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THE AXIOM OF INDEFINITE SURFACES IN SEMI-RIEMANNIAN MANIFOLDS

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Abstract. In this paper, we characterize a semi-Riemannian manifolds satisfies the axiom of indefinite surfaces. We obtain the following result: If a semi-Riemannian manifold satisfies the axiom of indefinite surfaces, then it is a real space form.

0. Introduction

The notion of axiom of planes for Riemannian manifolds was first introduced by Elie Cartan [1] in the middle of the 1940's as it follows: *A Riemannian manifold M of dimension $m \geq 3$ satisfies the axiom of ℓ-planes if for each point p in M and for every ℓ-dimensional linear subspace T of T_pM, there exists an ℓ-dimensional totally geodesic submanifold N of M containing p such that $T_pN = T$.* He proved the following:

Theorem A. *A Riemannian manifold of dimension $m \geq 3$ satisfies the axiom of ℓ-planes for some $\ell\,(2 \leq \ell < m)$ if and only if it is a real space form.*

Further in 1971 D. S. Leung and K. Nomizu [6] generalized this notion by introducing the axiom of ℓ-spheres as it follows: *A Riemannian manifold M of dimension $m \geq 3$ satisfies the axiom of ℓ-spheres if for each point p in M and for every ℓ-dimensional linear subspace T of T_pM, there exists a ℓ-dimensional totally umbilcal submanifold N of M with parallel mean curvature vector field of M such that $p \in N$ and $T_pN = T$.* They proved the following to characterize a real space form:

Theorem B. *A Riemannian manifold of dimension $m \geq 3$ satisfies the axiom of ℓ-spheres for some $\ell\,(2 \leq \ell < m)$ if and only if it is a real space form.*

In 1971, L. Graves and K. Nomizu [4] generalized these notions of axioms of planes and spheres for indefinite Riemannian manifolds.

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Recently, R. Kumar and others [5] studied the axioms of planes and spheres for semi-Riemannian manifolds with lightlike submanifolds. They gave these axioms as it follows: *A semi-Riemannian manifold* $\bar{M}$ *of dimension* $m+n \geq 3$ *satisfies the axiom of* ℓ*-planes* (ℓ*-spheres, respectively*) *if for each point* $p \in \bar{M}$ *and for every* ℓ*-dimensional linear subspace* T *of* $T_p\bar{M}$*, there exists a* ℓ*-dimensional totally geodesic lighlike submanifold* M (*totally umbilical lightlike submanifold* M *with parallel transversal curvature vector field respectively*) *such that* $p \in M$ *and* $T_pM = T\,(2 \leq \ell < m+n)$*.* However, we regret that several equations and some results in their paper [5] are mistaken or have serious errors. For example, the equations (21), (23), (25), (27) etc are not correct, and Lemma 1 and Lemma 2, which play an important role in [5], are flaws. In fact, under the assumptions of these lemmas, the induced connection ∇ is not metric(see, (1.13) in this paper). However, they used the fact ∇ is metric. Thus the main theorems(Theorem C and D) in [5] are mistaken.

The objective of this paper is also the study of a lightlike version of the above axioms of ℓ-planes or ℓ-spheres. We propose these axioms for semi-Riemannian manifolds as it follows:

Axiom of indefinite ℓ-surfaces. *A semi-Riemannian manifold* $\bar{M}$ *of dimension* $m+n \geq 3$ *satisfies the axiom of indefinite* ℓ*-planes* (*indefinite* ℓ*-spheres, respectively*) *if for each point* $p \in \bar{M}$ *and for every* ℓ*-dimensional linear subspace* T *of* $T_p\bar{M}$*, there exists an* ℓ*-dimensional totally geodesic lighlike submanifold* M (*totally umbilical lightlike submanifold* M *with parallel transversal curvature vector field and an induced metric connection respectively*) *such that* $p \in M$ *and* $T_pM = T$*.*

We have the following result:

Theorem 1. *If a semi-Riemannian manifold of dimension* $m+n \geq 3$ *satisfies the axiom of indefinite* ℓ*-surfaces for some* $\ell(2 \leq \ell < m+n)$*, then* $\bar{M}$ *is a real space form.*

1. Lightlike Submanifolds

Let $(\bar{M}, \bar{g})$ be a real $(m+n)$-dimensional semi-Riemannian manifold of constant index q such that $m, n \geq 1,\ 1 \leq q \leq m+n-1$, and let (M, g) be a submanifold of dimension m of $\bar{M}$. We follow Duggal-Jin [3] for notations and results used in this paper. Throughout this paper we denote by $F(M)$ the algebra of smooth functions on M and by $\Gamma(E)$ the $F(M)$ module of smooth sections of any vector bundle E over

