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DOMINATED SPLITTING WITH STABLY EXPANSIVE

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Abstract. In this paper, we show that if a transitive set Λ is C^1-stably expansive, then Λ admits a dominated splitting.

1. Introduction

In this paper, we study dominated splitting - a weak form of hyperbolicity. More precisely, using results of [2] and [3], we show that if a closed set have the some property then it admits dominated splitting.

Let M be a closed C^∞ manifold, and let $\mathrm{Diff}(M)$ be the space of diffeomorphisms of M endowed with the C^1-topology. Denote by d the distance on M induced from a Riemannian metric $\|\cdot\|$ on the tangent bundle TM. Let $f \in \mathrm{Diff}(M)$, and let $\Lambda \subset M$ be a closed f-invariant set. We say that $f|_\Lambda$ is *expansive* if there is a constant $e > 0$ such that for any pair of distinct points $x, y \in \Lambda$, $d(f^n(x), f^n(y)) > e$ for some $n \in \mathbb{Z}$. Let $f \in \mathrm{Diff}(M)$, and let $\Lambda \subset M$ ba a closed f-invariant set. We say that Λ is *locally maximal* if there is a compact neighborhood U of Λ such that

$$\bigcap_{n\in\mathbb{Z}} f^n(U) = \Lambda(U).$$

We say that Λ admits a *dominated splitting* if the tangent bundle $T_\Lambda M$ has a continuous Df-invariant splitting $E \oplus F$ and there exists constant $C > 0$ and $0 < \lambda < 1$ such that

$$\|D_x f^n|_{E(x)}\| \cdot \|D_x f^{-n}|_{F(f^n(x))}\| \le C\lambda^n$$

for all $x \in \Lambda$ and $n \ge 0$.

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The following definition is in [5].

Definition 1.1. We say that an f-invariant set Λ is C^1-*stably expansive* if there exists a C^1-neighborhood $\mathcal{U}(f)$ of f and a compact neighborhood U of Λ such that:

- $\Lambda(U) = \bigcap_{n\in\mathbb{Z}} f^n(U)$,
- for any $g \in \mathcal{U}(f)$, $g|_{\Lambda_g(U)}$ is expansive, where $\Lambda_g(U) = \bigcap_{n\in\mathbb{Z}} g^n(U)$ is called the *continuation* of Λ.

Let $f \in \mathrm{Diff}(M)$, and let $\Lambda \subset M$ be a closed f-invariant set. Then we say that Λ is called a *transitive set* if there exists a point $x \in \Lambda$ such that $\omega(x) = \Lambda$. Mañè [6] studied the case in which for $f \in \mathrm{Diff}(M)$ there is a C^1-neighborhood $\mathcal{U}(f)$ of f such that for any $g \in \mathcal{U}(f)$, g is expansive. He proved in the case f is quasi-Anosov, that is, for all $v \in TM, v \neq 0$, the set $\{Df^n(v)\| : n \in \mathbb{Z}\}$ is bounded. Thus we can restate the above facts are follows.

Theorem A. *M is C^1-stably expansive if and only if f satisfies quasi-Anosov.*

In this paper, we get a problem which if a transitive set Λ is C^1-stably expansive then is Λ is hyperbolic? Unfortunately, it is not true. Indeed, for any hyperbolic periodic points $p, q \in \Lambda$, we don't know that $W^s(p) \pitchfork W^u(q) \neq \phi$ and $W^u(p) \pitchfork W^s(q) \neq \phi$. Therefore, our aim is to characterize closed sets by making use of the C^1-stably expansive property. We are now in position to state main theorem.

Theorem B. *Let Λ be a transitive set. If Λ is C^1-stably expansive, then Λ admits a dominated splitting.*

2. INTRODUCTION SOME RESULTS.

We use Mãné's result which is on a uniformly family of periodic sequences of linear maps of $\mathbb{R}^n (n = \dim M)$. Let $GL(n)$ be the group of linear isomorphisms of $\mathbb{R}^n$. If a sequence $\xi : \mathbb{Z} \to GL(n)$ is *periodic* if there is $k > 0$ such that $\xi_{j+k} = \xi_j$ for $k \in \mathbb{Z}$. We call a finite subset $\mathcal{A} = \{\xi_i : 0 \le i \le k-1\} \subset GL(n)$ is a *periodic family* with period k. For a periodic family $\mathcal{A} = \{\xi_i : 0 \le i \le n-1\}$, we denote

$$\mathcal{C}_{\mathcal{A}} = \xi_{n-1} \circ \xi_{n-2} \circ \cdots \circ \xi_0.$$

