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AN APPLICATION OF COMPLICATEDNESS TO BH-ALGEBRAS

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Abstract. The notions of an initial section and a special set in BH-algebras are defined and some of their properties are obtained. The notion of a complicated BH-algebra is introduced and some related properties are obtained. Finally, the notion of essences in BH-algebras are defined, and many properties are investigated.

1. Introduction

Y. Imai and K. Iséki introduced two classes of abstract algebras: BCK-algebras and BCI-algebras ([2,3]). It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. BCK-algebras have some connections with other areas: D. Mundici [7] proved MV-algebras are categorically equivalent to bounded commutative algebra, and J. Meng [5] proved that implicative commutative semigroups are equivalent to a class of BCK-algebras. Y. B. Jun, E. H. Roh, and H. S. Kim [4] introduced the notion of a BH-algebra, which is a generalization of BCK/BCI-algebras. They defined the notions of ideal, maximal ideal and translation ideal and investigated some properties. E. H. Roh and S. Y. Kim [8] estimated the number of BH^*-subalgebras of order i in a transitive BH^*-algebras by using Hao's method. In [1], S. S. Ahn and J. H. Lee introduced the notion of strong ideals in BH-algebra and investigate some properties of it. They also defined the notion of a rough sets in BH-algebras. Using a strong ideal in BH-algebras, they obtained some relations between strong ideals and upper(lower) rough strong ideals in BH-algebras.

In this paper, we define the notions of an initial section and a special set and get of their properties. We also introduce the notion of complicated BH-algebra and

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obtain some related properties. Finally, the notion of essences in BH-algebras are defined, and many properties are investigated.

2. Preliminaries

By a *BH-algebra* ([4]), we mean an algebra $(X; *, 0)$ of type (2,0) satisfying the following conditions:

(I) $x * x = 0$,

(II) $x * 0 = x$,

(III) $x * y = 0$ and $y * x = 0$ imply $x = y$, for all $x, y \in X$.

For brevity, we also call X a BH-algebra. In X we can define an order relation " $\leq$ " by $x \leq y$ if and only if $x * y = 0$. Then $\leq$ is reflexive and antisymmetric. A non-empty subset S of a BH-algebra X is called a *subalgebra* of X if, for any $x, y \in S$, $x * y \in S$, i.e., S is closed under binary operation " $*$ ".

Definition 2.1 ([4]). A non-empty subset A of a BH-algebra X is called an *ideal* of X if it satisfies:

(I1) $0 \in A$,

(I2) $x * y \in A$ and $y \in A$ imply $x \in A$, $\forall x, y \in X$.

An ideal A of a BH-algebra X is said to be a *translation ideal* of X if it satisfies:

(I3) $x * y \in A$, $y * x \in A$ imply $(x * z) * (y * z), (z * x) * (z * y) \in A$ for any $x, y, z \in X$.

Obviously, $\{0\}$ and X are translation ideals of X

Definition 2.2 ([8]). A BH-algebra X is called a *BH^*-algebra* if it satisfies the identity $(x * y) * x = 0$ for all $x, y \in X$.

Example 2.3 ([4]). Let $X := \{0, 1, 2, 3\}$ be a BH-algebra which is not a BCK-algebra with the following Cayley table:

$*$	0	1	2	3
0	0	1	0	0
1	1	0	0	0
2	2	2	0	3
3	3	3	3	0

Then $A := \{0, 1\}$ is a translation ideal of X.

Lemma 2.4. *Let X be a BH^*-algebra. Then the following identity holds:*

$$0 * x = 0, \quad \forall x \in X.$$

Proof. If follows from (II) that $0*x = (0*x)*0 = 0$ for all $x \in X$. Hence $0*x = 0$. □

Definition 2.5. A BH-algebra $(X;*,0)$ is said to be *transitive* if $x*y = 0$ and $y*z = 0$ imply $x*z = 0$.

Lemma 2.6. *An ideal A of a BH-algebra X has the following property:*

$$(\forall x \in X)(\forall y \in A)(x \leq y \Rightarrow x \in A).$$

Proof. Straightforward. □

3. Complicated BH-algebras

Lemma 3.1. *Let A be an ideal of a BH^*-algebra X. Then A is a subalgebra of X.*

Proof. If $x, y \in A$, then $(x*y)*x = 0$ and so $x*y \in A$. This completes the proof. □

The converse of Lemma 3.1 need not be true in general as seen in the following example.

