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FUNCTIONAL RELATIONS INVOLVING SARAN'S HYPERGEOMETRIC FUNCTIONS F_E AND $F^{(3)}$

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Abstract. By simply splitting the hypergeometric Saran function F_E into eight parts, we show how some useful and generalized relations between F_E and Srivastava's hypergeometric function $F^{(3)}$ can be obtained. These main results are shown to be specialized to yield certain relations between functions ${}_0F_1$, ${}_1F_1$, ${}_0F_3$, Ψ_2, and their products including different combinations with different values of parameters and signs of variables.

1. Introduction

Investigation of multiple hypergeometric functions is essentially motivated by the fact that the solutions of many applied problems involving the thermal conductivity and dynamics, electromagnetic oscillation and aerodynamics, quantum mechanics and potential theory are obtainable with the help of hypergeometric (higher and special or transcendent) functions (see [7, 13, 27, 29]). Such functions are often referred to as special functions in mathematical physics. They are mainly appeared in the solution of partial differential equations by using harmonic analysis method [11]. In view of various applications, it is important as well as interesting in itself to conduct a continuous research on multiple hypergeometric functions. In fact, in Srivastava and Karlsson's work [35], there is an extensive list of as many as 205 hypergeometric functions of three variables together with their region of convergence. It is noted that Riemann's functions and the fundamental solutions of the degenerate second-order partial differential equations are expressible by using hypergeometric functions of several variables (see [2, 4, 5, 6, 14, 15, 16, 17, 18, 19, 28, 31, 38, 39, 40]). For solutions of the boundary-value problems for the involved partial differential

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equations, we need to investigate certain properties of hypergeometric functions of several variables (see [20, 21, 22, 23, 24, 30, 36]).

Lardner [25] gave some connections between Bessel functions and hypergeometric ${}_0F_3$-series, for example,

$$ {}_0F_3\left(\frac{1}{2},\frac{1}{2},1;z\right)=\frac{1}{2}\left[J_0\left(4z^{\frac{1}{4}}\right)+I_0\left(4z^{\frac{1}{4}}\right)\right] \tag{1.1}$$

and

$$ \operatorname{ber}(x)={}_0F_3\left(\frac{1}{2},\frac{1}{2},1;-\frac{x^4}{256}\right),\quad \text{and}\quad \operatorname{bei}(x)=\frac{x^2}{4}{}_0F_3\left(\frac{3}{2},\frac{3}{2},1;-\frac{x^4}{256}\right), \tag{1.2}$$

where J_ν and I_ν denote a Bessel function and a modified Bessel function of order ν (see [1]; also [37]) defined by

$$ J_\nu(z)=\frac{\left(\frac{z}{2}\right)^\nu}{\Gamma(\nu+1)}\,{}_0F_1\left(-\,;\,\nu+1\,;\,-\frac{z^2}{4}\right)\text{ and }I_\nu(z)=\frac{\left(\frac{z}{2}\right)^\nu}{\Gamma(\nu+1)}\,{}_0F_1\left(-\,;\,\nu+1\,;\,\frac{z^2}{4}\right), \tag{1.3}$$

and ber(x) and bei(x) (x real) denote the Kelvin's functions (see [12, p. 6]) defined by

$$ \operatorname{ber}(x)+i\operatorname{bei}(x)=J_0\left(x\,e^{i\,\frac{3}{4}\,\pi}\right)=I_0\left(x\,e^{i\,\frac{1}{4}\,\pi}\right). \tag{1.4}$$

Carlson [8] generalized these results for arbitrary parameters to give the following results

$$ {}_0F_3\left(\frac{1}{2},c,c+\frac{1}{2};z\right)=\frac{1}{2}\,\Gamma(2c)\,\left(2z^{\frac{1}{4}}\right)^{1-2c}\left[I_{2c-1}\left(4z^{\frac{1}{4}}\right)+J_{2c-1}\left(4z^{\frac{1}{4}}\right)\right] \tag{1.5}$$

and

$$ {}_0F_3\left(\frac{3}{2},c,c+\frac{1}{2};z\right)=\frac{1}{2}\,\Gamma(2c)\,\left(2z^{\frac{1}{4}}\right)^{-2c}\left[I_{2c-2}\left(4z^{\frac{1}{4}}\right)-J_{2c-2}\left(4z^{\frac{1}{4}}\right)\right]. \tag{1.6}$$

Saran(1954) initiated a systematic study of these ten triple hypergeometric functions of Lauricella's set. One of them is presented as follows:

$$ F_E\left(a;b_1,b_2;c_1,c_2,c_3;x,y,z\right)=\sum_{m,n,p=0}^{\infty}\frac{(a)_{m+n+p}\,(b_1)_m\,(b_2)_{n+p}}{(c_1)_m\,(c_2)_n\,(c_3)_p\,m!n!p!}x^m y^n z^p \tag{1.7}$$

where $\mathbb{C}$ and $\mathbb{Z}_0^-$ denote the set of complex numbers and the set of nonpositive integers, respectively, and $(\lambda)_n$ is the Pochhammer symbol defined (for $\lambda\in\mathbb{C}$) by (see [34]):

$$ (\lambda)_n:=\begin{cases}1 & (n=0)\\ \lambda(\lambda+1)\ldots(\lambda+n-1) & (n\in\mathbb{N}:=\{1,2,3,\ldots\})\end{cases}$$
$$ =\frac{\Gamma(\lambda+n)}{\Gamma(\lambda)}\,(\lambda\in\mathbb{C}\backslash\mathbb{Z}_0^-),$$

$\Gamma(x)$ is the well-known Gamma function. Obviously, F_E is a new and interesting generalization of Appell's function F_2 and F_4. The three-dimensional region of convergence of (1.7) is given by Srivastava and Karlsson [35]: $(|x| < r,\ |y| < s,\ |z| < t, r + (\sqrt{s} + \sqrt{t})^2 = 1)$, where the positive quantities r, s and t are associated with radii of convergence.

Here, by simply splitting Saran's hypergeometric function F_E into eight parts, we show how some useful and generalized relations between Saran's hypergeometric function F_E and Srivastava's hypergeometric function $F^{(3)}$ can be obtained. As a particular case, some decomposition formulas for the generalized Saran's hypergeometric function $F^{(3)}$ were obtained by means of the generalized hypergeometric function and vice-versa(see [9, 10]). The other main results are shown to be specialized to yield certain relations between functions Ψ_2 and ${}_0F_1$, ${}_1F_1$, ${}_0F_3$.

