

ENTROPY RIGIDITY FOR METRIC SPACES

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ABSTRACT. This is a survey on the volume entropy and its rigidity of various metric spaces. This survey is aimed to summarize recent results as well as remaining open questions and possible directions on this subject.

1. INTRODUCTION

For any Riemannian manifold, the volume entropy is defined as the exponential growth rate of volumes of balls in the universal cover (see Definition 2.1). It is named as an entropy since it is equal to the topological entropy of the geodesic flow for a large class of Riemannian manifolds, namely all non-positively curved ones. Volume entropy is related to many other geometric and algebraic invariants such as Gromov's simplicial volume, the bottom of the spectrum of Laplacian, the Cheeger isoperimetric constant, the growth of fundamental groups, etc.

The minimal entropy rigidity conjecture due to Gromov and Katok states that among all Riemannian metrics on a closed n -dimensional Riemannian manifold of non-positive curvature, the locally symmetric metrics minimize the normalized volume entropy. (Here, the normalized entropy is the volume entropy multiplied by $Vol(M)^{(1/n)}$. Alternatively, one may consider all metrics of volume 1 and use unnormalized volume entropy.)

This conjecture was first shown by Katok for surfaces. Fifteen years later, Besson, Courtois and Gallot proved that for a manifold which carries a rank one symmetric metric, the normalized entropy is minimal if and only if the metric is rank one symmetric [1]. The conjecture is still open for general symmetric spaces of rank at

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least two, except the case where the manifold is locally a product of rank-1 symmetric spaces [11].

Another type of volume entropy rigidity, which we will call maximal entropy rigidity, works in the opposite direction, i.e. some special metrics can be characterized as those having maximal (non-normalized) entropy, among all Riemannian metrics with some constraint. A classical volume comparison shows that the volume entropy of a Riemannian manifold, with Ricci curvature bounded below by $-(n-1)$, is bounded above by $n-1$. Recently, Ledrappier and Wang showed that the equality case occurs if and only if the given space is hyperbolic [21].

There are two types of metric spaces more general than Riemannian manifolds for which we want to consider entropy rigidity in this paper. One is Hilbert geometry and the other is Tits building.

Hilbert geometry was introduced by David Hilbert related to his fourth problem on characterizing the metric spaces whose geodesics are straight lines. It is a metric space in a bounded convex open set $\Omega \subset \mathbb{R}^n$. When Ω is an ellipsoid, the Hilbert geometry is isometric to the hyperbolic space. Even when Ω is not an ellipsoid, if $\partial\Omega$ is sufficiently smooth, then some analog of the sectional curvature, namely the flag curvature, is equal to -1 [32]. More recently, Y. Benoist showed that if a discrete group Γ divides some properly convex set Ω in \mathbb{S}^n , then Γ is Gromov-hyperbolic (i.e. geodesic triangles are δ -thin for some $\delta > 0$) if and only if Ω is locally the graph of a quasisymmetric convex function [4]. In these senses, Hilbert geometries are considered as metric spaces similar to hyperbolic spaces. Crampon showed that if M is a compact n -dimensional manifold with a strictly convex projective structure, then its topological entropy is less than or equal to $n-1$, where the equality holds if and only if the structure is hyperbolic (see Section 3.3 for details). The results of Crampon and of Berck-Bernig-Vernicos which are similar to the volume comparison theorem indicate further that Hilbert geometries can be considered as spaces with Ricci curvature bounded from below. Note that there can be several good choices of volume on a Finsler manifold, for example an n -dimensional Hausdorff measure. (See [3] for more details.)

The second type of spaces we consider are Tits buildings. It is remarkable that not much attention has been paid so far to the natural question whether there exists an entropy rigidity for singular spaces. We may extend the definition of volume entropy in the previous paragraph to any piecewise Riemannian manifold without any change, for example to any finite-dimensional polyhedral complexes, especially

Tits buildings. The entropy rigidity question on buildings is a natural question in the sense that Bruhat-Tits building is an analog of symmetric space for Lie groups over non-archimedean local fields. As CAT(0)-metric spaces which are piecewise Riemannian but not Riemannian, they often give new insights in solving various problems in geometric group theory. See [8], [9] for results on quasi-isometry rigidity or conformal dimension of some buildings. See also [16] for a survey on problems related to automorphism group of buildings and more generally nonpositively curved polyhedral complexes.

