if one of the following equations holds in S: $x^2 + y^3 + z^5 = 0$, $x^2 + y^3 + z^4 = 0$, $x^2 + y^2 + z^n = 0$ $(n \geq 2)$.

Theorem 2.2 ([3, Theorem 2.2]). *Let F be a field and $a \in F$. Then $f = x^2 + y^3 + az^4$ is not a determinant in $R = F[[x, y, z]]$ if and only if $\sqrt{-a} \notin F$. Equivalently, $S = F[[x, y, z]]/(x^2 + y^3 + az^4)$ is factorial.*

If $\alpha = \sqrt{-a} \in F$, then $x^2 + y^3 + az^4$ is a determinant in R. Note that

$$x^2 + y^3 + az^4 = \begin{pmatrix} x + \alpha z^2 & -y \\ y^2 & x - \alpha z^2 \end{pmatrix}$$

In this case the divisor class group $Cl(S)$ of S is generated by the prime ideal $(x + \alpha z^2, y)S$ ([3, Theorem 3.2]).

Now consider the surface S over a finite field F. Then the divisor class group $Cl(S)$ of S depends on the characteristic of F.

Theorem 2.3. *Let F be a finite field of characteristic p and $S = F[[x, y, z]]/(x^2 + y^3 + az^4)$ for $a \in F$.*

(1) *If $p = 2$, then S is not factorial for any a.*
(2) *If p is an odd prime, then S is factorial only for $|F^*|/2$ elements $a \in F^*$.*

Proof. Consider the group-homomorphism, $\epsilon_2 : F^* \longrightarrow F^*$, $\epsilon_2(a) = a^2$.

(1) If $p = 2$, then $|F^*|$ is odd. So ϵ_2 is an automorphism. So for any $a \in F^*$, $\alpha = \sqrt{-a} \in F$. Thus $x^2 + y^3 + az^4$ is a determinant in $F[[x, y, z]]$ and S is not factorial by Theorem 2.2.
(2) If p is odd, then $|F^*|$ is even. So ϵ_2 is not surjective. Also only for a half of elements in F^*, $a \notin im(\epsilon_2)$. For those a, $\sqrt{-a} \notin F$ and $S = F[[x, y, z]]/(x^2 + y^3 + az^4)$ is factorial by Theorem 2.2.

$\square$

Now consider the surface $S = \overline{\mathbb{Z}}_p[[x,y,z]]/(x^2+y^3+z^4)$ over the prime field $\overline{\mathbb{Z}}_p$. Note that the equation $T^2+1=0$ has no roots in $\overline{\mathbb{Z}}_p$ if and only if p is a prime integer of the form $4n+3$.

If $p=2$, then $\alpha=1$ is a solution.

If p is a prime integer of the form $4n+1$, then $\alpha = (\frac{p-1}{2})!$ is a solution.

On the other hand, if p is a prime integer of the form $4n+3$, then p is irreducible in the ring of Gaussian integers $\mathbb{Z}[i]$. So the equation $T^2+1=0$ has no roots in $\overline{\mathbb{Z}}_p$.

Corollary 2.4. *Let p be a prime integer and $S = \overline{\mathbb{Z}}_p[[x,y,z]]/(x^2+y^3+z^4)$. Then S is factorial if and only if p is a prime integer of the form $4n+3$.*

Theorem 2.5 ([3, Theorem 2.2])**.** *Let F be a field and $a \in F$. Then $f = x^2+y^3+az^3$ is not a determinant in $R = F[[x,y,z]]$ if and only if $\sqrt[3]{a} \notin F$. Equivalently, $S = F[[x,y,z]]/(x^2+y^3+az^3)$ is factorial.*

If $\beta = \sqrt[3]{a} \in F$, then $x^2+y^3+az^3$ is a determinant in R. Note that

$$x^2+y^3+az^3 = \begin{pmatrix} x & -(y+\beta z) \\ y^2-\beta yz+\beta^2z^2 & x \end{pmatrix}$$

Also if $\beta = \sqrt[3]{a} \in F$, then the divisor class group $Cl(S)$ of S is generated by the prime ideal $(x, y+\beta z)S$ ([3, Theorem 3.2]).