2. Relationships between Saran's Hypergeometric Functions F_E and $F^{(3)}$

In this section we establish some interesting and useful identities associated with Saran's functions F_E and $F^{(3)}$. For this purpose we simply separate the summations in (1.7) into odd and even powers of each of x^m, y^n, and z^p. In fact, for any complex c_1, c_2, $c_3 \in \mathbb{C} \setminus \mathbb{Z}_0^-$, and any finite complex x, y, and z, the series F_E converges absolutely in the region of convergence and can therefore be rearranged as in the following eight summations:

$$
\begin{aligned}
F_E\left(a; b_1, b_2; c_1, c_2, c_3; x, y, z\right) = & \sum_{i,j,k=0}^{\infty} \frac{(a)_{2(i+j+k)}\,(b_1)_{2i}\,(b_2)_{2(j+k)}}{(c_1)_{2i}\,(c_2)_{2j}\,(c_3)_{2k}\,(2i)!\,(2j)!\,(2k)!} x^{2i} y^{2j} z^{2k} \\
& + x \sum_{i,j,k=0}^{\infty} \frac{(a)_{2(i+j+k)+1}\,(b_1)_{2i+1}\,(b_2)_{2(j+k)}}{(c_1)_{2i+1}\,(c_2)_{2j}\,(c_3)_{2k}\,(2i+1)!\,(2j)!\,(2k)!} x^{2i} y^{2j} z^{2k} \\
& + y \sum_{i,j,k=0}^{\infty} \frac{(a)_{2(i+j+k)+1}\,(b_1)_{2i}\,(b_2)_{2(j+k)+1}}{(c_1)_{2i}\,(c_2)_{2j+1}\,(c_3)_{2k}\,(2i)!\,(2j+1)!\,(2k)!} x^{2i} y^{2j} z^{2k} \\
& + z \sum_{i,j,k=0}^{\infty} \frac{(a)_{2(i+j+k)+1}\,(b_1)_{2i}\,(b_2)_{2(j+k)+1}}{(c_1)_{2i}\,(c_2)_{2j}\,(c_3)_{2k+1}\,(2i)!\,(2j)!\,(2k+1)!} x^{2i} y^{2j} z^{2k} \\
& + xy \sum_{i,j,k=0}^{\infty} \frac{(a)_{2(i+j+k+1)}\,(b_1)_{2i+1}\,(b_2)_{2(j+k)+1}}{(c_1)_{2i+1}\,(c_2)_{2j+1}\,(c_3)_{2k}\,(2i+1)!\,(2j+1)!\,(2k)!} x^{2i} y^{2j} z^{2k} \\
& + xz \sum_{i,j,k=0}^{\infty} \frac{(a)_{2(i+j+k+1)}\,(b_1)_{2i+1}\,(b_2)_{2(j+k)+1}}{(c_1)_{2i+1}\,(c_2)_{2j}\,(c_3)_{2k+1}\,(2i+1)!\,(2j)!\,(2k+1)!} x^{2i} y^{2j} z^{2k}
\end{aligned}
\tag{2.1}
$$

$$+yz\sum_{i,j,k=0}^{\infty}\frac{(a)_{2(i+j+k+1)}\,(b_1)_{2i}\,(b_2)_{2(j+k+1)}}{(c_1)_{2i}\,(c_2)_{2j+1}\,(c_3)_{2k+1}\,(2i)!\,(2j+1)!\,(2k+1)!}x^{2i}y^{2j}z^{2k}$$
$$+xyz\sum_{i,j,k=0}^{\infty}\frac{(a)_{2(i+j+k+1)+1}\,(b_1)_{2i+1}\,(b_2)_{2(j+k+1)}}{(c_1)_{2i+1}\,(c_2)_{2j+1}\,(c_3)_{2k+1}\,(2i+1)!\,(2j+1)!\,(2k+1)!}x^{2i}y^{2j}z^{2k}.$$

Now making use of the following well-known (or easily derivable) identities for the Pochhammer symbol (see [25, 26]):

$$(2m)! = 2^{2m}\left(\frac{1}{2}\right)_m m!,\ \ (2m+1)! = 2^{2m}\left(\frac{3}{2}\right)_m m!;$$

$$(\alpha)_{2m} = 2^{2m}\left(\frac{\alpha}{2}\right)_m\left(\frac{\alpha+1}{2}\right)_m,\ (\alpha)_{2m+1} = \alpha 2^{2m}\left(\frac{\alpha+1}{2}\right)_m\left(\frac{\alpha+2}{2}\right)_m$$

$(m \in \mathbb{N}_0 := \mathbb{N}\cup\{0\})$, after some simplification, we obtain

Theorem 1. *The following relationship between F_E and $F^{(3)}$ holds true.*

$$\begin{aligned}
&F_E\left(a;b_1,b_2;c_1,c_2,c_3;x,y,z\right)\\
&= F^{(3)}\left[\begin{array}{l} \frac{a}{2},\frac{a+1}{2} :: -;\ \frac{b_2}{2},\frac{b_2+1}{2};\ -;\ \frac{b_1}{2},\frac{b_1+1}{2};\ -;\ - :\\ - :: -;\ -;\ -;\ \frac{c_1}{2},\frac{c_1+1}{2},\frac{1}{2};\ \frac{c_2}{2},\frac{c_2+1}{2},\frac{1}{2};\ \frac{c_3}{2},\frac{c_3+1}{2},\frac{1}{2} :\end{array} x^2,y^2,z^2\right]\\
&+\frac{ab_1}{c_1}xF^{(3)}\left[\begin{array}{l} \frac{a+1}{2},\frac{a+2}{2} :: -;\ \frac{b_2}{2},\frac{b_2+1}{2};\ -;\ \frac{b_1+1}{2},\frac{b_1+2}{2};\ -;\ - :\\ - :: -;\ -;\ -;\ \frac{c_1+1}{2},\frac{c_1+2}{2},\frac{3}{2};\ \frac{c_2}{2},\frac{c_2+1}{2},\frac{1}{2};\ \frac{c_3}{2},\frac{c_3+1}{2},\frac{1}{2} :\end{array} x^2,y^2,z^2\right]\\
&+\frac{ab_2}{c_2}yF^{(3)}\left[\begin{array}{l} \frac{a+1}{2},\frac{a+2}{2} :: -;\ \frac{b_2+1}{2},\frac{b_2+2}{2};\ -;\ \frac{b_1}{2},\frac{b_1+1}{2};\ -;\ - :\\ - :: -;\ -;\ -;\ \frac{c_1}{2},\frac{c_1+1}{2},\frac{1}{2};\ \frac{c_2+1}{2},\frac{c_2+2}{2},\frac{3}{2};\ \frac{c_3}{2},\frac{c_3+1}{2},\frac{1}{2} :\end{array} x^2,y^2,z^2\right]\\
&+\frac{ab_2}{c_3}zF^{(3)}\left[\begin{array}{l} \frac{a+1}{2},\frac{a+2}{2} :: -;\ \frac{b_2+1}{2},\frac{b_2+2}{2};\ -;\ \frac{b_1}{2},\frac{b_1+1}{2};\ -;\ - :\\ - :: -;\ -;\ -;\ \frac{c_1}{2},\frac{c_1+1}{2},\frac{1}{2};\ \frac{c_2}{2},\frac{c_2+1}{2},\frac{1}{2};\ \frac{c_3+1}{2},\frac{c_3+2}{2},\frac{3}{2} :\end{array} x^2,y^2,z^2\right]+\frac{a\,(a+1)\,b_1b_2}{c_1c_2}x\,y\\
&\cdot F^{(3)}\left[\begin{array}{l} \frac{a+2}{2},\frac{a+3}{2} :: -;\ \frac{b_2+1}{2},\frac{b_2+2}{2};\ -;\ \frac{b_1+1}{2},\frac{b_1+2}{2};\ -;\\ - :: -;\ -;\ -;\ \frac{c_1+1}{2},\frac{c_1+2}{2},\frac{3}{2};\ \frac{c_2+1}{2},\frac{c_2+2}{2},\frac{3}{2};\end{array}\right.
\end{aligned}\tag{2.2}$$