Example 3.2. Let $X := \{0,1,2\}$ be a set with the following Cayley table:

$*$	0	1	2
0	0	0	0
1	1	0	2
2	2	0	0

Then $(X;*,0)$ is a BH^*-algebra ([8]). Let $A := \{0,2\}$. Then A is a subalgebra of X but not an ideal of X since $1*2 = 2 \in A$ and $1 \notin A$.

Definition 3.3. Let X be a BH-algebra. X is said to be *positive implicative* if it satisfies the following identity:

$$(x*y)*z = (x*z)*(y*z), \forall x, y, z \in X.$$

Example 3.4. Let $X := \{0, a, b, c\}$ be a BH-algebra with the Cayley table as follows:

$*$	0	a	b	c
0	0	0	0	0
a	a	0	a	0
b	b	b	0	0
c	c	c	c	0

Then $(X;*,0)$ is a positive implicative BH-algebra.

Proposition 3.5. *If X is a positive implicative BH-algebra, then the following condition holds:*

$$x*y = (x*y)*y, \forall x, y \in X.$$

Proof. Since X is positive implicative, we have

$$x * y = (x * y) * 0 = (x * y) * (y * y) = (x * y) * y.$$

This completes the proof. $\square$

Lemma 3.6. *Let X be a positive implicative BH^*-algebra. If $x \leq y$, then $x*z \leq y*z$ for any $x, y, z \in X$.*

Proof. Let $x, y \in X$ with $x \leq y$. Then $x * y = 0$. Since X is positive implicative, we have

$$(x * z) * (y * z) = (x * y) * z = 0 * z = 0.$$

Hence $x * z \leq y * z$. This completes the proof. $\square$

Proposition 3.7. *If X is a positive implicative BH^*-algebra, then X is a transitive BH^*-algebra.*

Proof. Assume that $x * y = 0$ and $y * z = 0$ for any $x, y, z \in X$. Then we have

$$\begin{aligned} x * z &= (x * z) * 0 \\ &= (x * z) * (y * z) \\ &= (x * y) * z \\ &= 0 * z = 0. \end{aligned}$$

Hence X is a transitive BH^*-algebra. $\square$

For an element a of a BH-algebra X, the set $\{x \in X | x \leq a\}$, denoted by $A(a)$, is called the *initial section* of an element a. Since $a \in A(a)$, $A(a)$ is not empty.

Proposition 3.8. *Let X be a transitive BH-algebra and $x \leq y$. If $y \in A(a)$, then $x \in A(a)$.*

Proof. Since $y \in A(a)$, we have $y \leq a$. Hence $x \leq y \leq a$, i.e., $x \leq a$. This implies $x \in A(a)$. $\square$

Theorem 3.9. *For any a in a positive implicative BH^*-algebra X, $A(a)$ is the least ideal of X containing a.*

Proof. It follows from that $0*a = 0$ for any $a \in X$. Hence $0 \in A(a)$. Let $x*y \in A(a)$ and $y \in A(a)$ for any $x, y \in X$. Then $(x * y) * a = 0$ and $y * a = 0$. Since X is positive implicative, we have $0 = (x * y) * a = (x * a) * (y * a) = (x * a) * 0 = x * a$. Hence $x \in A(a)$. Therefore $A(a)$ is an ideal of X. Clearly, $a \in A(a)$.

Let H be any ideal of X containing a. Let $x \in A(a)$. Then $x * a = 0 \in H$. Since $a \in H$ and H is an ideal of X, we have $x \in H$. Therefore $A(a) \subseteq H$. Thus $A(a)$ is the least ideal of X containing a. □

Corollary 3.10. *Let X be a positive implicative BH^*-algebra. Then $A(a)$ is a subalgebra of X.*

Proof. By Lemma 3.1 and Theorem 3.9, $A(a)$ is a subalgebra of X. □

Theorem 3.11. *Let X be a BH-algebra and let A be an ideal of X and $x \in A$. Then $A(x) \subseteq A$.*

Proof. If $y \in A(x)$, then we have $y \leq x$. Hence $y * x = 0$. Since A is an ideal of X and $x \in A$, we obtain $y \in A$. Therefore $A(x) \subseteq A$. □

For any a, b in a BH-algebra X, the set $\{x \in X | (x * a) * b = 0\}$, denoted by $A(a, b)$, is called a *special set* of X. Note that $A(a, b)$ is not an ideal of X in general as seen in the following example.