M. We say that M is a *lightlike submanifold* of $\bar{M}$ if it admits a degenerate metric g induced from $\bar{g}$. In this case the radical distribution $Rad(TM) = TM \cap TM^{\perp}$ of M is a vector subbundle of both the tangent bundle TM and the normal bundle $TM^{\perp}$, of rank r. In general, there exist two complementary non-degenerate distributions $S(TM)$ and $S(TM^{\perp})$ of $Rad(TM)$ in TM and $TM^{\perp}$ respectively, called the *screen* and *co-screen distributions* on M, such that

$$TM = Rad(TM) \oplus_{orth} S(TM)\,, \quad TM^{\perp} = Rad(TM) \oplus_{orth} S(TM^{\perp}), \tag{1.1}$$

where $\oplus_{orth}$ denotes the orthogonal direct sum. We denote such a lightlike submanifold by $(M, g, S(TM), S(TM^{\perp}))$.

We say that a lightlike submanifold of $\bar{M}$ is

(1) *r-lightlike* if $1 \leq r < \min\{m,\, n\}$;
(2) *co-isotropic* if $1 \leq r = n < m$;
(3) *isotropic* if $1 \leq r = m < n$;
(4) *totally lightlike* if $1 \leq r = m = n$.

The above three classes (2)∼(4) are particular cases of the class (1) as it follows: $S(TM^{\perp}) = \{0\}$, $S(TM) = \{0\}$ and $S(TM) = S(TM^{\perp}) = \{0\}$ respectively. The geometry of r-lightlike submanifolds is more general form than that of the other three type submanifolds. For this reason, we consider only r-lightlike submanifolds $(M, g, S(TM), S(TM^{\perp}))$. *For the rest of this paper, by a lightlike submanifold M we shall mean an r-lightlike submanifold $(M, g, S(TM), S(TM^{\perp}))$, unless specified.*

Let $tr(TM)$ and $ltr(TM)$ be complementary (but not orthogonal) vector bundles to TM in $T\bar{M}_{|M}$ and $TM^{\perp}$ in $S(TM)^{\perp}$ respectively. Then we have

$$tr(TM) = ltr(TM) \oplus S(TM^{\perp}), \tag{1.2}$$

$$\begin{aligned} T\bar{M}|_M &= TM \oplus tr(TM) \\ &= (Rad(TM) \oplus ltr(TM)) \oplus S(TM) \oplus S(TM^{\perp}). \end{aligned} \tag{1.3}$$

We call $tr(TM)$ and $ltr(TM)$ *transversal* and *lightlike transversal vector bundle* of M, respectively. Consider the following local quasi-orthonormal field of frames of $\bar{M}$ along M:

$$\{\xi_1, \dots, \xi_r,\; N_1, \dots, N_r,\; X_{r+1}, \dots, X_m,\; W_{r+1}, \dots, W_n\} \tag{1.4}$$

where $\{\xi_1, \dots, \xi_r\}$ is a lightlike basis of $\Gamma\,(Rad(TM))$, $\{N_1, \dots, N_r\}$ a lightlike basis of $\Gamma\,(ltr(TM))$, $\{X_{r+1}, \dots, X_m\}$ and $\{W_{r+1}, \dots, W_n\}$ orthonormal basis of

$\Gamma(S(TM))$ and $\Gamma(S(TM^{\perp}))$ respectively. Then we have

$$\bar{g}\,(N_i,\,\xi_j) = \delta_{ij}, \quad \bar{g}\,(N_i,\,N_j) = 0.$$

Let $\bar{\nabla}$ be the Levi-Civita connection on $\bar{M}$. Due to (1.3) we put

$$\bar{\nabla}_X Y = \nabla_X Y \,+\, h(X,Y),\ \forall\, X,\, Y \in \Gamma(TM), \tag{1.5}$$

$$\bar{\nabla}_X V = -A_V X \,+\, \nabla^{\perp}_X V,\ \forall\, X \in \Gamma(TM),\ V \in \Gamma(tr(TM)), \tag{1.6}$$

where $\{\nabla_X Y, A_V\}$ and $\{h(X,Y), \nabla^{\perp}_X V\}$ belong to $\Gamma(TM)$ and $\Gamma(tr(TM))$ respectively. ∇ and $\nabla^{\perp}$ are linear connections on M and $tr(TM)$ respectively. According to (1.2) we consider the projection morphisms L and S of $tr(TM)$ on $ltr(TM)$ and $S(TM^{\perp})$ respectively. Then (1.5) and (1.6) become

$$\bar{\nabla}_X Y \,=\, \nabla_X Y \,+\, h^{\ell}(X,\,Y) \,+\, h^{s}(X,\,Y), \tag{1.7}$$