Definition 2.1. We say that the periodic family $\mathcal{A} = \{\xi_i : 0 \le i \le n-1\}$ admits a l-dominated splitting, if there is a splitting $\mathbb{R}^n = E \oplus F$ which satisfies:

(a) E and F are $\mathcal{C}_{\mathcal{A}}$ invariant, i.e., $\mathcal{C}_{\mathcal{A}}(E) = E$ and $\mathcal{C}_{\mathcal{A}}(F) = F$,
(b) For any $k = 0, 1, 2, \ldots,$

$$\frac{\|\xi_{k+l-1} \circ \cdots \circ \xi_{k+1} \circ \xi_k|_{E_k}\|}{m(\xi_{k+l-1} \circ \cdots \circ \xi_{k+1} \circ \xi_k|_{F_k})} \leq \frac{1}{2},$$

where

$$E_k = \xi_{k-1} \circ \xi_{k-2} \circ \cdots \circ \xi_0(E)$$

and

$$F_k = \xi_{k-1} \circ \xi_{k-2} \circ \cdots \circ \xi_0(F).$$

We know that the following theorems for periodic family from [3].

Theorem 2.2. *Given any $\epsilon > 0$ and $K > 0$, there is $n_1 \geq 0$ which satisfies the following property: given any periodic family $\mathcal{A} = \{\xi_i : 0 \leq i \leq n-1\}$ which satisfies the period $n \geq n_1$ and $\max\{\|\xi_i\|, \|\xi_i^{-1}\|\} \leq K$, for all $i = 0, 1, \cdots, n-1$, one can find a periodic family $\mathcal{B} = \{\zeta_i : 0 \leq n-1\}$ such that $\max\{\|\zeta_i - \xi_i\|, \|\zeta_i^{-1} - \xi_i^{-1}\|\} < \epsilon$, for any $i = 0, 1, \cdots, n-1$, and $\det(\mathcal{C}_{\mathcal{A}}) = \det(\mathcal{C}_{\mathcal{B}})$ and the eigenvalues of $\mathcal{C}_{\mathcal{B}}$ are all real, multiplicity one and different moduli.*

Theorem 2.3. *Given any $\epsilon > 0$ and $K > 0$, there is positive integers $n_2 \geq 0$ and $l \geq 0$ which satisfies the following property: given any periodic family $\mathcal{A} = \{\xi_i : 0 \leq i \leq n-1\}$ which satisfies the period $n \geq n_2$ and $\max\{\|\xi_i\|, \|\xi_i^{-1}\|\} \leq K$, for all $i = 0, 1, \cdots, n-1$, if $\mathcal{A}$ does not admits any l-dominated splitting, then one can find a periodic family $\mathcal{B} = \{\zeta_0, \zeta_1, \cdots, \zeta_{n-1}\}$ such that $\max\{\|\zeta_i - \xi_i\|, \|\zeta_i^{-1} - \xi_i^{-1}\|\} < \epsilon$, for any $i = 0, 1, \cdots, n-1$, and $\det(\mathcal{C}_{\mathcal{A}}) = \det(\mathcal{C}_{\mathcal{B}})$ and the eigenvalues of $\mathcal{C}_{\mathcal{B}}$ are all real, and have same modulus.*

To prove Theorem B, we need another lemma about uniformly contracting family. Let $\mathcal{A} = \{\xi_i : 0 \leq i \leq k-1\} \subset GL(n)$ be a periodic family. We say the sequence $\mathcal{A}$ is *uniformly contracting family* if there is a constant $\delta > 0$ such that for any δ-perturbation of $\mathcal{A}$ are sink,.i.e, for any $\mathcal{B} = \{\zeta_i : 0 \leq i \leq k-1\}$ with $\|\zeta_i - \xi_i\| < \delta$, all eigenvalue of $\mathcal{C}_{\mathcal{B}}$ have moduli less than 1. Similarly, we can define the *uniformly expanding periodic family.* The following theorem is well known.

Theorem 2.4 ([7]). *For any $\delta > 0$ and $K > 0$, there are constants $C > 0, 0 < \lambda < 1$ and positive integer m such that if $\mathcal{A} = \{A_0, A_1, \ldots, A_{n-1}\}$ is a uniformly contracting periodic family which satisfies*

$$\max_{i=0,1,\ldots,n-1} \left\{\|A_i\|, \|A_i^{-1}\|\right\} < K$$

for $n > m$, then

$$\prod_{j=0}^{k-1}\left\|\prod_{i=0}^{m-1} A_{i+mj}\right\| \leq C\lambda^k,$$

where $k = [n/m]$.