$$\left.\begin{array}{l} -: \\ \frac{c_3}{2},\frac{c_3+1}{2},\frac{1}{2}: \end{array} x^2,y^2,z^2\right] + \frac{a(a+1)\,b_1b_2}{c_1c_3}\,x\,z$$

$$\cdot F^{(3)}\left[\begin{array}{llllll} \frac{a+2}{2},\frac{a+3}{2} :: & -; & \frac{b_2+1}{2},\frac{b_2+2}{2}; & -; & \frac{b_1+1}{2},\frac{b_1+2}{2}; & -; \\ - :: & -; & -; & -; & \frac{c_1+1}{2},\frac{c_1+2}{2},\frac{3}{2}; & \frac{c_2}{2},\frac{c_2+1}{2},\frac{1}{2}; \end{array}\right.$$

$$\left.\begin{array}{l} -: \\ \frac{c_3+1}{2},\frac{c_3+2}{2},\frac{3}{2}: \end{array} x^2,y^2,z^2\right] + \frac{a(a+1)\,b_2\,(b_2+1)}{c_2c_3}\,y\,z$$

$$\cdot F^{(3)}\left[\begin{array}{llllll} \frac{a+2}{2},\frac{a+3}{2} :: & -; & \frac{b_2+2}{2},\frac{b_2+3}{2}; & -; & \frac{b_1}{2},\frac{b_1+1}{2}; & -; \\ - :: & -; & -; & -; & \frac{c_1}{2},\frac{c_1+1}{2},\frac{1}{2}; & \frac{c_2+1}{2},\frac{c_2+2}{2},\frac{3}{2}; \end{array}\right.$$

$$\left.\begin{array}{l} -: \\ \frac{c_3+1}{2},\frac{c_3+2}{2},\frac{3}{2}: \end{array} x^2,y^2,z^2\right] + \frac{a(a+1)(a+2)\,b_1b_2\,(b_2+1)}{c_1c_2c_3}\,x\,y\,z$$

$$\cdot F^{(3)}\left[\begin{array}{llllll} \frac{a+3}{2},\frac{a+4}{2} :: & -; & \frac{b_2+2}{2},\frac{b_2+3}{2}; & -; & \frac{b_1+1}{2},\frac{b_1+2}{2}; & -; \\ - :: & -; & -; & -; & \frac{c_1+1}{2},\frac{c_1+2}{2},\frac{3}{2}; & \frac{c_2+1}{2},\frac{c_2+2}{2},\frac{3}{2}; \end{array}\right.$$

$$\left.\begin{array}{l} -: \\ \frac{c_3+1}{2},\frac{c_3+2}{2},\frac{3}{2}: \end{array} x^2,y^2,z^2\right],$$

where $F^{(3)}$ *is Srivastava's generalized hypergeometric function* (*see* [35]):

$$F^{(3)}\left[\begin{array}{lllllll} - :: & b_1,b_2; & b_1',b_2'; & b_1'',b_2''; & -; & -; & -; \\ - :: & -; & g_1,g_2; & -; & h_1,h_2,h_3; & h_1'; & h_1''; \end{array} x,y,z\right]$$

(2.3)

$$= \sum_{i,j,k=0}^{\infty} \frac{(b_1)_{i+j}\,(b_2)_{i+j}\,\left(b_1'\right)_{j+k}\left(b_2'\right)_{j+k}\left(b_1''\right)_{i+k}\left(b_2''\right)_{i+k}}{(g_1)_{j+k}\,(g_2)_{j+k}\,(h_1)_i\,(h_2)_i\,(h_3)_i\,\left(h_1'\right)_j\left(h_1''\right)_k\,i!j!k!}\,x^i y^j z^k.$$

Conversely, combining the signs of x, y and z in the definition of F_E, from (2.2) we readily express $F^{(3)}$ in terms of F_E's.

Theorem 2. *The following eight relationships between* $F^{(3)}$ *and* F_E *hold true.*

$$8F^{(3)}\left[\begin{array}{llllll} \frac{a}{2},\frac{a+1}{2} :: & -; & \frac{b_2}{2},\frac{b_2+1}{2}; & -; & \frac{b_1}{2},\frac{b_1+1}{2}; & -; \\ - :: & -; & -; & -; & \frac{c_1}{2},\frac{c_1+1}{2},\frac{1}{2}; & \frac{c_2}{2},\frac{c_2+1}{2},\frac{1}{2}; \end{array}\right.$$

(2.4)

$$\left.\begin{array}{l} -: \\ \frac{c_3}{2},\frac{c_3+1}{2},\frac{1}{2}: \end{array} x^2,y^2,z^2\right]$$

$$= F_E(x,y,z) + F_E(-x,y,z) + F_E(x,y,-z) + F_E(x,-y,z)$$
$$+F_E(-x,-y,z) + F_E(-x,y,-z) + F_E(x,-y,-z) + F_E(-x,-y,-z);$$

$$\frac{8ab_1x}{c_1}F^{(3)}\left[\begin{array}{llllll} \frac{a+1}{2},\frac{a+2}{2} :: & -; & \frac{b_2}{2},\frac{b_2+1}{2}; & -; & \frac{b_1+1}{2},\frac{b_1+2}{2}; & -; \\ - :: & -; & -; & -; & \frac{c_1+1}{2},\frac{c_1+2}{2},\frac{3}{2}; & \frac{c_2}{2},\frac{c_2+1}{2},\frac{1}{2}; \end{array}\right.$$