Example 3.12. Let $X := \{0, 1, 2, 3\}$ be a set with the following Cayley table:

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	2	2	0	3
3	3	3	3	0

Then $(X; *, 0)$ is a BH-algebra which is not a BCK/BCI-algebra. Let $I := \{0, 1\}$. Then I is an ideal of X. Moreover, it is easy to check that $I = A(0, 1)$. It is easy to show that $A(0, 3) = \{x \in X | (x * 0) * 3 = 0\} = \{0, 1, 3\}$, but it is not an ideal of X, since $2 * 3 = 3 \in A(0, 3)$, but $2 \notin A(0, 3)$.

An ideal I of a BH-algebra X is said to be *closed* if $0 * x \in I$ for any $x \in I$.

Proposition 3.13. *Let I be a subset of a BH-algebra X with the following conditions:*

(1) $0 \in I$
(2) $x * z \in I$, $y * z \in I$ *and* $z \in I$ *imply* $x * y \in I$ *for any* $x, y, z \in X$.

Then I is both a subalgebra and a closed ideal of X.

Proof. Let $x, y \in I$. By (II), we have $x = x * 0$ and $y = y * 0$. It follows from (2) that $x * y \in I$. Hence I is a subalgebra of X.
Assume that I satisfies (1) and (2). We claim that I is a closed ideal of X. Let $x * y, y \in I$. Since $0 * 0, y * 0, 0 \in I$, it follows from (2) that $0 * y \in I$ which proves

that I is closed. Since $x * y, 0 * y, y \in I$, by applying (2) again, we obtain that $x = x * 0 \in I$, so that I is an ideal of X. $\square$

Definition 3.14. Let X be a BH^*-algebra. If for any $a, b \in X$, the set $A(a, b)$ has the greatest element, then the BH^*-algebra is said to be *complicated*.

Note that $A(a, b)$ is a non-empty set, since $0, a, b \in A(a, b)$, where X is a BH^*-algebra. The greatest element of $A(a, b)$ is denoted by $a \odot b$.

Example 3.15. Let $X := \{0, 1, 2, 3\}$ be a set with the following Cayley table:

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	2	2	0	0
3	3	3	3	0

It is easy to show that $(X; *, 0)$ is a complicated BH^*-algebra.

Theorem 3.16. *Let X be a positive implicative complicated BH^*-algebra and let $a, b \in X$. Then the set*

$$\mathcal{H}(a, b) := \{x \in X | a \leq b \odot x\}$$

has the least element, and it is $a * b$.

Proof. The inequality $a * b \leq a * b$ implies that $a \leq b \odot (a * b)$ and so $a * b \in \mathcal{H}(a, b)$. Let $z \in \mathcal{H}(a, b)$. Then $a \leq b \odot z$, which implies from Lemma 3.6 and Definition 3.14 that $a * b \leq (b \odot z) * b \leq z$. Hence $a * b \leq z$ by Proposition 3.7. Thus $a * b$ is the least element of $\mathcal{H}(a, b)$. $\square$

Proposition 3.17. *Let X be a complicated BH^*-algebra. Then for any $a, b \in X$, the following hold:*

(i) $a \leq a \odot b$ *and* $b \leq a \odot b$,
(ii) $a \odot 0 = a = 0 \odot a$.

Proof. Straightforward. $\square$

Proposition 3.18. *Let X be a positive implicative complicated BH^*-algebra.*

(i) *If $x \leq y$, then $x \odot y = y$ for any $x, y \in X$.*
(ii) *$x \odot x = x$ for any $x \in X$.*

Proof. (i) If $x \leq y$, then

$$\begin{aligned} 0 &= ((x \odot y) * x) * y \\ &= ((x \odot y) * y) * (x * y) \end{aligned}$$

$$= ((x \odot y) * y) * 0$$
$$= (x \odot y) * y,$$

which means that $x \odot y \leq y$. By Proposition 3.17(i), we have $x \odot y = y$.
(ii) Let $y := x$ in (i). Then $x \odot x = x$. □

We provide some characterizations of ideals in a complicated BH^*-algebra.