3. Proof of Theorem B

Let M be as before, and let $f \in \mathrm{Diff}(M)$. In this section, we will use the notation of *pre-sink* (resp. *pre-source*). A periodic point p is called a *pre-sink* (resp. *pre-source*) if $Df^{\pi(p)}(p)$ has an multiplicity one eigenvalue equal to $+1$ or -1 and the other eigenvalues has norm less than 1 (resp. bigger than 1).

Remark 3.1 ([1, Theorem 2.2.23 and 2.2.26])**.** Let $f \in \mathrm{Diff}(M)$.

- Let $\mathcal{I}$ be an small arc. Then $f : \mathcal{I} \to \mathcal{I}$ is not expansive.
- Let $\mathcal{C}$ be a small circle. Then $f : \mathcal{C} \to \mathcal{C}$ is not expansive.

Recall that if Λ is C^1-stably expansive then there are a C^1-neighborhood $\mathcal{U}(f)$ and a compact neighborhood U of Λ such that for any $g \in \mathcal{U}(f)$, $\Lambda_g(U) = \bigcap_{n\in\mathbb{Z}} g^n(U)$ is expansive for g.

Lemma 3.2. *Let Λ be a closed set of $f \in \mathrm{Diff}(M)$, and let $\mathcal{U}(f)$ and U be as above. If Λ is C^1-stably expansive, then for any $g \in \mathcal{U}(f)$, g has neither pre-sink nor pre-source with the orbit staying in U.*

Proof. Suppose that f is C^1-stably expansive on Λ. Then there are a C^1-neighborhood $\mathcal{U}(f)$ of f and a compact neighborhood U of Λ such that for any $g \in \mathcal{U}(f)$, g is expansive on $\Lambda_g(U) = \bigcap_{n\in\mathbb{Z}} g^n(U)$. Assume that there is $g \in \mathcal{U}(f)$ such that g has a pre-sink p with $\mathcal{O}(p) \subset U$. For simplicity, we may assume p is fixed point of g (other case is similar).

By making use of the Franks' Lemma, we linearize g at p with respect to the exponential coordinates $\exp_p$, i.e, choose $\epsilon_1 > 0$ and $\alpha > 0$ with $B_\alpha(p) \subset U$ and there exists g_1 C^1-$\epsilon_1$nearby g such that

$$g_1(x) = \begin{cases} \exp_p \circ D_p g(p) \circ \exp_p^{-1}(x) & \text{if } x \in B_\alpha(p), \\ g(x) & \text{if } x \notin B_{4\alpha}(p). \end{cases}$$

Then $g_1(p) = g(p) = p$.

Since p is pre-sink of g, $D_p g$ has a multiplicity one eigenvalue such that $|\lambda| = 1$ and other eigenvalues of $D_p g$ are with modulus less than 1. Denote by E_p^c the eigenspace corresponding to λ, and E_p^s the eigenspace corresponding to the eigenvalues with modulus less than 1. Thus $T_pM = E_p^c \oplus E_p^s$. Then we get two cases: λ is real or complex.

Case 1: λ is real. Then $\dim E^c_p = 1$. For simplicity, we suppose that $\lambda = 1$. There is a small arc $\mathcal{I}_p \subset B_\alpha(p) \cap \exp_p(E^c_p(\alpha))$ center at p such that $g_1|_{\mathcal{I}_p} = id$, where id is identity map. Here $E^c_p(\alpha)$ is the α-ball in E^c_p center at the origin O_p. Clearly, $\mathcal{I}_p \subset \Lambda_{g_1}(U)$.

Note that for a set $A \subset M$, if M is expansive then A have to expansive. By the definition of the C^1-stably expansivity, $g_1|_{\Lambda_{g_1}(U)}$ is expansive. Moreover, by Remark 3.1, $g_1|_{\mathcal{I}_p}$ is not expansive. This is a contradiction. Therefore, if Λ is C^1-stably expansive of f then it does not have pre-sink.

Case 2: λ is complex. Then $\dim E^c_p = 2$. Since the eigenvalue λ is complex, there is a small circle $\mathcal{C}_p \subset B_\alpha(p) \cap \exp_p(E^c_p(\alpha))$ center at p such that $g_1|_{\mathcal{C}_p}$ is conjugate to an irrational rotation map. Here $E^c_p(\alpha)$ is the α-ball in E^c_p center at the origin O_p. Clearly, $\mathcal{C}_p \subset \Lambda_{g_1}(U)$. Thus by the notion of C^1-stably expansivity, $g_1|_{\mathcal{C}_p}$ has to be expansive. Again by Remark 3.1, the rotation map $g_1 : \mathcal{C}_p \to \mathcal{C}_p$ is not expansive. This is a contradiction.

Therefore, if Λ is C^1-stably expansive of f then it does not have pre-sink. Similarly, f does not have pre-source. $\square$

The following lemma is well known result. In fact, we make using the C^1-closing lemma and property of transitive set. Hereafter, we consider transitive sets is non-trival, that is, the set is not one orbit.