$$\bar{\nabla}_X N \,=\, -A_N X \,+\, \nabla^{\ell}_X N \,+\, D^{s}(X,\,N), \tag{1.8}$$

$$\bar{\nabla}_X W \,=\, -A_W X \,+\, \nabla^{s}_X W \,+\, D^{\ell}(X,\,W), \tag{1.9}$$

for any $X,\, Y \in \Gamma(TM),\ N \in \Gamma(ltr(TM))$ and $W \in \Gamma(S(TM^{\perp}))$, where we set

$$\begin{aligned} h^{\ell}(X,Y) &= L(h(X,Y)) \quad ; \quad & h^{s}(X,Y) &= S(h(X,Y)),\\ \nabla^{\ell}_X N &= L(\nabla^{\perp}_X N) \quad ; \quad & D^{s}(X,N) &= S(\nabla^{\perp}_X N),\\ D^{\ell}(X,W) &= L(\nabla^{\perp}_X W) \quad ; \quad & \nabla^{s}_X W &= S(\nabla^{\perp}_X W). \end{aligned}$$

As h^{ℓ} and h^{s} are $\Gamma(ltr(TM))$-valued and $\Gamma(S(TM)^{\perp})$-valued symmetric $F(M)$-bilinear forms on $\Gamma(TM)$, we called them the *second fundamental forms* on M. Also, as A_N and A_W are linear operators on $\Gamma(TM)$, we call them the *shape operators* of M. By using (1.5) $\sim$ (1.9), for any $X,\, Y \in \Gamma(TM),\ \xi \in \Gamma(Rad(TM))$ and $W \in \Gamma(S(TM^{\perp})$, we have

$$\bar{g}(h^{s}(X,Y),\,W) + \bar{g}(Y,\,D^{\ell}(X,W)) = g(A_W X,\,Y), \tag{1.10}$$

$$\bar{g}(h^{\ell}(X,Y),\,\xi) + \bar{g}(Y,\,h^{\ell}(X,\xi)) + g(Y,\,\nabla_X \xi) = 0, \tag{1.11}$$

$$\bar{g}(D^{s}(X,N),\,W) = \bar{g}(N,\,A_W X). \tag{1.12}$$

The induced connection ∇ on TM is not metric and satisfies

$$(\nabla_X g)(Y,Z) = \bar{g}(h^{\ell}(X,Y),Z) + \bar{g}(h^{\ell}(X,Z),Y),\ \forall\, X,\, Y,\, Z \in \Gamma(TM). \tag{1.13}$$

By using the above linear connections, we use the following covariant derivatives:

$$(\nabla_X h^{\ell})(Y,\,Z) = \nabla^{\ell}_X(h^{\ell}(Y,\,Z)) \,-\, h^{\ell}(\nabla_X Y,\,Z) \,-\, h^{\ell}(Y,\,\nabla_X Z), \tag{1.14}$$

$$(\nabla_X h^{s})(Y,\,Z) = \nabla^{s}_X(h^{s}(Y,\,Z)) \,-\, h^{s}(\nabla_X Y,\,Z) \,-\, h^{s}(Y,\,\nabla_X Z), \tag{1.15}$$

for any $X, Y, Z \in \Gamma(TM)$, $\xi \in \Gamma(Rad(TM)))$, $N \in \Gamma(ltr(TM))$ and $W \in \Gamma(S(TM^{\perp}))$. Denote by $\bar{R}$ and R the curvature tensors of $\bar{\nabla}$ and ∇ respectively. Then, using (1.7), (1.8), (1.9), (1.14) and (1.15), for any $X, Y, Z \in \Gamma(TM)$, we obtain

$$\begin{aligned}
&\bar{R}(X, Y)Z = R(X, Y)Z \\
&+A_{h^{\ell}(X,Z)}Y - A_{h^{\ell}(Y,Z)}X + A_{h^{s}(X,Z)}Y - A_{h^{s}(Y,Z)}X \\
&+(\nabla_X h^{\ell})(Y, Z) - (\nabla_Y h^{\ell})(X, Z) + D^{\ell}(X, h^{s}(Y, Z)) - D^{\ell}(Y, h^{s}(X, Z)) \\
&+(\nabla_X h^{s})(Y, Z) - (\nabla_Y h^{s})(X, Z) + D^{s}(X, h^{\ell}(Y, Z)) - D^{s}(Y, h^{\ell}(X, Z)).
\end{aligned} \tag{1.16}$$

2. Proof of Theorem 1

Definition 1. A lightlike submanifold M of $(\bar{M}, \bar{g})$ is said to be *totally umbilical* if there is a smooth transversal vector field $H \in \Gamma(tr(TM))$ on M, called the *transversal curvature vector field of* M, such that

$$h(X,Y) = H\, g(X,Y), \quad \forall\, X, Y \in \Gamma(TM). \tag{2.1}$$

In case $H = 0$, we say that M is *totally geodesic*.