Proposition 3.19. *Let A be a non-empty subset of a complicated BH^*-algebra X. If A is an ideal of X, then it satisfies the following conditions:*

(i) $(\forall x \in A)(\forall y \in X)(y \leq x \Rightarrow y \in A)$.
(ii) $(\forall x, y \in A)(\exists z \in A)(x \leq z, y \leq z)$.

Proof. Assume that A is an ideal of X. Let $x \in A, y \in X$ with $y \leq x$. Then $y*x = 0$. Since A is an ideal of X, we have $y \in A$, i.e., (i) is valid.

Let $x, y \in A$. Since $(x \odot y)*x \leq y$ and $y \in A$, it follows from (i) that $(x \odot y)*x \in A$ so that $x \odot y \in A$ because A is an ideal of X. If we take $z := x \odot y$, then $x \leq z$ and $y \leq z$ by Proposition 3.17(i). This completes the proof. □

Proposition 3.20. *Let A be a non-empty subset of a positive implicative complicated BH^*-algebra X. Then A is an ideal of X if and only if it satisfies the following conditions:*

(i) $(\forall x \in A)(\forall y \in X)(y \leq x \Rightarrow y \in A)$.
(ii) $(\forall x, y \in A)(\exists z \in A)(x \leq z, y \leq z)$.

Proof. The necessity follows from Proposition 3.19.

Conversely, let A be a non-empty subset of X satisfying conditions (i) and (ii). Since A is non-empty, we have $0 \in A$ by (i). Let $x, y \in X$ with $y \in A$ and $x * y \in A$. Then, by (ii), there exists $z \in A$ such that $y \leq z$ and $x * y \leq z$. Since X is positive implicative, we have $x * z = (x * z) * 0 = (x * z) * (y * z) = (x * y) * z = 0$ and so $x \leq z$. Since $z \in A$, it follows from (i) that $x \in A$. Hence A is an ideal of X. □

In Proposition 3.19, the condition, "complicated", is very necessary. See the following example.

Example 3.21. Let $X := \{0, 1, 2, 3, 4\}$ be a set with the following Cayley table:

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	2	0
3	3	3	3	0	0
4	4	4	3	2	0

It is easy to show that X is a BH^*-algebra which is not a BCK/BCI-algebra. Moreover, X is not complicated, since $A(1,2) = \{z \in X | (z * 1) * 2 = 0\} = \{0, 1, 2\}$ has no greatest element. It is easy to see that $\{0, 1, 2\}$ is an ideal of X, but there is no element $z \in \{0, 1, 2\}$ such that $x \leq z, y \leq z$ in the set $\{0, 1, 2\}$, proving that the condition, "complicated", is necessary in Proposition 3.19.

Theorem 3.22. *Let A be a non-empty subset of a positive implicative complicated BH^*-algebra X. Then A is an ideal of X if and only if it satisfies the following conditions:*

(i) $(\forall x \in A)(\forall y \in X)(y \leq x \Rightarrow y \in A)$.
(ii) $(\forall x, y \in A \Rightarrow x \odot y \in A)$.

Proof. The necessity follows immediately from Proposition 3.19.

Conversely, let A be a non-empty subset of X satisfying conditions (i) and (ii). Obviously, $0 \in A$ by (i). Let $x, y \in X$ satisfying $y \in A$ and $x * y \in A$. Then $y \odot (x * y) \in A$ by (ii). Since $x \leq y \odot (x * y)$ by Theorem 3.16, it follows from (i) that $x \in A$. Thus A is an ideal of X. □

4. ESSENCE OF BH-ALGEBRAS

Let X be a BH-algebra. For any subsets G and H of X, we define

$$G * H := \{x * y | x \in G, y \in H\}.$$

Lemma 4.1. *Let X be a BH-algebra. If $0 \in H \subseteq X$, then*

$$(\forall G \subseteq X)(G \subseteq G * H).$$

Proof. Let $x \in G$. Then $x = x * 0 \in G * H$ by (II), and so $G \subseteq G * H$. □