(2.5)

$$\left.\begin{array}{l} -: \\ \frac{c_3}{2},\frac{c_3+1}{2},\frac{1}{2}: \end{array} x^2,y^2,z^2\right]$$

$$= F_E(x,y,z) - F_E(-x,y,z) + F_E(x,-y,z) + F_E(x,y,-z)$$
$$-F_E(-x,-y,z) - F_E(-x,y,-z) + F_E(x,-y,-z) - F_E(-x,-y,-z);$$

$$\frac{8ab_2y}{c_2}F^{(3)}\left[\begin{array}{l} \frac{a+1}{2},\frac{a+2}{2}::-;\ \frac{b_2+1}{2},\frac{b_2+2}{2};\ -;\ \frac{b_1}{2},\frac{b_1+1}{2};\ -;\ -: \\ -::-;\ -;\ -;\ \frac{c_1}{2},\frac{c_1+1}{2},\frac{1}{2};\ \frac{c_2+1}{2},\frac{c_2+2}{2},\frac{3}{2};\ \frac{c_3}{2},\frac{c_3+1}{2},\frac{1}{2}: \end{array} x^2,y^2,z^2\right] \tag{2.6}$$

$$= F_E(x,y,z)+F_E(-x,y,z)-F_E(x,-y,z)+F_E(x,y,-z)$$
$$-F_E(-x,-y,z)+F_E(-x,y,-z)-F_E(x,-y,-z)-F_E(-x,-y,-z);$$

$$\frac{8ab_2z}{c_3}F^{(3)}\left[\begin{array}{l} \frac{a+1}{2},\frac{a+2}{2}::-;\ \frac{b_2+1}{2},\frac{b_2+2}{2};\ -;\ \frac{b_1}{2},\frac{b_1+1}{2};\ -;\ -: \\ -::-;\ -;\ -;\ \frac{c_1}{2},\frac{c_1+1}{2},\frac{1}{2};\ \frac{c_2}{2},\frac{c_2+1}{2},\frac{1}{2};\ \frac{c_3+1}{2},\frac{c_3+2}{2},\frac{3}{2}: \end{array} x^2,y^2,z^2\right] \tag{2.7}$$

$$= F_E(x,y,z)+F_E(-x,y,z)+F_E(x,-y,z)-F_E(x,y,-z)$$
$$+F_E(-x,-y,z)-F_E(-x,y,-z)-F_E(x,-y,-z)-F_E(-x,-y,-z);$$

$$\frac{8a(a+1)b_1b_2xy}{c_1c_2}F^{(3)}\left[\begin{array}{l} \frac{a+2}{2},\frac{a+3}{2}::-;\ \frac{b_2+1}{2},\frac{b_2+2}{2};\ -;\ \frac{b_1+1}{2},\frac{b_1+2}{2};\ -;\ -: \\ -::-;\ -;\ -;\ \frac{c_1+1}{2},\frac{c_1+2}{2},\frac{3}{2};\ \frac{c_2+1}{2},\frac{c_2+2}{2},\frac{3}{2};\ \frac{c_3}{2},\frac{c_3+1}{2},\frac{1}{2}: \end{array} x^2,y^2,z^2\right] \tag{2.8}$$

$$= F_E(x,y,z)-F_E(-x,y,z)-F_E(x,-y,z)+F_E(x,y,-z)$$
$$+F_E(-x,-y,z)-F_E(-x,y,-z)-F_E(x,-y,-z)+F_E(-x,-y,-z);$$

$$\frac{8a(a+1)b_1b_2xz}{c_1c_3}F^{(3)}\left[\begin{array}{l} \frac{a+2}{2},\frac{a+3}{2}::-;\ \frac{b_2+1}{2},\frac{b_2+2}{2};\ -;\ \frac{b_1+1}{2},\frac{b_1+2}{2};\ -;\ -: \\ -::-;\ -;\ -;\ \frac{c_1+1}{2},\frac{c_1+2}{2},\frac{3}{2};\ \frac{c_2}{2},\frac{c_2+1}{2},\frac{1}{2};\ \frac{c_3+1}{2},\frac{c_3+2}{2},\frac{3}{2}: \end{array} x^2,y^2,z^2\right] \tag{2.9}$$

$$= F_E(x,y,z)-F_E(-x,y,z)+F_E(x,-y,z)-F_E(x,y,-z)$$
$$-F_E(-x,-y,z)+F_E(-x,y,-z)-F_E(x,-y,-z)+F_E(-x,-y,-z);$$

$$\frac{8a(a+1)b_2(b_2+1)yz}{c_2c_3}F^{(3)}\left[\begin{array}{l} \frac{a+2}{2},\frac{a+3}{2}::-;\ \frac{b_2+2}{2},\frac{b_2+3}{2};\ -;\ \frac{b_1}{2},\frac{b_1+1}{2};\ -;\ -: \\ -::-;\ -;\ -;\ \frac{c_1}{2},\frac{c_1+1}{2},\frac{1}{2};\ \frac{c_2+1}{2},\frac{c_2+2}{2},\frac{3}{2};\ \frac{c_3+1}{2},\frac{c_3+2}{2},\frac{3}{2}: \end{array} x^2,y^2,z^2\right] \tag{2.10}$$

$$= F_E(x,y,z)+F_E(-x,y,z)-F_E(x,-y,z)-F_E(x,y,-z)$$
$$-F_E(-x,-y,z)-F_E(-x,y,-z)+F_E(x,-y,-z)+F_E(-x,-y,-z);$$

$$\frac{8a\,(a+1)\,(a+2)\,b_1b_2\,(b_2+1)}{c_1c_2c_3}x\,y\,zF^{(3)}\left[\begin{matrix} \frac{a+3}{2},\frac{a+4}{2} :: & -; & \frac{b_2+2}{2},\frac{b_2+3}{2}; & -; \\ - :: & -; & -; & -; \end{matrix}\right.$$

$$\left.\begin{matrix} \frac{b_1+1}{2},\frac{b_1+2}{2}\;; & -\;; & -\;: \\ \frac{c_1+1}{2},\frac{c_1+2}{2},\frac{3}{2}; & \frac{c_2+1}{2},\frac{c_2+2}{2},\frac{3}{2}; & \frac{c_3+1}{2},\frac{c_3+2}{2},\frac{3}{2}: \end{matrix}\; x^2,y^2,z^2\right] \tag{2.11}$$

$$= F_E\,(x,y,z) - F_E\,(-x,y,z) - F_E\,(x,-y,z) - F_E\,(x,y,-z)$$

$$+F_E\,(-x,-y,z) + F_E\,(-x,y,-z) + F_E\,(x,-y,-z) - F_E\,(-x,-y,-z)\,,$$

where, for simplicity, $F_E(x,y,z) := F_E\,(a;b_1,b_2;c_1,c_2,c_3;x,y,z)\,.$

3. Limiting Cases

Here we want to express the triple hypergeometric functions in terms of simpler hypergeometric functions. For this purpose we begin by providing functional relationships between a little simpler function of F_E and $F^{(3)}$ as in Corollary 1. Indeed, in order to use the method suggested in [8], employing the following transformations $a \sim 1/\varepsilon,\ x \sim \varepsilon x,\ y \sim \varepsilon y, z \sim \varepsilon z$ in identities (2.1) and (2.4) to (2.11), and taking the limit of the resulting identities as $\varepsilon \to 0$, we obtain

