J. Korean Soc. Math. Educ. Ser. B: Pure Appl. Math. ISSN 1226-0657
Volume 19, Number 1 (February 2012), Pages 1–5

SURFACES OF REVOLUTION WITH MORE THAN ONE AXIS

Dong-Soo Kim [a,*] and Young Ho Kim [b]

Abstract. We study surfaces of revolution in the three dimensional Euclidean space $\mathbb{R}^3$ with two distinct axes of revolution. As a result, we prove that if a connected surface in the three dimensional Euclidean space $\mathbb{R}^3$ admits two distinct axes of revolution, then it is either a sphere or a plane.

1. Introduction

Spheres and planes are the most basic geometric objects in the theory of surfaces in the three dimensional Euclidean space $\mathbb{R}^3$. They are also members of the family of surfaces of revolution in $\mathbb{R}^3$. Most of all surfaces of revolution have just one axis of revolution, but spheres and planes admit two distinct (in fact, infinitely many) axes of revolution.

In this regards, it is natural to raise a question:

"Are there any other surfaces of revolution with more than one axis?"

In this note, we give a negative answer in an elementary manner as follows (cf. [1], p.10):

Theorem A. *Suppose that a connected surface M in the three dimensional Euclidean space $\mathbb{R}^3$ admits two distinct axes of revolution. Then, M is either a sphere or a plane.*

As a corollary of our theorem and a theorem in [2] (see also Theorem 3 in Section 2.), we immediately obtain

Theorem B. *Let M be a complete surface in $\mathbb{R}^3$. Then, M is either a sphere or a plane if and only if there exist three points of M such that every geodesic through them is a normal section.*

Received by the editors September 1, 2011. Revised January 17, 2012. Accepted Jan. 21, 2012.
2000 *Mathematics Subject Classification.* 53A05.
Key words and phrases. surface of revolution, axis, normal section, geodesic, sphere.
*Corresponding author.

2. Surfaces of Revolution

We now give some well-known definitions; surfaces of revolution, axes of revolution, vertices on a surface of revolution, and normal sections of a surface.

Definition 1. Let M be a connected surface in the 3-dimensional Euclidean space $\mathbb{R}^3$ and L a straight line in $\mathbb{R}^3$. For any point $p \in M$, let $\Pi(p)$ denote the plane passing through p and orthogonal to L. L is called an *axis of revolution* of M and M a surface of revolution with axis L if for any point $p \in M$, $M \bigcap \Pi(p)$ is a union of some circles centered at O, which is the intersection point of L and $\Pi(p)$. Here we regard the point O as a circle of radius 0.

The circles centered at O of positive radius are called the *parallels* of M. The intersection of M and the plane $\Pi(L, p)$ which contains L and passes through p are called a *meridian* of M through p. Every meridian of M is a plane curve and symmetric with respect to the axis. By revolving a meridian around the axis, we get the surface of revolution M.

Definition 2. Let M be a surface of revolution. A point $p \in M$ is called a *vertex* of M if p is an intersection point of M and an axis of revolution of M.

If a surface of revolution M is smooth, the tangent plane of M at $p \in M$ is generated by the orthogonal pair of tangent lines of the parallel and the meridian through p. In particular, at a vertex p of M, the tangent plane of M at p is the plane passing through p and orthogonal to the corresponding axis.

But, in this note, surfaces of revolution need not be smooth. As the surfaces of revolution generated by revolving polygonal lines around an axis show, they might have many singularities.

Now let M be a smooth surface in $\mathbb{R}^3$ and let p be a point of M. Let t be a nonzero tangent vector at p and n the normal vector at p. The curve given by intersection of M and the plane through p spanned by t and n is called a *normal section* of M at p in the t direction ([3]).

Finally we introduce a characterization of a smooth surface of revolution in terms of normal sections through a fixed point, which is proved by an elementary calculation.

Theorem 3 ([2]). *Let M be a complete surface of $\mathbb{R}^3$. Then, M is a surface of revolution with vertex p if and only if every geodesic through p is a normal section.*

3. Surfaces of Revolution with More Than One Axis

First, we give some lemmas which play a crucial role to prove the main theorem as follows.

Lemma 4. *Let L_1 and L_2 be two distinct axes of revolution of a surface M. Let R denote a rotation with respect to L_2. Then, $L_3 = R(L_1)$ is also an axis of revolution of M.*

Proof. Let p_3 be a point in M and Π_3 the plane orthogonal to L_3 passing through p_3. We denote by O_3 the intersection point of L_3 and Π_3. If we let $R^{-1}(O_3) = O_1$, $R^{-1}(\Pi_3) = \Pi_1$, and $R^{-1}(p_3) = p_1$, then we have $p_1 \in M$. Since Π_1 passes through p_1 and orthogonal to L_1, by definition $\Pi_1 \bigcap M$ is a union of some circles centered at O_1.