Theorem 2.6. *Let F be a finite field of characteristic p and $S = F[[x,y,z]]/(x^2+y^3+az^3)$ for $a \in F$.*

(1) *If $p=3$, then S is not factorial for any a.*
(2) *If p is a prime integer of the form $3n+1$, then S is factorial only for $\frac{2}{3}|F^*|$ elements $a \in F^*$.*
(3) *For a prime integer p of the form $3n+2$, put $m = [F : \overline{\mathbb{Z}}_p]$. If m is odd, then S is not factorial for any a. If m is even, then S is factorial only for $\frac{2}{3}|F^*|$ elements $a \in F^*$.*

Proof. Consider the group-homomorphism, $\epsilon_3 : F^* \longrightarrow F^*$, $\epsilon_3(a) = a^3$.

(1) If $p=3$, then $\gcd(|F^*|, 3) = 1$. So ϵ_3 is an automorphism. So for any $a \in F^*$, $\beta = \sqrt[3]{a} \in F$. Thus $x^2+y^3+az^3$ is a determinant in $F[[x,y,z]]$ and S is not factorial by Theorem 2.5.
(2) If $p \equiv 1 \pmod 3$, then $3 || F^*|$. So ϵ_3 is not surjective. Also only for $\frac{2}{3}F$ of elements a in F^*, $a \notin im(\epsilon_3)$. For those a, $\sqrt[3]{a} \notin F$ and $S = F[[x,y,z]]/(x^2+y^3+az^3)$ is factorial by Theorem 2.2.

(3) If $p \equiv 2(\bmod 3)$ and m is odd, ϵ_3 is surjective. Also if $p \equiv 2(\bmod 3)$ and m is even, ϵ_3 is not surjective. Hence the conclusion follows.

$\square$

For the surface $S = \overline{\mathbb{Z}}_p[[x, y, z]]/(x^2 + y^3 + az^3)$ over the prime field $\overline{\mathbb{Z}}_p$, Theorem 2.6 can be restated as follows.

Corollary 2.7. *Let p be a prime integer and $S = \overline{\mathbb{Z}}_p[[x, y, z]]/(x^2 + y^3 + az^3)$ for $a \in \overline{\mathbb{Z}}_p$. Then S is factorial if and only if $p \equiv 1(\bmod 3)$ and $a \notin im(\epsilon_3)$ where $\epsilon_3 : \overline{\mathbb{Z}}_p \longrightarrow \overline{\mathbb{Z}}_p$, $\epsilon_3(a) = a^3$.*

Let F be a field and $m \geq 2$. If $i = \sqrt{-1} \in F$, then $x^2 + y^2 + az^m$ is a determinant in $F[[x, y, z]]$ for any $a \in F$. Note that

$$x^2 + y^2 + az^m = \begin{pmatrix} x + iy & -z \\ az^{m-1} & x - iy \end{pmatrix}.$$

If m is even, $\sqrt{-1} \notin F$ and $\sqrt{-a} \notin F$, then $x^2 + y^2 + az^m$ is not a determinant in $F[[x, y, z]]$ ([3, Theorem 2.4]). So $S = F[[x, y, z]]/(x^2 + y^2 + az^m)$ is factorial.

If m is odd, $\sqrt{-1} \notin F$, $\sqrt{-a} \notin F$ and F is real, then $S = F[[x, y, z]]/(x^2+y^2+az^m)$ is factorial ([3, Theorem 2.4]).

If a field F is of characteristic $p = 2$ or $p \equiv 1(\bmod 4)$, then $\sqrt{-1} \in F$. So we can formulate the above result as follows.

Theorem 2.8. *Let F be a field of characteristic $p = 2$ or $p \equiv 1(\bmod 4)$. Then $S = F[[x, y, z]]/(x^2 + y^2 + az^m)$ is not factorial for any $a \in F$ and $m \geq 2$.*

If a prime integer p is of the form $4n + 3$, then $\sqrt{-1} \notin \overline{\mathbb{Z}}_p$ and $|\overline{\mathbb{Z}}_p^*|$ is even. So we obtain the following theorem.

Theorem 2.9. *Let p be a prime integer of the form $4n + 3$, and m even. Then $S = \overline{\mathbb{Z}}_p[[x, y, z]]/(x^2 + y^2 + az^m)$ is factorial only for $|\overline{\mathbb{Z}}_p^*|/2$ elements $a \in \overline{\mathbb{Z}}_p^*$.*

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