This survey complements a series of survey papers on other various rigidity results for singular spaces, which are well collected in the lecture notes of the summer school in Grenoble held in 2004, published by SMF, including that of Calabi-Weil infinitesimal rigidity by G. Besson, Quasi-conformal rigidity and Mostow rigidity by M. Bourdon, quasi-isometry rigidity of groups by C. Drutu, and deformation rigidity on bounded cohomology by M. Burger and A. Iozzi [6].

We finish this survey with open questions, and we would like to conclude that even when restricted to these spaces, there are many open questions yet to be solved.

2. VARIOUS INVARIANTS RELATED TO ENTROPY

In this section, we recall various invariants of Riemannian manifolds related to volume entropy, which we will use to generalize the volume entropy rigidity question to metric spaces more general than Riemannian manifolds.

2.1. Topological entropy, Hausdorff dimension and critical exponent. Let M be a Riemannian manifold of non-positive curvature. Denote its universal cover by \widetilde{M} , a base point $x \in \widetilde{M}$, and a ball of radius T based at x by $B(x, T)$.

Definition 2.1. The volume entropy of M is defined as

$$h_{vol}(M) = \liminf_{T \rightarrow \infty} \frac{\log vol(B(x, T))}{T}.$$

It is easy to see that the above limit inf is a limit and the limit does not depend on the base point x .

Now instead of $vol(B(x, T))$, we take the cardinality $|B(x, T) \cap \pi_1(M).x|$ of the orbit of $\pi_1(M)$ in the ball, to get the growth of $\pi_1(M)$, called the *critical exponent* of $\pi_1(M)$,

$$\delta(\pi_1(M)) = \liminf_{T \rightarrow \infty} \frac{\log |B(x, T) \cap \pi_1(M).x|}{T}.$$

Let $X = T^1(M)$ be the universal cover of M , which is a metric space again with the metric, denoted by \bar{d} and defined by

$$\bar{d}((x, \vec{v}), (y, \vec{w})) = \sqrt{d(x, y)^2 + \sphericalangle(\vec{v}, t(\vec{w}))^2},$$

where $t(\vec{w})$ is the parallel transport of \vec{w} to x and \sphericalangle is the spherical angle between two vectors \vec{v} and $t(\vec{w})$. Let g_t be the geodesic flow on it, i.e. g_t assigns to a unit vector (x, \vec{v}) another unit vector (x_t, \vec{v}_t) where x_t is a point of distance t from x along the geodesic passing through x and have unit tangent vector \vec{v} at time zero, and \vec{v}_t is the unit tangent vector of the geodesic at x_t .

For positive constants δ and T , a subset $Y \subset X$ is called (T, δ) -spanning if for any point $x \in X$ there exists $y \in Y$ such that $d(g_t(x), g_t(y)) \leq \delta$ for all $0 \leq t \leq T$.

Definition 2.2. The *topological entropy* of the geodesic flow g_t on $X = T^1(M)$ is defined by

$$h_{top}(M) = \lim_{\delta \rightarrow 0} \limsup_{T \rightarrow \infty} \frac{\log N(T, \delta)}{T},$$

where $N(T, \delta)$ is the minimal cardinality of a (T, δ) -spanning set.

Finally, let $d_{\mathcal{H}}(\pi_1(M))$ be the Hausdorff dimension of the limit set of $\pi_1(M)$, where the limit set is the set of orbits $\overline{\pi_1(M).x} \cap \widetilde{\partial M}$ on the boundary of the universal cover.

Theorem 2.3. *If M is a compact manifold of negative curvature, then*

$$h_{vol}(M) = \delta(\pi_1(M)) = h_{top}(M) = d_{\mathcal{H}}(\pi_1(M)).$$

(See [28], [13], [33] for proofs.)