Lemma 4.2. *For any subsets A, B and E of a BH-algebra X, we have the following properties:*

(i) $A \subseteq B \Rightarrow A * E \subseteq B * E, E * A \subseteq E * B$.
(ii) $(A \cap B) * E \subseteq (A * E) \cap (B * E)$.
(iii) $E * (A \cap B) \subseteq (E * A) \cap (E * B)$.
(iv) $(A \cup B) * E = (A * E) \cup (B * E)$.
(v) $E * (A \cup B) = (E * A) \cup (E * B)$.

Proof. (i) Let $x \in A * E$. Then $x = a * e$ for some $a \in A$ and $e \in E$. Since $A \subseteq B$, it follows that $x = a * e$ for some $a \in B$ and $e \in E$ so that $x \in B * E$. Therefore $A * E \subseteq B * E$. Similarly, we get $E * A \subseteq E * B$.

(ii) Since $A \cap B \subseteq A, B$, it follows from (i) that $(A \cap B) * E \subseteq A * E$ and $(A \cap B) * E \subseteq B * E$ so that $(A \cap B) * E \subseteq (A * E) \cap (B * E)$. Similarly, (iii) is valid.
(iv) Since $A, B \subseteq A \cup B$, we have $A * E \subseteq (A \cup B) * E$ and $B * E \subseteq (A \cup B) * E$ by (i), and so $(A * E) \cup (B * E) \subseteq (A \cup B) * E$. If $x \in (A \cup B) * E$, then $x = y * e$ for some $y \in A \cup B$ and $e \in E$. It follows that $x = y * e$ for some $y \in A$ and $e \in E$; or $x = y * e$ for some $y \in B$ and $e \in E$ so that $x = y * e \in A * E$ or $x = y * e \in B * E$. Hence $(A \cup B) * E \subseteq (A * E) \cup (B * E)$. Thus $(A \cup B) * E = (A * E) \cup (B * E)$. Similarly we can prove that (v) is valid. □

Definition 4.3. If a non-empty subset G of a BH-algebra X satisfies $G * X = G$, then G is called an *essence* of X.

Obviously, $\{0\}$ is an essence of a BH^*-algebra X which is called a *trivial essence of* X, and X itself is an essence of a BH-algebra X. Note that if a is an element of a BH-algebra X such that $\{a\} * X = X$, then any proper subset G of X containing a can be not an essence of X.

Example 4.4. Let $X := \{0, 1, 2, 3\}$ be a set with the following Cayley table:

$*$	0	1	2	3
0	0	0	0	0
1	1	0	1	0
2	2	2	0	0
3	3	2	1	0

Then X is a BH-algebra. It is easy to check that $G_1 = \{0, 1\}$, $G_2 := \{0, 2\}$ and $G_3 := \{0, 1, 2\}$ are essences of X. But $H := \{0, 3\}$ is not an essence of X, since $3 * 2 = 1 \notin H$.

Theorem 4.5. *Let X be a BH-algebra. Then the following properties hold:*

(i) *Every essence of X contains the zero element* 0.
(ii) *Every essence of X is a subalgebra of X.*
(iii) *Every ideal of a BH^*-algebra X is an essence of X.*

Proof. (i) Let G be an essence of X. Then $\emptyset \neq G = G * X$, and so there exists $x \in G$ and thus $0 = x * x \in G * X = G$.
(ii) Let G be an essence of X and let $x, y \in G$. Then $x * y \in G * G \subseteq G * X = G$ by Lemma 4.2(i) and thus G is a subalgebra of X.
(iii) Let I be an ideal of X. Then $0 \in I$, and so $I \neq \emptyset$. By Lemma 2.6, for any $x \in X$ and $y \in I$, we have $y * x \in I$ since $y * x \leq y$. Thus $I * X \subseteq I$. Obviously, $I \subseteq I * X$ by Lemma 4.1. Therefore $I * X = I$, i.e., I is an essence of X. □

The converse of (ii) and (iii) of Theorem 4.5 may not be true as seen the following example.