But, we have $R^{-1}(\Pi_3 \bigcap M) = R^{-1}(\Pi_3) \bigcap R^{-1}(M) = \Pi_1 \bigcap M$. Thus, we see that $\Pi_3 \bigcap M = R(\Pi_1 \bigcap M)$ is a union of some circles centered at O_3. Consequently, L_3 is an axis of revolution of M. □

Lemma 5. *Let L_1 and L_2 be two axes of revolution of M. If L_1 and L_2 are not parallel, then they meet each other.*

Proof. Suppose that the axes L_1 and L_2 are in skew position. We denote by $d_0 > 0$ and $\phi_0 \in (0, \frac{\pi}{2})$ the distance and the angle between them, respectively. By revolving L_1 around L_2 by angle $\theta \in [0, 2\pi]$, we get a revolutionary hyperboloid of one sheet. Let R_θ denote the rotation by angle θ. Then it follows from Lemma 4 that the straight line $R_\theta(L_1)$ becomes an axis of revolution of M. The distance (the angle, resp.) between L_1 and the new axis $R_\theta(L_1)$ increase monotonically from 0 to $2d_0$ (0 to $2\phi_0$, resp.) as θ increases from 0 to π. Repeating this process $n-$times so that $2n\phi_0 > \pi/2$, we can choose an axis of revolution L_3 of M such that the angle and the distance between L_1, L_3 are $\pi/2$ and $d > 0$, respectively.

Now let Π denote the plane containing L_1 and orthogonal to L_3, O the intersection point of Π and L_3, respectively. Then $C = \Pi \bigcap M$ is a meridian of M, hence it is symmetric with respect to L_1. On the other hand, since Π is orthogonal to the axis L_3, $C = \Pi \bigcap M$ is a union of circles centered at O. Hence we see that the axis L_1 must pass through the center O. Thus, the axes L_1 and L_3 intersect at O. This contradiction completes the proof. □

Lemma 6. *Let L_1 and L_2 be two distinct axes of revolution of a surface M such that they meet each other perpendicularly. Then, M is a sphere.*

Proof. Let Π_2 be the plane containing L_2 and perpendicular to L_1. Then, $\Pi_2 \bigcap M$ is nonempty, because Π_2 contains an axis of revolution. Hence it is a union of some circles centered at $O = L_1 \bigcap L_2$. Since M is a 2-dimensional connected surface and L_2 is an axis of revolution, $\Pi_2 \bigcap M$ must be a single circle of positive radius. Revolving the circle around L_2 gives rise to a sphere. □

Now we prove Theorem A. Suppose that a surface of revolution M admits two distinct axes of revolution L_1 and L_2. By Lemma 5, L_1 and L_2 are either parallel or meet each other.

First suppose that they meet each other. Then by Lemma 6, we may assume that the angle between L_1 and L_2 is $\theta_0 \in (0, \pi/2)$. Lemma 4 implies that any straight line L_3 on the cone generated by revolving L_1 around L_2 is an axis of revolution of M. Therefore, for every $\theta \in (0, 2\theta_0)$, there exists a straight line L_3 that is an axis of revolution of M such that the angle between L_1 and L_3 is θ. Repeating this process $n-$times so that $2n\theta_0 > \pi/2$, we can choose an axis of revolution L of M such that the angle between L_1 and L is $\pi/2$. Therefore, by Lemma 5, M is a sphere.

Now suppose that L_1 and L_2 are parallel axes of revolution of M. Let U be a vector parallel to L_1 and let $d_0 = d(L_1, L_2)$ denote the distance between L_1 and L_2. By Lemma 4, every straight line L_3 parallel to U on the circular cylinder generated by revolving L_1 around L_2 becomes an axis of revolution of M. Therefore for every $d \in (0, 2d_0)$, there exists an axis L_3 of M which is parallel to U with $d = d(L_1, L_3)$. Repeating this process, for any $d > 0$ we may choose an axis of revolution L of M with $d = d(L_1, L)$ such that L is parallel to U. Hence Lemma 4 again shows that every straight line parallel to U becomes an axis of revolution of M.

For a fixed point $p \in M$, let Π be the plane passing through p and orthogonal to U. For any point $q \in \Pi$, let L be the straight line parallel to U and passing through the midpoint O of the segment pq. Since L is an axis of revolution of M and Π is the plane orthogonal to L and passing through $p \in M$, the parallel of M passing through p with respect to L is just the circle passing through p and q and centered at O. Hence q is contained in M. Thus M is nothing but the plane Π. This completes the proof of Theorem A.

Finally note that any axes of revolution of M can contain at most two vertices. Thus, Theorem A together with Theorem 3 implies Theorem B.

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[a]Department of Mathematics, Chonnam National University, Kwangju 500-757, Korea
Email address: dosokim@chonnam.ac.kr

[b]Department of Mathematics, College of Natural Sciences, Kyungpook National University, Taegu, 702-701, Korea
Email address: yhkim@knu.ac.kr