More recently, F. Dalbo, M. Peigne, J. Picaud, and A. Sambusetti showed that if M is non-compact but $1/4$ -pinched (i.e. when $-b^2 \leq \kappa(M) \leq -a^2$ and $b^2/a^2 \leq 4$), then the above theorem still holds. They also showed that $1/4$ is optimal by constructing examples of non-compact $(1/4 + \varepsilon)$ -pinched manifold for which the volume entropy is strictly larger than the critical exponent [13].

From Theorem 2.3, there are many possibilities to extend the question of finding minimal volume entropy in some parameter space of metrics. For example, when M is equipped with a noncompact finite volume Riemannian metric, we take the critical exponent as the entropy functional. For contact flows, Finsler metric and magnetic field flows, we take the topological entropy, and for geometrically finite manifold M , we can take the Hausdorff dimension of the limit set of $\pi_1(M)$ [12].

2.2. Measure-theoretic entropy and variational principle. In this subsection, we recall a fundamental relation between the topological entropy and measure-theoretic entropies, the measure-theoretic entropies being the entropy most often considered in dynamics, information theory and physics.

Suppose we are given a measure μ on the space $X = T^1(M)$ with Borel σ -algebra \mathcal{B} and the flow g_t . For a given finite partition \wp , the entropy of g_t with respect to \wp is

$$h_\mu(g_t, \wp) = \lim_{n \rightarrow \infty} \frac{1}{n} H(\wp_{-n}^g),$$

where $g = g_1$ is the time one map of the geodesic flow, and

$$H(\wp_{-n}^g) = - \sum_{\alpha \in \wp_{-n}^g} \mu(\alpha) \log \mu(\alpha)$$

is the entropy of the join partition $\wp_{-n}^g = \wp \vee g^{-1}(\wp) \vee \dots \vee g^{-n+1}(\wp)$.

Definition 2.4. The *entropy* of $(X, \mathcal{B}, g_t, \mu)$ is

$$h_\mu(g_t) = \sup\{h_\mu(g_t, \wp) : \wp \text{ is a measurable partition with } H(\xi) < \infty\}.$$

The variational principle says that the topological entropy is supremum of measure-theoretic entropies of all Borel probability measures invariant under the flow g_t .

$$h_{top}(g_t) = \sup\{h_\mu(g_t) : \mu \text{ is a } g_t\text{-invariant Borel probability measure}\}.$$

In some classical cases, a measure attaining the maximum entropy is known: notably in our situation when the given manifold is of non-positively curved, the measure is called Bowen-Margulis measure. Bowen-Margulis measure was later extensively used for all $CAT(-1)$ -spaces by T. Roblin for equidistribution problems.

Note that for Riemannian manifolds, there is another measure invariant under the geodesic flow: Liouville measure which is locally the product of the canonical volume form with the angular measure. See the last question in Section 4 for a question related to Liouville measure and Bowen-Margulis measure.

2.3. Other topological invariant related to growth. There is a long history of counting number of primitive closed geodesics. The main theme is that the exponential growth rate of such number is equal to the volume entropy, and more precisely the number of primitive closed geodesics of length $\leq t$ is asymptotically proportional to $\frac{\exp(h_{vol}t)}{h_{vol}t}$ where t tends to infinity. (See the thesis of G. Margulis [29] for compact negatively curved manifolds, and for more recent results, papers of A. Eskin-M. Mirzakhani [15] for Moduli spaces with Teichmüller flow, E. Makover

- J. McGowan [26] for random manifolds, Z. Lian - L.S. Young [24] for mappings of Hilbert spaces, etc.)

As the fundamental group is often quasi-isometric to the given manifold, the question of volume growth of the manifold can be asked for groups as well. There is a paper by Igor Rivin with an excellent survey on the growth of free groups [?].