Corollary 1. *Each of the following relationships holds true.*

$${}_1F_1\,(b_1;c_1;x)\,\Psi_2\,(b_2;c_2,c_3;y,z)$$

$$= F^{(3)}\left[\begin{matrix} - :: & -; & \frac{b_2}{2},\frac{b_2+1}{2}; & -; & \frac{b_1}{2},\frac{b_1+1}{2}\;; & -; & -: \\ - :: & -; & -; & -; & \frac{c_1}{2},\frac{c_1+1}{2},\frac{1}{2}; & \frac{c_2}{2},\frac{c_2+1}{2},\frac{1}{2}; & \frac{c_3}{2},\frac{c_3+1}{2},\frac{1}{2}: \end{matrix}\;\frac{x^2}{4},\frac{y^2}{4},\frac{z^2}{4}\right]$$

$$+\frac{b_1x}{c_1}F^{(3)}\left[\begin{matrix} - :: & -; & \frac{b_2}{2},\frac{b_2+1}{2}; & -; & \frac{b_1+1}{2},\frac{b_1+2}{2}\;; & -; & -: \\ - :: & -; & -; & -; & \frac{c_1+1}{2},\frac{c_1+2}{2},\frac{3}{2}; & \frac{c_2+1}{2},\frac{c_2+2}{2},\frac{1}{2}; & \frac{c_3}{2},\frac{c_3+1}{2},\frac{1}{2}: \end{matrix}\right.$$

$$\left.\frac{x^2}{4},\frac{y^2}{4},\frac{z^2}{4}\right] \tag{3.1}$$

$$+\frac{b_2y}{c_2}F^{(3)}\left[\begin{matrix} - :: & -; & \frac{b_2+1}{2},\frac{b_2+2}{2}; & -; & \frac{b_1}{2},\frac{b_1+1}{2}\;; & -; & -: \\ - :: & -; & -; & -; & \frac{c_1}{2},\frac{c_1+1}{2},\frac{1}{2}; & \frac{c_2+1}{2},\frac{c_2+2}{2},\frac{3}{2}; & \frac{c_3}{2},\frac{c_3+1}{2},\frac{1}{2}: \end{matrix}\right.$$

$$\left.\frac{x^2}{4},\frac{y^2}{4},\frac{z^2}{4}\right]$$

$$+\frac{b_2z}{c_3}F^{(3)}\left[\begin{matrix} - :: & -; & \frac{b_2+1}{2},\frac{b_2+2}{2}; & -; & \frac{b_1}{2},\frac{b_1+1}{2}\;; & -; & -: \\ - :: & -; & -; & -; & \frac{c_1}{2},\frac{c_1+1}{2},\frac{1}{2}; & \frac{c_2}{2},\frac{c_2+1}{2},\frac{1}{2}; & \frac{c_3+1}{2},\frac{c_3+2}{2},\frac{3}{2}: \end{matrix}\right.$$

$$\left.\frac{x^2}{4},\frac{y^2}{4},\frac{z^2}{4}\right]$$

$$+\frac{b_1b_2xy}{c_1c_2}F^{(3)}\left[\begin{array}{llllll} -:: & -; & \frac{b_2+1}{2},\frac{b_2+2}{2}; & -; & \frac{b_1+1}{2},\frac{b_1+2}{2}\ ; & -; \quad -: \\ -:: & -; & -; & -; & \frac{c_1+1}{2},\frac{c_1+2}{2},\frac{3}{2}; & \frac{c_2+1}{2},\frac{c_2+2}{2},\frac{3}{2};\quad \frac{c_3}{2},\frac{c_3+1}{2},\frac{1}{2}: \end{array}\frac{x^2}{4},\frac{y^2}{4},\frac{z^2}{4}\right]$$

$$+\frac{b_1b_2xz}{c_1c_3}F^{(3)}\left[\begin{array}{llllll} -:: & -; & \frac{b_2+1}{2},\frac{b_2+2}{2}; & -; & \frac{b_1+1}{2},\frac{b_1+2}{2}\ ; & -; \quad -: \\ -:: & -; & -; & -; & \frac{c_1+1}{2},\frac{c_1+2}{2},\frac{3}{2}; & \frac{c_2}{2},\frac{c_2+1}{2},\frac{1}{2};\quad \frac{c_3+1}{2},\frac{c_3+2}{2},\frac{3}{2}: \end{array}\frac{x^2}{4},\frac{y^2}{4},\frac{z^2}{4}\right]$$

$$+\frac{b_2\left(b_2+1\right)yz}{c_2c_3}$$

$$\cdot F^{(3)}\left[\begin{array}{llllll} -:: & -; & \frac{b_2+2}{2},\frac{b_2+3}{2}; & -; & \frac{b_1}{2},\frac{b_1+1}{2}\ ; & -; \quad -: \\ -:: & -; & -; & -; & \frac{c_1}{2},\frac{c_1+1}{2},\frac{1}{2}; & \frac{c_2+1}{2},\frac{c_2+2}{2},\frac{3}{2};\quad \frac{c_3+1}{2},\frac{c_3+2}{2},\frac{3}{2}: \end{array}\frac{x^2}{4},\frac{y^2}{4},\frac{z^2}{4}\right]$$

$$+\frac{b_1b_2\left(b_2+1\right)xyz}{c_1c_2c_3}$$

$$\cdot F^{(3)}\left[\begin{array}{llllll} -:: & -; & \frac{b_2+2}{2},\frac{b_2+3}{2}; & -; & \frac{b_1+1}{2},\frac{b_1+2}{2}\ ; & -; \quad -: \\ -:: & -; & -; & -; & \frac{c_1+1}{2},\frac{c_1+2}{2},\frac{3}{2}; & \frac{c_2+1}{2},\frac{c_2+2}{2},\frac{3}{2};\quad \frac{c_3+1}{2},\frac{c_3+2}{2},\frac{3}{2}: \end{array}\frac{x^2}{4},\frac{y^2}{4},\frac{z^2}{4}\right];$$