It is easy to see that M is totally umbilical if and only if there exist smooth vector fields $H^{\ell} \in \Gamma(ltr(TM))$ and $H^{s} \in \Gamma(S(TM^{\perp}))$ such that

$$h^{\ell}(X,Y) = H^{\ell}\, \bar{g}(X,Y),\ h^{s}(X,Y) = H^{s}\, \bar{g}(X,Y),\ \forall\, X, Y \in \Gamma(TM), \tag{2.2}$$

In case M is totally umbilical, using (1.10) and (2.2), we have

$$h^{\ell}(X,\xi) = 0,\ h^{s}(X,\xi) = 0,\ D^{\ell}(X,W) = 0,\ \forall\, X \in \Gamma(TM). \tag{2.3}$$

Definition 2. We say that the transversal curvature vector field H is *parallel* in the transversal vector bundle $tr(TM)$ if $\nabla^{t}_{X} H = 0$ for all $X \in \Gamma(TM)$.

For a totally umbilical M, using (1.6), (1.8), (1.9) and (2.2), we show that the transversal curvature vector field H is parallel in $tr(TM)$ if and only if

$$\nabla^{\ell}_{X} H^{\ell} = 0 \quad \& \quad \nabla^{s}_{X} H^{s} + D^{s}(X, H^{\ell}) = 0,\ X \in \Gamma(TM). \tag{2.4}$$

Assume that M is totally umbilical. Then (1.6) deduces to

$$\begin{aligned}
\bar{R}(X, Y)Z = R(X, Y)Z &- g(Y,Z)A_H X + g(X,Z)A_H Y \\
&+ \{(\nabla_X g)(Y,Z) - (\nabla_Y g)(X,Z)\}H \\
&+ g(Y,Z)\{\nabla^{\ell}_{X} H^{\ell} + D^{s}(X,H^{\ell}) + \nabla^{s}_{X} H^{s}\} \\
&- g(X,Z)\{\nabla^{\ell}_{Y} H^{\ell} + D^{s}(Y,H^{\ell}) + \nabla^{s}_{Y} H^{s}\}
\end{aligned} \tag{2.5}$$

for any X, Y, $Z \in \Gamma(TM)$. From (2.4) and (2.5), we have

Lemma 1. *Let M be a totally umbilical submanifold of a semi-Riemannian manifold $\bar{M}$ with parallel transversal curvature vector field H. Then we have*

$$\begin{aligned}\bar{R}(X, Y)Z = R(X, Y)Z - g(Y, Z)A_H X + g(X, Z)A_H Y \\ + \{(\nabla_X g)(Y, Z) - (\nabla_Y g)(X, Z)\}H, \ \forall X, Y, Z \in \Gamma(TM).\end{aligned} \tag{2.6}$$

Lemma 2 ([4])**.** *Let $(\bar{M}, \bar{g})$ be a semi-Riemannian manifold. If $g(\bar{R}(X, Y)Z, X) = 0$ for all X, Y, $Z \in \Gamma(TM)$, then $\bar{M}$ has the constant sectional curvature.*

Proof of Theorem 1. Case 1. At an arbitrary point $p \in M$, let X, ξ and V be orthonormal at p. Let T be an ℓ-dimensional subspace of $T_p\bar{M}|_M$ containing X and ξ, transversal to V. Now $\bar{M}$ satisfies the axiom of indefinite ℓ-spheres and hence there exists an ℓ-dimensional totally umbilical lightlike submanifold M with parallel transversal curvature vector field H such that $T_pM = T$ for any point $p \in M$. Now from Lemma 2, for $X, \xi \in \Gamma(TM)$ the transversal form of $\bar{R}(X, \xi)X$ is given by

$$(\bar{R}(X, \xi)X)^N = \{(\nabla_X g)(\xi, X) - (\nabla_\xi g)(X, X)\}H.$$

As the induced connection ∇ of M is metric, we have $g(\bar{R}(X, \xi)X, V) = 0$. Then the Theorem follows from Lemma 2.

Case 2. Since there exists an ℓ-dimensional totally geodesic lightlike submanifold on M, from Theorem 2.1, we have $(\bar{R}(X, \xi)X)^N = 0$, and hence the theorem follows from the Lemma 2. □

Remark. It is clear that a semi-Riemannian manifold with coisotropic, isotropic or totally lightlike submanifolds is a real space form if it satisfies the axiom of ℓ-planes and ℓ-spheres.

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