Remark. Another invariant related to volume entropy, which we will not consider in the next sections, is Lyapunov exponent. (See Ruelle's inequality, and also papers by A. Manning [27], L. S. Young [33] for early developments.)

3. RECENT PROGRESSES

The famous barycenter map machinery of Besson-Courtois-Gallot was extended to a class of singular metric spaces called convex Riemannian amalgam, including cone-manifolds and the metric doubling of hyperbolic convex cores, by P. Storm. In this paper, we restrict ourselves to singular spaces which are buildings.

3.1. Volume entropy for graphs. Locally finite graphs have universal covering trees, which are 1-dimensional buildings. As a preliminary section to the next section, let us briefly summarize entropy rigidity for graphs.

Let X be a finite (unoriented) graph and \tilde{X} its universal cover, which is a locally finite tree. The volume entropy of X is the exponential growth rate of volume of metric balls as in Definition 2.1, where the volume is now the sum of length of edges (or part of the edges) in the metric balls. Let us denote the degree at each vertex x , i.e. the number of edges with vertex x , by $k_x + 1$. By a normalized length distance, we mean that the sum of lengths of edges equals one.

Theorem 3.1 ([25]). *Let X be a finite connected graph with degree at least three, i.e. $k_x \geq 2$. Then there is a unique normalized length distance minimizing the volume entropy $h_{vol}(d)$. The minimal volume entropy is*

$$h_{\min} = \sum_{x \in VX} (k_x + 1) \log k_x,$$

and the entropy minimizing length distance $d = d_\ell$ is given by

$$\forall e \in EX, \quad l(e) = \frac{\log(k_{i(e)} k_{t(e)})}{2 \sum_{x \in VX} (k_x + 1) \log k_x}.$$

The case when the degree is constant, i.e. when $k_x = c$, the above theorem was

already proved in a paper of I. Rivin [30] in 1999 and also independently by I. Kapovich and T. Nagnibeda [18] around the same time as the author.

I. Kapovich and T. Nagnibeda wanted the above result only for regular graphs as they worked in the context of Outer space. Recall that Outer space was introduced by Culler and Vogtmann as a free group analog of Teichmüller space of a Riemann surface. The group $Out(F_n)$ of outer automorphisms of free groups F_n on n generators naturally acts on the n -Outer space. A point of the n -Outer space is a metric graph together with a base point, whose fundamental group isomorphic to free group F_n .

Theorem 3.2 ([18]). *In n -Outer space, the volume entropy is minimized by any (regular) trivalent graph in the space, with the metric assigning the same length for every edge.*

Let us also remark that in general, a group acting on a tree does not necessarily act freely, so that the quotient is not a graph anymore but a (Bass-Serre) graph of groups. Theorem 3.1 is stated in this general context, where you need to have some extra weights depending on the cardinality of the vertex stabilizers. We refer to [25] for details.

Application. Note also that Theorem 3.1 was used for finding traffic equilibrium state of the transport network, where it is important to consider weights on edges [35].

3.2. Volume entropy for buildings. Let P be a Coxeter polyhedron in \mathbb{X}^n , where \mathbb{X}^n is either \mathbb{H}^n or \mathbb{E}^n (with its standard metric of constant curvature -1 and 0 , respectively). It is a compact, convex regular polyhedron each of whose dihedral angle is of the form π/m for some integer $m \geq 2$. Let (W, S) be the *Coxeter system* consisting of the set S of reflections of \mathbb{X}^n with respect to the faces of codimension 1 of P , and the group W of isometries of \mathbb{X}^n generated by S . It has the following finite presentation:

$$W = \langle s_i : s_i^2 = 1, [s_i s_j]^{m_{ij}} = 1 \rangle,$$

where the dihedral angle between s_i and s_j is π/m_{ij} .

A *polyhedral complex* Δ of type $(W, S) = (W(P), S(P))$ is a CW-complex such that there exists a morphism of CW-complex, called a *function type*, $\tau : \Delta \rightarrow P$, for which its restriction to any maximal cell is an isometry, and that for all $x_\infty \in \Delta_\infty$, any cell σ of Δ containing x_∞ is of finite volume or isometric to \mathbb{R} or \mathbb{R}^+ .