(3.2)

$$8F^{(3)}\left[\begin{array}{llllll} -:: & -; & \frac{b_2}{2},\frac{b_2+1}{2}; & -; & \frac{b_1}{2},\frac{b_1+1}{2}\ ; & -; \quad -: \\ -:: & -; & -; & -; & \frac{c_1}{2},\frac{c_1+1}{2},\frac{1}{2}; & \frac{c_2}{2},\frac{c_2+1}{2},\frac{1}{2};\quad \frac{c_3}{2},\frac{c_3+1}{2},\frac{1}{2}: \end{array}\frac{x^2}{4},\frac{y^2}{4},\frac{z^2}{4}\right]$$

$$=\left[{}_1F_1\left(b_1;c_1;x\right)+{}_1F_1\left(b_1;c_1;-x\right)\right]\left[\Psi_2\left(b_2;c_2,c_3;y,z\right)+\Psi_2\left(b_2;c_2,c_3;y,-z\right)\right.$$
$$\left.+\Psi_2\left(b_2;c_2,c_3;-y,z\right)+\Psi_2\left(b_2;c_2,c_3;-y,-z\right)\right];$$

$$\frac{8b_1x}{c_1}F^{(3)}\left[\begin{array}{llllll} -:: & -; & \frac{b_2}{2},\frac{b_2+1}{2}; & -; & \frac{b_1+1}{2},\frac{b_1+2}{2}\ ; & -; \quad -: \\ -:: & -; & -; & -; & \frac{c_1+1}{2},\frac{c_1+2}{2},\frac{3}{2}; & \frac{c_2}{2},\frac{c_2+1}{2},\frac{1}{2};\quad \frac{c_3}{2},\frac{c_3+1}{2},\frac{1}{2}: \end{array}\frac{x^2}{4},\frac{y^2}{4},\frac{z^2}{4}\right]$$

(3.3)

$$=\left[{}_1F_1\left(b_1;c_1;x\right)-{}_1F_1\left(b_1;c_1;-x\right)\right]\left[\Psi_2\left(b_2;c_2,c_3;y,z\right)+\Psi_2\left(b_2;c_2,c_3;y,-z\right)\right.$$
$$\left.+\Psi_2\left(b_2;c_2,c_3;-y,z\right)+\Psi_2\left(b_2;c_2,c_3;-y,-z\right)\right];$$

(3.4)

$$\frac{8b_2y}{c_2}F^{(3)}\left[\begin{array}{llllll} -:: & -; & \frac{b_2+1}{2},\frac{b_2+2}{2}; & -; & \frac{b_1}{2},\frac{b_1+1}{2}\ ; & -; \quad -: \\ -:: & -; & -; & -; & \frac{c_1}{2},\frac{c_1+1}{2},\frac{1}{2}; & \frac{c_2+1}{2},\frac{c_2+2}{2},\frac{3}{2};\quad \frac{c_3}{2},\frac{c_3+1}{2},\frac{1}{2}: \end{array}\frac{x^2}{4},\frac{y^2}{4},\frac{z^2}{4}\right]$$

$$= \left[{}_1F_1\left(b_1;c_1;x\right)+{}_1F_1\left(b_1;c_1;-x\right)\right]\Big[\Psi_2\left(b_2;c_2,c_3;y,z\right)+\Psi_2\left(b_2;c_2,c_3;y,-z\right)$$
$$-\Psi_2\left(b_2;c_2,c_3;-y,z\right)-\Psi_2\left(b_2;c_2,c_3;-y,-z\right)\Big];$$

$$\frac{8b_2z}{c_3}F^{(3)}\left[\begin{array}{llllll} -::\ -; & \frac{b_2+1}{2},\frac{b_2+2}{2}; & -; & \frac{b_1}{2},\frac{b_1+1}{2}; & -; & -: \\ -::\ -; & -; & -; & \frac{c_1}{2},\frac{c_1+1}{2},\frac{1}{2}; & \frac{c_2}{2},\frac{c_2+1}{2},\frac{1}{2}; & \frac{c_3+1}{2},\frac{c_3+2}{2},\frac{3}{2}: \end{array}\ \frac{x^2}{4},\frac{y^2}{4},\frac{z^2}{4}\right] \tag{3.5}$$
$$= \left[{}_1F_1\left(b_1;c_1;x\right)+{}_1F_1\left(b_1;c_1;-x\right)\right]\Big[\Psi_2\left(b_2;c_2,c_3;y,z\right)+\Psi_2\left(b_2;c_2,c_3;y,-z\right)$$
$$-\Psi_2\left(b_2;c_2,c_3;-y,z\right)-\Psi_2\left(b_2;c_2,c_3;-y,-z\right)\Big];$$

$$\frac{8b_1b_2xy}{c_1c_2}\cdot F^{(3)}\left[\begin{array}{llllll} -::\ -; & \frac{b_2+1}{2},\frac{b_2+2}{2}; & -; & \frac{b_1+1}{2},\frac{b_1+2}{2}; & -; & -: \\ -::\ -; & -; & -; & \frac{c_1+1}{2},\frac{c_1+2}{2},\frac{3}{2}; & \frac{c_2+1}{2},\frac{c_2+2}{2},\frac{3}{2}; & \frac{c_3}{2},\frac{c_3+1}{2},\frac{1}{2}: \end{array}\ \frac{x^2}{4},\frac{y^2}{4},\frac{z^2}{4}\right] \tag{3.6}$$
$$= \left[{}_1F_1\left(b_1;c_1;x\right)-{}_1F_1\left(b_1;c_1;-x\right)\right]\Big[\Psi_2\left(b_2;c_2,c_3;y,z\right)+\Psi_2\left(b_2;c_2,c_3;y,-z\right)$$
$$-\Psi_2\left(b_2;c_2,c_3;-y,z\right)-\Psi_2\left(b_2;c_2,c_3;-y,-z\right)\Big];$$

$$\frac{8b_1b_2xz}{c_1c_3}F^{(3)}\left[\begin{array}{llllll} -::\ -; & \frac{b_2+1}{2},\frac{b_2+2}{2}; & -; & \frac{b_1+1}{2},\frac{b_1+2}{2}; & -; & -: \\ -::\ -; & -; & -; & \frac{c_1+1}{2},\frac{c_1+2}{2},\frac{3}{2}; & \frac{c_2}{2},\frac{c_2+1}{2},\frac{1}{2}; & \frac{c_3+1}{2},\frac{c_3+2}{2},\frac{3}{2}: \end{array}\ \frac{x^2}{4},\frac{y^2}{4},\frac{z^2}{4}\right] \tag{3.7}$$
$$= \Big[{}_1F_1\left(b_1;c_1;x\right)-{}_1F_1\left(b_1;c_1;-x\right)\Big]\Big[\Psi_2\left(b_2;c_2,c_3;y,z\right)+\Psi_2\left(b_2;c_2,c_3;y,-z\right)$$
$$-\Psi_2\left(b_2;c_2,c_3;-y,z\right)-\Psi_2\left(b_2;c_2,c_3;-y,-z\right)\Big];$$