Lemma 3.3 ([8]). *Let Λ be a transitive set. There exist a sequence $\{g_n\}_{n\in N}$ of diffeomorphism and a periodic orbit P_n of g_n with period $\pi(P_n) \to \infty$ as $n \to \infty$ such that $g_n \to f$ in the C^1-topology and $\lim_H P_n = \Lambda$, where $\lim_H$ is the Hausdorff limit and $\pi(P_n)$ is the period of P_n.*

From Lemma 3.3, we can choose $p_n \in P_n$ such that we get a periodic family $\mathcal{A}_n = \{D_{p_n}f, D_{f(p_n)}f, \ldots, D_{f^{\pi(p_n)-1}(p_n)}f\}$.

Lemma 3.4 ([4]). *Let Λ, P_n be as in Lemma 3.3, and $\mathcal{A}_n$ be given as above. Then for any $\epsilon > 0$ there exists an integer $n_0(\epsilon) > 0$ such that for any $n > n_0(\epsilon), \mathcal{A}_n$ is neither ϵ-uniformly contracting nor ϵ-uniformly expanding.*

Let $\mathcal{U}_0(f)$ be given by Lemma 3.2, and let $g \in \mathcal{U}_0(f)$. We consider the periodic family of linear maps $\mathcal{A} = \{D_p g$: for any $p \in P(g) \cap \Lambda_g(U)\}$. Let $\mathcal{B} = \{\xi_p$: for any $p \in P(g) \cap \Lambda_g(U)\}$ be a family of periodic sequence of linear maps closed to $\mathcal{A}$, and for any $p \in P(g) \cap \Lambda_g(U)$, consider the linear map

$$\mathcal{C}_{\mathcal{B}} = \xi_{g^{\pi(p)-1}(p)} \circ \cdots \circ \xi_p,$$

and denote by $\lambda_s(\mathcal{C}_\mathcal{B})$, $\lambda_u(\mathcal{C}_\mathcal{B})$ its eigenvalues. Here $\xi_{g^i(p)}$ is a linear map nearby $D_{g^i(p)}g$ for $0 \le i \le \pi(p) - 1$ and $|\lambda_s(\mathcal{C}_\mathcal{B})| \le |\lambda_u(\mathcal{C}_\mathcal{B})|$.

Lemma 3.5 ([4]). *Let Λ, P_n be as in Lemma 3.3. Then for any $\epsilon > 0$ there are $n(\epsilon), l(\epsilon) > 0$ such that for any $n > n(\epsilon)$ if P_n does not admits a $l(\epsilon)$ dominated splitting, then choose g C^1-nearby f and preserving the orbit of P_n such that P_n is pre-sink or pre-source respecting g.*

From Lemma 3.2 and Lemma 3.5, we can get the following Proposition 3.6.

Proposition 3.6. *Let Λ be a transitive set. Then if Λ is C^1-stably expansive, then we can choose $N, l > 0$ such that for any $n > N, P_n$ admits a l-dominated splitting.*

Proof. Let Λ be a transitive set. Suppose that Λ is C^1-stably expansive. Then by Franks' Lemma, and by the notion of the C^1-stably expansivity, there are a C^1-neighborhood $\mathcal{U}(f)$ of f and a compact neighborhood U of Λ such that for any $g \in \mathcal{U}_0(f) \subset \mathcal{U}(f)$, $g|_{\Lambda_g(U)}$ is expansive. By Lemma 3.2, g has neither pre-sink nor pre-source. And, by Lemma 3.5, P_n is neither pre-sink nor pre-source respecting g. Therefore, by Lemma 3.5, P_n admits a l-dominated splitting. □

By Proposition 3.6 and the following proposition, we directly obtain Theorem B.

Proposition 3.7 ([2]). *Let g_n convergent to f and if Λ_{g_n} be a closed g_n-invariant set of g_n and $\lim \Lambda_{g_n} = \Lambda$. Then if Λ_{g_n} admits a l-dominated splitting respecting g_n, then Λ admits a l-dominated splitting respecting f.*

End of proof of Theorem B. Let Λ be a transitive set of $f \in \mathrm{Diff}(M)$. Then by Lemma 3.3, there exists a sequence $\{g_n\}_{n\in\mathbb{Z}}$ of diffeomorphism and a periodic orbit P_n of g_n such that $g_n \to f$ in the C^1-topology and $P_n \to \Lambda$ in the Hausdorff limit. By Proposition 3.6, P_n admits a l-dominated splitting. Thus by Proposition 3.7, Λ admits a l-dominated splitting. □

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