Definition 3.3. Let (W, S) be a Coxeter system of \mathbb{X}^n . A *building* Δ of type (W, S) is a polyhedral complex of type (W, S) , equipped with a maximal family of subcomplexes, called apartments, polyhedrally isometric to the tessellation of \mathbb{X}^n by P under W , satisfying the usual axioms for a building:

- (1) for any two cells of Δ , there is an apartment containing them,
- (2) for any two apartments $\mathcal{A}, \mathcal{A}'$, there exists a polyhedral isometry of \mathcal{A} to \mathcal{A}' fixing $\mathcal{A} \cap \mathcal{A}'$.

The *link* of a vertex x is a $(n - 1)$ -spherical building, whose vertices are the edges of Δ containing x , and two vertices (two edges of Δ) are connected by an edge if there is a 2-dimensional cell containing both edges of Δ , etc. The building Δ is a $CAT(\kappa)$ -space, with κ the curvature of \mathbb{X}^n and its links are $CAT(1)$ -spherical buildings.

Using some dynamics of suspension flow of a shift of finite type, F. Ledrappier and the author showed that the volume entropy equals the topological pressure on an apartment.

Now let Δ be a right-angled hyperbolic building. For example, the building Δ is called a Bourdon's building if P is a regular hyperbolic right-angled polyhedron. By showing that Liouville measure is not entropy maximizing, we obtained a strict lower bound as a consequence.

Theorem 3.4 ([20]). *Let X be a compact quotient of an n -dimensional right-angled hyperbolic building of type $(W(P), S(P))$. The Liouville measure does not coincide with the Bowen-Margulis measure, for any hyperbolic metric. Consequently, the following strict inequality holds:*

$$h_{\text{vol}}(X) > (n - 1) + \frac{c_n}{\text{vol}(T^1P)} \sum_F \log q(F) \text{vol}(F),$$

where c_n is a constant depending only on n and the sum is over all $(n-1)$ -dimensional faces of the polyhedron P .

Note that the non-strict inequality holds both for any building by the variational principle for topological pressure, similar to the variational principle mentioned in Section 2.2.

Volume entropies of more classical buildings are also explored, notably by E. Leuzinger. He characterized the volume entropy of Bruhat-Tits buildings in terms of the sum of the roots of the associated Lie group. Let us recall how a building is associated to some given Lie group.

For any prime p , the p -adic field \mathbb{Q}_p is the completion of \mathbb{Q} with respect to the metric induced by the discrete valuation $\nu(p^n a/b) = n$, where a, b are coprime, and $p \nmid a, b$:

$$d(x, y) = |x - y| = e^{-\nu(x-y)}.$$

For any power $q = p^n$ of a prime p , the field $\mathbb{F}_q((t))$ of formal Laurent series with coefficients in the finite field \mathbb{F}_q has discrete valuation

$$\nu\left(\sum_{j=-m}^{\infty} a_j t^j\right) = -m$$

where $a_{-m} \neq 0$. It is known that any non-archimedean local field (i.e. a local field such that $|x + y| \leq \max\{|x|, |y|\}$) is a finite extension of either \mathbb{Q}_p or $\mathbb{F}_q((t))$.

Let \mathbb{F} be a non-archimedean local field. Let \mathbf{G} be a connected, simply connected, semisimple linear algebraic group over \mathbb{F} and let $G = \mathbf{G}(\mathbb{F})$ be the group of \mathbb{F} -rational points of \mathbf{G} . We assume that the \mathbb{F} -rank, i.e. the dimension of any maximal \mathbb{F} -split torus, is at least 1. Let \mathbf{S} be such a maximal \mathbb{F} -split torus, and let \mathbf{N}, \mathbf{Z} be the normalizer and the centralizer of \mathbf{S} in \mathbf{G} . Let $N = \mathbf{N}(\mathbb{F})$.