$$\frac{8b_2\left(b_2+1\right)yz}{c_2c_3}\cdot F^{(3)}\left[\begin{array}{llllll} -::\ -; & \frac{b_2+2}{2},\frac{b_2+3}{2}; & -; & \frac{b_1}{2},\frac{b_1+1}{2}; & -; & -: \\ -::\ -; & -; & -; & \frac{c_1}{2},\frac{c_1+1}{2},\frac{1}{2}; & \frac{c_2+1}{2},\frac{c_2+2}{2},\frac{3}{2}; & \frac{c_3+1}{2},\frac{c_3+2}{2},\frac{3}{2}: \end{array}\ \frac{x^2}{4},\frac{y^2}{4},\frac{z^2}{4}\right] \tag{3.8}$$
$$= \Big[{}_1F_1\left(b_1;c_1;x\right)+{}_1F_1\left(b_1;c_1;-x\right)\Big]\Big[\Psi_2\left(b_2;c_2,c_3;y,z\right)-\Psi_2\left(b_2;c_2,c_3;y,-z\right)$$
$$-\Psi_2\left(b_2;c_2,c_3;-y,z\right)+\Psi_2\left(b_2;c_2,c_3;-y,-z\right)\Big];$$

(3.9)

$$\frac{8b_1b_2\left(b_2+1\right)xyz}{c_1c_2c_3} \cdot F^{(3)}\left[\begin{array}{c} -::\ -;\ \frac{b_2+2}{2},\frac{b_2+3}{2};\ -;\ \frac{b_1+1}{2},\frac{b_1+2}{2};\ -;\ -: \\ -::\ -;\ -;\ -;\ \frac{c_1+1}{2},\frac{c_1+2}{2},\frac{3}{2};\ \frac{c_2+1}{2},\frac{c_2+2}{2},\frac{3}{2};\ \frac{c_3+1}{2},\frac{c_3+2}{2},\frac{3}{2}: \end{array} \frac{x^2}{4},\frac{y^2}{4},\frac{z^2}{4}\right]$$

$$= \left[{}_1F_1\left(b_1;c_1;x\right) - {}_1F_1\left(b_1;c_1;-x\right)\right]\left[-\Psi_2\left(b_2;c_2,c_3;y,z\right) + \Psi_2\left(b_2;c_2,c_3;y,-z\right) + \Psi_2\left(b_2;c_2,c_3;-y,z\right) - \Psi_2\left(b_2;c_2,c_3;-y,-z\right)\right],$$

where

(3.10)

$$\begin{aligned} &{}_1F_1\left(b_1;c_1;x\right)\Psi_2\left(b_2;c_2,c_3;y,z\right) \\ &= \lim_{\varepsilon\to 0} F_E\left(\frac{1}{\varepsilon};b_1,b_2;c_1,c_2,c_3;\varepsilon x,\varepsilon y,\varepsilon z\right) \\ &= \sum_{m,n,p=0}^{\infty} \frac{\left(b_1\right)_m\left(b_2\right)_{n+p}}{\left(c_1\right)_m\left(c_2\right)_n\left(c_3\right)_p m!n!p!} x^m y^n z^p, \end{aligned}$$

Ψ_2 is Humbert's confluent hypergeometric function of two variables [33, pp. 26, Eq. (22)].

For further specializations we start with observing the following limits:

Setting $b_2 \sim 1/\varepsilon,\ y \sim \varepsilon y, z \sim \varepsilon z,$ in (3.1) to (3.9), and taking the limit of the resulting identities as $\varepsilon \to 0$, we get

Corollary 2. *Each of the following relationships holds true.*

(3.11)

$$\begin{aligned} &{}_1F_1\left(b_1;c_1;x\right){}_0F_1\left(-;c_2;y\right){}_0F_1\left(-;c_3;z\right) \\ &= \left[{}_2F_3\left(\frac{b_1}{2},\frac{b_1+1}{2};\frac{c_1}{2},\frac{c_1+1}{2},\frac{1}{2};\frac{x^2}{4}\right)\right. \\ &\quad \left. + \frac{b_1x}{c_1}{}_2F_3\left(\frac{b_1+1}{2},\frac{b_1+2}{2};\frac{c_1+1}{2},\frac{c_1+2}{2},\frac{3}{2};\frac{x^2}{4}\right)\right] \\ &\quad \times\left[{}_0F_3\left(-;\frac{c_2}{2},\frac{c_2+1}{2},\frac{1}{2};\frac{x^2}{16}\right) + \frac{y}{c_2}{}_0F_3\left(-;\frac{c_2+1}{2},\frac{c_2+2}{2},\frac{3}{2};\frac{y^2}{16}\right)\right] \\ &\quad \times\left[{}_0F_3\left(-;\frac{c_3}{2},\frac{c_3+1}{2},\frac{1}{2};;\frac{z^2}{16}\right) + \frac{z}{c_3}{}_0F_3\left(-;\frac{c_3+1}{2},\frac{c_3+2}{2},\frac{3}{2};\frac{z^2}{16}\right)\right]; \end{aligned}$$

(3.12)

$$8\,{}_2F_3\left(\frac{b_1}{2},\frac{b_1+1}{2};\frac{c_1}{2},\frac{c_1+1}{2},\frac{1}{2};\frac{x^2}{4}\right){}_0F_3\left(-;\frac{c_2+1}{2},\frac{c_2+2}{2},\frac{3}{2};\frac{y^2}{16}\right)$$

$$\times {}_0F_3\left(-;\frac{c_3+1}{2},\frac{c_3+2}{2},\frac{3}{2};\frac{z^2}{16}\right)$$
$$=\Big[{}_1F_1(b_1;c_1;x)+{}_1F_1(b_1;c_1;-x)\Big]\Big[{}_0F_1(-;c_2;y)+{}_0F_1(-;c_2;-y)\Big]$$
$$\times\Big[{}_0F_1(-;c_3;z)+{}_0F_1(-;c_3;-z)\Big];$$

(3.13)
$$8\frac{b_1x}{c_1}\,{}_2F_3\left(\frac{b_1+1}{2},\frac{b_1+2}{2};\frac{c_1+1}{2},\frac{c_1+2}{2},\frac{3}{2};\frac{x^2}{4}\right){}_0F_3\left(-;\frac{c_2}{2},\frac{c_2+1}{2},\frac{1}{2};\frac{y^2}{16}\right)$$
$$\times {}_0F_3\left(-;\frac{c_3}{2},\frac{c_3+1}{2},\frac{1}{2};\frac{z^2}{16}\right)$$
$$=\Big[{}_1F_1(b_1;c_1;x)-{}_1F_1(b_1;c_1;-x)\Big]\Big[{}_0F_1(-;c_2;y)+{}_0F_1(-;c_2;-y)\Big]$$
$$\times\Big[{}_0F_1(-;c_3;z)+{}_0F_1(-;c_3;-z)\Big];$$