Example 4.6. In Example 4.4, $G_3 := \{0, 1, 2\}$ is an essence which is not an ideal, and $H := \{0, 3\}$ is a subalgebra which is not an essence of X.

Proposition 4.7. *Let G and H be essence of a BH-algebra X. Then $G \cap H$ and $G \cup H$ are essences of X.*

Proof. By Lemma 4.1 and Lemma 4.2(ii),

$$\begin{aligned} G \cap H &\subseteq (G \cap H) * X \\ &\subseteq (G * X) \cap (H * X) \\ &= G \cap H, \end{aligned}$$

and so $(G \cap H) * X = G \cap H$, i.e., $G \cap H$ is an essence of X. Now by Lemma 4.1 and Lemma 4.2(iv), we get

$$\begin{aligned} G \cup H &\subseteq (G \cup H) * X \\ &= (G * X) \cup (H * X) \\ &= G \cup H, \end{aligned}$$

and thus $(G \cup H) * X = G \cup H$, i.e., $G \cup H$ is an essence of X. □

In general, we have the following observation.

Corollary 4.8. *If $\{G_i | i \in \Lambda \subset \mathbb{N}\}$ is a family of essences of a BH-algebra X, then $\cap_{i \in \Lambda} G_i$ and $\cup_{i \in \Lambda} G_i$ are essences of X.*

Generally, the union of two ideals of a BH-algebra may not be an ideal of X. For example, in Example 4.4, $G_1 := \{0, 1\}$ and $G_2 := \{0, 2\}$ are ideal of X, but $G_1 \cup G_2 = \{0, 1, 2\}$ is not an ideal of X, since $3 * 1 = 2 \in G_1 \cup G_2$ and $3 \notin G_1 \cup G_2$. But we know that the above Theorem 4.5 and Proposition 4.7 lead to the following result.

Corollary 4.9. *The intersection and union of two ideals of a BH^*-algebra X are essences of X.*

Proposition 4.10. *Let X and Y be BH-algebras. If G and H are essences of X and Y, respectively, then $G \times H$ is an essence of $X \times Y$.*

Proof. Since $(G \times H) * (X \times Y) = (G * X) \times (H * X) = G \times H$, we know that $G \times H$ is an essence of $X \times Y$. □

Let G be an essence and H be a subalgebra of a BH-algebra X. Then $G \cup H$ is not an essence of X in general as seen in the following example.

Example 4.11. (i) In Example 4.4, $G_1 := \{0, 1\}$ is an essence and $H := \{0, 3\}$ is a subalgebra of X, but $G \cup H = \{0, 1, 3\}$ is not an essence of X, since $3 * 1 = 2 \notin G \cup H$.
(ii) Let $X := \{0, 1, 2, 3, 4\}$ be a set with the following Cayley table:

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	1
2	2	2	0	0	2
3	3	2	1	0	3
4	4	4	4	4	0

Then $G := \{0, 1, 4\}$ is an essence of X and $H := \{0, 3\}$ is a subalgebra of X, but $G \cup H = \{0, 1, 3, 4\}$ is not an essence of X, since $3 * 1 = 2 \notin G \cup H$.

Proposition 4.12. *Let X be a BH-algebra. If G is an essence of X and H is a subalgebra of X, then $G \cap H$ is an essence of H.*

Proof. Using Lemma 4.1 and Lemma 4.2(i)-(ii), we have $(G \cap H) * H \subseteq (G * H) \cap (H * H) \subseteq (G * X) \cap H = G \cap H \subseteq (G \cap H) * H$, and so $(G \cap H) * H = G \cap H$. Therefore $G \cap H$ is an essence of H. □

Theorem 4.13. *Let X be a positive implicative BH^*-algebra. For any $a \in X$, $A(a)$ is an essence of X containing a.*

Proof. Obviously, $a \in A(a)$. Note that $A(a) \subseteq A(a) * X$ by Lemma 4.1. For any $y \in X$ and any $x \in A(a)$, we have $x \leq a$. By Lemma 3.6, $x * y \leq a * y$ and $a * y \leq a$. Hence $x * y \leq a$, i.e., $x * y \in A(a)$. Therefore $A(a) * X \subseteq A(a)$. Thus $A(a) * X = A(a)$, i.e., $A(a)$ is an essence of X. □

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