Then there exists a subgroup $B \subset G$ called Iwahori subgroup such that

- (1) B and N generate G ,
- (2) the subgroup $B \cap N$ is normal in N , and
- (3) the quotient $W = N/T$ admits a set of generators S such that, (W, S) is a Coxeter system.

Such a BN-pair is called Euclidean, if the group W is a Euclidean Coxeter group. The Bruhat-Tits building associated to a Euclidean BN-pair is the poset of cosets of special subgroups. It satisfies the axioms of building in Definition 3.3.

Theorem 3.5. *The volume entropy of a Bruhat-Tits building quotiented by a discrete group of G with the combinatorial metric satisfies*

$$h_{\text{vol}}(\Delta) = 2\|\rho\|,$$

where 2ρ is the sum of positive roots of \mathbf{G} with respect to some Weyl chamber in a maximal \mathbb{F} -split torus.

See [23] for details. Note that in this theorem, the volume entropy is calculated for the canonical combinatorial metric, and that it might vary as one varies the metrics on Δ . On the other hand, Bruhat-Tits buildings are Euclidean buildings, and they have much less freedom on which kind of Euclidean metrics it can admit, compared to hyperbolic buildings.

3.3. Volume entropy rigidity for Finsler metric spaces. In this section, we summarize entropy rigidity results in Hilbert geometries which are examples of Finsler metric spaces.

Let Ω be a compact convex set of $\mathbb{P}^n(\mathbb{R})$, $n \geq 2$. The distance on Ω is defined by the cross ratio:

$$d_{\Omega}(x, y) = \frac{1}{2} |\ln[a, b, x, y]| = \frac{1}{2} \left| \ln \frac{d(a, x)d(b, x)}{d(a, y)d(b, y)} \right|,$$

where a, b are the intersection points of the line (xy) with the boundary $\partial\Omega$. Then (Ω, d_{Ω}) is a complete metric space. It has a Hilbert geometry, i.e. its tangent space has only a convex norm which is not necessarily a quadratic form.

The cross ratio is invariant under the projective transformation. Ω is an ellipse if and only if d_{Ω} is Riemannian, and that case reduces to the Klein model of hyperbolic geometry. On the other extreme, Ω is a convex polytope if and only if it is bi-Lipshitz equivalent to the Euclidean space ([10]). Now the volume entropy is defined by the exponential growth rate of volume of balls as before, where the volume of a ball is defined as the Hausdorff measure associated to the Hilbert metric. The proof of Manning in Theorem 2.3 works for strictly convex domain as well and one concludes that volume entropy is equal to the topological entropy of the geodesic flow.

Recall that geodesics are straight lines, thus there is a unique geodesic between any two given points if there is no plane section which contains two straight segments (see Appendix A in [14]). The geodesic flow g_t simply assigns to a point with direction $(x, [x\vec{x}^+])$ (here $x^+ \in \partial\Omega$), another point with direction $(x_t, [x_t\vec{x}^+])$, where $d_{\Omega}(x, x_t) = t$ and x_t is closer to x^+ than x is.

On the side of minimal volume entropy rigidity, Verovic noticed that the Gromov-Katok conjecture does not hold anymore in the set of Finsler metrics, i.e. there exists a Finsler metric whose volume entropy is smaller than the volume entropy of the symmetric metrics. It is still an open question how to characterize Finsler metrics which attain minimal volume entropy.

As for maximal volume entropy, Berck, Bernig and Vernicos showed that among all plane Hilbert geometries, the hyperbolic plane has maximal volume entropy. They showed that the volume entropy is bounded above by $2/(3-d)$, where d is the Minkowski dimension of the extremal set of K . They also proved higher dimensional version with some additional technical hypothesis [3].

On the other hand, for strictly convex domain, maximal entropy rigidity is solved by Crampon:

Theorem 3.6. *Let Ω be a strictly proper convex open set in $\mathbb{P}^n(\mathbb{R})$ and $M = \Gamma \backslash \Omega$ be a compact quotient by a discrete group $\Gamma \subset PGL(\mathbb{R}^n)$. Then the volume entropy $h_{vol}(\Omega, d_\Omega) \leq n - 1$ with equality if and only if Ω is an ellipsoid.*

Recall that strictly convex open sets with Hilbert geometry are exactly the ones that are Gromov-hyperbolic.