(3.14)
$$8\frac{y}{c_2}\,{}_2F_3\left(\frac{b_1}{2},\frac{b_1+1}{2};\frac{c_1}{2},\frac{c_1+1}{2},\frac{1}{2};\frac{x^2}{4}\right){}_0F_3\left(-;\frac{c_2+1}{2},\frac{c_2+2}{2},\frac{3}{2};\frac{y^2}{16}\right)$$
$$\times {}_0F_3\left(-;\frac{c_3}{2},\frac{c_3+1}{2},\frac{1}{2};\frac{z^2}{16}\right)$$
$$=\Big[{}_1F_1(b_1;c_1;x)+{}_1F_1(b_1;c_1;-x)\Big]\Big[{}_0F_1(-;c_2;y)-{}_0F_1(-;c_2;-y)\Big]$$
$$\times\Big[{}_0F_1(-;c_3;z)+{}_0F_1(-;c_3;-z)\Big];$$

(3.15)
$$8\frac{z}{c_3}\,{}_2F_3\left(\frac{b_1}{2},\frac{b_1+1}{2};\frac{c_1}{2},\frac{c_1+1}{2},\frac{1}{2};\frac{x^2}{4}\right){}_0F_3\left(-;\frac{c_2}{2},\frac{c_2+1}{2},\frac{1}{2};\frac{y^2}{16}\right)$$
$$\times {}_0F_3\left(-;\frac{c_3+1}{2},\frac{c_3+2}{2},\frac{3}{2};\frac{z^2}{16}\right)$$
$$=\Big[{}_1F_1(b_1;c_1;x)+{}_1F_1(b_1;c_1;-x)\Big]\Big[{}_0F_1(-;c_2;y)+{}_0F_1(-;c_2;-y)\Big]$$
$$\times\Big[{}_0F_1(-;c_3;z)-{}_0F_1(-;c_3;-z)\Big];$$

(3.16)
$$8\frac{b_1xy}{c_1c_2}\,{}_2F_3\left(\frac{b_1+1}{2},\frac{b_1+2}{2};\frac{c_1+1}{2},\frac{c_1+2}{2},\frac{3}{2};\frac{x^2}{4}\right){}_0F_3\left(-;\frac{c_2+1}{2},\frac{c_2+2}{2},\frac{3}{2};\frac{y^2}{16}\right)$$
$$\times\,{}_0F_3\left(-;\frac{c_3}{2},\frac{c_3+1}{2},\frac{1}{2};\frac{z^2}{16}\right)$$
$$=\left[{}_1F_1\left(b_1;c_1;x\right)-{}_1F_1\left(b_1;c_1;-x\right)\right]\left[{}_0F_1\left(-;c_2;y\right)-{}_0F_1\left(-;c_2;-y\right)\right]$$
$$\times\left[{}_0F_1\left(-;c_3;z\right)+{}_0F_1\left(-;c_3;-z\right)\right];$$

(3.17)
$$8\frac{b_1xz}{c_1c_3}\,{}_2F_3\left(\frac{b_1+1}{2},\frac{b_1+2}{2};\frac{c_1+1}{2},\frac{c_1+2}{2},\frac{3}{2};\frac{x^2}{4}\right){}_0F_3\left(-;\frac{c_2}{2},\frac{c_2+1}{2},\frac{1}{2};\frac{y^2}{16}\right)$$
$$\times\,{}_0F_3\left(-;\frac{c_3+1}{2},\frac{c_3+2}{2},\frac{3}{2};\frac{z^2}{16}\right)$$
$$=\left[{}_1F_1\left(b_1;c_1;x\right)-{}_1F_1\left(b_1;c_1;-x\right)\right]\left[{}_0F_1\left(-;c_2;y\right)+{}_0F_1\left(-;c_2;-y\right)\right]$$
$$\times\left[{}_0F_1\left(-;c_3;z\right)-{}_0F_1\left(-;c_3;-z\right)\right];$$

(3.18)
$$8\frac{yz}{c_2c_3}\,{}_2F_3\left(\frac{b_1}{2},\frac{b_1+1}{2};\frac{c_1}{2},\frac{c_1+1}{2},\frac{1}{2};\frac{x^2}{4}\right){}_0F_3\left(-;\frac{c_2+1}{2},\frac{c_2+2}{2},\frac{3}{2};\frac{y^2}{16}\right)$$
$$\times\,{}_0F_3\left(-;\frac{c_3+1}{2},\frac{c_3+2}{2},\frac{3}{2};\frac{z^2}{16}\right)$$
$$=\left[{}_1F_1\left(b_1;c_1;x\right)+{}_1F_1\left(b_1;c_1;-x\right)\right]\left[{}_0F_1\left(-;c_2;y\right)-{}_0F_1\left(-;c_2;-y\right)\right]$$
$$\times\left[{}_0F_1\left(-;c_3;z\right)-{}_0F_1\left(-;c_3;-z\right)\right];$$

(3.19)
$$8\frac{b_1xyz}{c_1c_2c_3}\,{}_2F_3\left(\frac{b_1+1}{2},\frac{b_1+2}{2};\frac{c_1+1}{2},\frac{c_1+2}{2},\frac{3}{2};\frac{x^2}{4}\right){}_0F_3\left(-;\frac{c_2+1}{2},\frac{c_2+2}{2},\frac{1}{2};\frac{y^2}{16}\right)$$
$$\times\,{}_0F_3\left(-;\frac{c_3+1}{2},\frac{c_3+2}{2},\frac{3}{2};\frac{z^2}{16}\right)=\left[{}_1F_1\left(b_1;c_1;x\right)-{}_1F_1\left(b_1;c_1;-x\right)\right]$$
$$\times\left[{}_0F_1\left(-;c_2;y\right)+{}_0F_1\left(-;c_2;-y\right)\right]\left[{}_0F_1\left(-;c_3;z\right)-{}_0F_1\left(-;c_3;-z\right)\right].$$

4. Concluding Remarks

We note that in a specialized parameters we can easily obtain many interesting functional relations from the identities established here. For instance, at $x = 0$ or $z = 0$ from (2.2) and (2.4) to (2.11) we can get decompositions for Appell's functions F_2 and F_4 in terms of Srivastava's function $F^{(3)}$.
Applying this method to some other special functions, instead of Saran's hypergeometric function F_E and Srivastava's hypergeometric function $F^{(3)}$, defined by power series, interested readers can find certain other unexpected functional relations.

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