4. OPEN QUESTIONS

In this last section, we collect open questions related to entropy rigidity of various metric spaces. Most of them were mentioned in the AIM-ETH workshop on volume entropy rigidity mentioned in the acknowledgement. Unfortunately, I do not have records about which participants suggested the following questions.

4.1. Barycenter map for buildings. Recall that the proof of Besson-Courtois-Gallot uses the barycenter map and the Patterson-Sullivan measure on the boundary of \widetilde{M} . We have Patterson-Sullivan measure for hyperbolic buildings since they are $CAT(-1)$ -spaces. It is an open question whether there is a combinatorial barycenter map for buildings. Can one use it to prove Mostow rigidity?

4.2. Minimal volume entropy for Hilbert geometries. As summarized in Section 3.3, most results for Hilbert geometries are on maximal entropy rigidity. Minimal entropy rigidity is still open, and it is still not settled which kind of volume one needs to take for the question. Even any good lower bound on the volume entropy should be interesting. Note that J. Boland and F. Newberger has a result in this direction [7], where they consider the normalized volume entropy with another factor which is the n -th root of a measure of the distortion of the Finsler structure that equals 1 for Riemannian manifolds.

4.3. Topological mixing. Note that although positivity of volume entropy is not strictly related to mixing property of the geodesic flow, we know that for Riemannian manifolds, the ones with positive entropy are the ones with mixing geodesic flow. Start with a pinched negatively-curved Hadamard manifold and a discrete convex cocompact isometry group. On the quotient, is the geodesic flow, restricted to the non-wondering set of the quotient, topologically mixing?

4.4. Isospectrality. It is open whether isospectrality implies the same volume

entropy. We can paraphrase the question as “Can you hear the volume entropy of a compact negatively curved manifold?”.

4.5. Minimal volume entropy and Betti numbers. One aspect of volume entropy omitted in this survey is Gromov’s simplicial volume. He showed that the simplicial volume of a closed orientable Riemannian manifold is the volume of M divided by a constant that depends only on \widetilde{M} . For hyperbolic manifolds, he further showed that this constant is the volume of the ideal regular geodesic simplices in \mathbb{H}^n , which depends only on the dimension. As the simplicial volume can be considered as the volume of a cycle given by the ideal triangle, one can ask questions related to bounded cohomology. For example, given $n \geq 2$, does there exist a constant $c(n)$, depending only on n , such that all the l -Betti numbers of any n -dimensional closed aspherical manifold M are $\leq c(n) \min h_{vol}(M)$?

4.6. Liouville measure, harmonic measure and Bowen-Margulis measure.

Apart from Liouville measure and Bowen-Margulis measure defined already in Section 2.1, there is yet another canonical measure to consider : harmonic measure which is defined as the probability measures solving Dirichlet problem on \widetilde{M} . More precisely, it is known that there exists a unique function u_f on $\widetilde{M} \cup \widetilde{M}(\infty)$ such that $\Delta u_f = 0$ on \widetilde{M} and $u_f(z) \rightarrow f(\xi)$ when $z \rightarrow \xi, \xi \in \widetilde{M}(\infty)$. For any $x \in \widetilde{M}$, the map $f \mapsto u_f(x)$ is a positive linear functional on $C(\widetilde{M}(\infty))$. It defines a probability measure on $\widetilde{M}(\infty)$, the harmonic measure $\widetilde{\mu}_x$. It was shown by Ledrappier that for surfaces, harmonic measure and Bowen-Margulis measure coincide only when the curvature is constant. It is an open question for hyperbolic buildings whether two of any of these three measures coincide if and only if the volume entropy is minimized. (This conjecture is sometimes attributed to A. Katok, Sullivan and Kaimanovich.)

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